GRAPH THEORY

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PREFACE

Many of the technical papers that have appeared in recent years contain words related to linear graph theory such as "topological," "graph theoretical," "cut-sets," and "trees" in their titles. This is no accident. The theory of linear graphs, itself currently in the process of mathematical development, is being applied in a variety of apparently unrelated fields such as engineering system science, social science and human relations, business administration and scientific management, political science, chemistry, and psychology.

The purpose of this book is not only to provide an introduction to the fascinating study of linear graph theory but to bring the reader far enough along the way to enable him to embark on a research problem of his own, whether it be in the theory of linear graphs itself or in one of its manifold applications.

It would be impossible to discuss all of the applications of linear graphs in one book; instead we will concentrate on electrical network theory, switching theory, communication net and transportation theory, and system diagnosis.

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INTRODUCTION

There are many physical systems whose performance depends not only on the characteristics of the components but also on the relative locations of the elements. An obvious example is an electrical network. If we change a resistor to a capacitor, generally some of the properties (such as an input impedance of the network) also change. This indicates that the performance of a system depends on the characteristics of the components. If, on the other hand, we change the location of one resistor, the input impedance again may change, which shows that the topology of the system is influencing the system's performance. There are systems constructed of only one kind of component so that the system's performance depends only on its topology. An example of such a system is a single-contact switching circuit. Similar situations can be seen in nonphysical systems such as structures of administration. Hence it is important to represent a system so that its topology can be visualized clearly.

One simple way of displaying a structure of a system is to draw a diagram consisting of points called "vertices" and line segments called "edges" which connect these vertices so that the vertices and edges indicate components and relationships between these components. Such a diagram is a linear graph. A linear graph often is known by another name, depending on the kind of physical system we are dealing with; it may be called a network, a net, a graph, a circuit, a diagram, a structure, and so on.

Instead of indicating the physical structure of a system, we frequently indicate its mathematical model or its abstract model by a linear graph. Under such a circumstance, a linear graph is referred to as a flow graph, a signal flow graph, a flow chart, a state diagram, a simplical complex, a sociogram, an organization diagram, and so forth.

The earliest known paper on linear graph theory (1736) is due to Euler, who gave a solution to the Könisberger bridge problem by introducing the concept of linear graphs. In 1847, Kirchhoff employed linear graph theory for an analysis of electrical networks, known today as the topological formulas for driving point impedances and transfer admittances. This probably is the first paper that applies the theory of linear graphs to engineering problems. However, it is not Kirchhoff's paper but Möbins' conjecture (about 1840) concerning the four-color problem that seems to attract many scholars to devote themselves to linear graph theory.

The four-color problem is to prove or disprove that four colors are sufficient to color any planar map such that no two adjacent regions have the same color. Place one vertex inside of each region of a planar map, and then connect two vertices by an edge if and only if the regions containing these vertices are adjacent to each other. These operations yield what is known as a planar graph. In other words, for a given planar map, there is a planar graph such that a region of the map corresponds to a vertex in the linear graph and the boundary between two regions corresponds to the edge connected between two vertices which represent these two regions. We may restate the four-color problem as follows: Prove or disprove that four colors are sufficient to color vertices of any planar graph such that no two vertices connected by an edge have the same color.

If you try to attack the four-color problem, you will immediately face the difficulty of distinguishing planar graphs from nonplanar graphs, and you will start to study the properties of planar graphs. In spite of the work done by Kuratowski and Whitney, who discovered fundamental properties of planar graphs, the four-color problem is still unsolved and attracting many scholars to devote themselves to search for more properties of linear graphs. Of course, some properties have been found because of the necessity of specific applications.

The first part of this book is the study of properties of linear graphs for beginners. This does not mean that we are studying only elementary and simple properties. In fact, it covers the most advanced materials such as the following:

1. The property of collection $\{P_{ij}\}$ of paths, suitable for generating all possible paths, and properties among collections of paths.
2. How to generate cut-sets; especially, how to generate all possible cut-sets separating two specific vertices. These cut-sets are very important in communication nets and traffic systems.
3. Proofs of realizability conditions of cut-set matrices including Tutte's condition.
4. A proof of Kuratowski's conditions for nonplanar graphs and a proof of Whitney's condition for planar graphs (duality).
5. An introduction to pseudo-cuts, which become the dual of paths when a linear graph is planar.
6. An algorithm for testing the existence of directed circuits in oriented linear graphs.

7. The development of two types of generation of all possible trees without duplications.

When we discuss applications of linear graphs, we often use weighted linear graphs where edges and (or) vertices have specifications known as weights. For example, we can represent a passive linear bilateral lumped electrical network by a linear graph where each edge has three weights (e.g., voltage, current, and a proportionality factor) as discussed in Chapter 7. To study maximum flow in a communication net or a traffic system, the corresponding linear graph often needs only one weight on each edge indicating the maximum capability of handling traffic by the edge (in Chapter 12). We can see that suitable weighted linear graphs can represent many other systems such as switching circuits, logic circuits, air traffic networks, and computer systems.

There are cases when such a weighted linear graph may be used only for representation of a system. However, in this book, we study how to employ weighted linear graphs in order to analyze systems, particularly one such method known as the topological method of analysis. There are two distinct types of topological method; one is to use rules known as topological formulas so that the property of the system that is in question can be studied directly from a weighted linear graph, and the other is to employ so-called equivalent transformations successively so that a weighted linear graph of a system will be simplified to consist of only one edge whose weight indicates the property. Examples of the first type include the following: (1) calculate electrical network functions by finding all possible special subgraphs of the weighted linear graph corresponding to a given linear lumped electrical network (given in Chapters 7 and 8); (2) find the maximum flow by locating a so-called minimum cut in a communication net and a traffic system (shown in Chapter 12); and (3) obtain a switching function by finding all possible paths between specified terminals in a switching circuits (discussed in Chapter 11). Some examples of the second type are (1) the node elimination technique to obtain a simpler signal flow graph (indicated in Chapter 10) and (2) equivalent transformations for a linear electrical network (given in Chapter 8).

The topological analysis of the first type gives a clear relationship between a property of a system and the locations of edges (components). In several cases, this relationship is enough to design or improve a system which satisfies a given specification, and this is called a topological synthesis of systems. The synthesis of switching functions and that of communication nets (in Chapters 11, 12, and 13) are good examples. The system diagnosis discussed in Chapter 14 again indicates that linear graph theory is an essential tool in the system theory area.

CHAPTER 1
NONORIENTED LINEAR GRAPH

1-1 INTRODUCTION

In this chapter, some properties of paths and circuits of a nonoriented linear graph are discussed. The paths and circuits are rather important subgraphs of linear graphs. For example, we will see later that paths determine the properties of switching networks, and circuits are related to Kirchhoff's voltage law in electrical network theory.

For defining linear graphs, it would be easier if we consider the familiar tetrahedron shown in Fig. 1-1-1. There are four vertices 1, 2, 3, and 4 and six

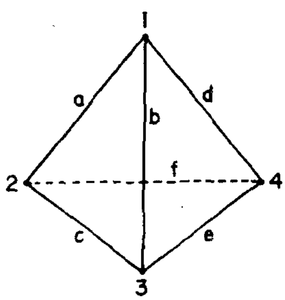

Fig. 1-1-1. A *tetrahedron.*

edges, a, b, c, d, e, and f. Each edge is located between two vertices; edge a is between vertices 1 and 2, edge b is between 1 and 3, edge c is between 2 and 3, and so on. In combinatorial topology, a collection of vertices and edges is known as a simplical one-dimensional (*line*ar) complex, which we call a linear graph. However, the definitions of vertices and edges are more general than those of polyhedrons.

To expand the concept of vertices and edges geometrically, consider an n-dimensional Euclidean space. First, a vertex is a point in the space. With a given set Ω of vertices, an edge e is a curve between two vertices v and v' in Ω which passes no other vertices in Ω. The vertices v and v' are called the endpoints of edge e. When v and v' are the same, then edge e as shown in Fig. 1-1-2*a* is called a "self-loop." If we give a direction to a curve as shown in Fig. 1-1-2*b*, then an edge represented by the curve is called an oriented edge. Otherwise, it is a nonoriented edge.

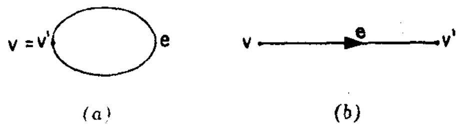

Fig. 1-1-2. Edges. (*a*) Self-loops; (*b*) oriented edge.

We may now use the preceding geometrical concept to derive an abstract definition of edges, vertices, and linear graphs. Instead of chosen points (in a space) being vertices, we take a set such that members in the set are given vertices. Geometrically, an edge is a curve between two vertices. Since there are no other vertices in the curve, we can consider an edge to correspond to a pair of vertices. On the other hand, we would like to allow several edges having the same endpoints. Hence the definition of vertices edges and a linear graph become abstractly as follows.

Definition 1-1-1. Let $\mathscr{E}$ and Ω be sets. If every $e \in \mathscr{E}$* corresponds to exactly one pair (v, v') where $v, v' \in \Omega$, then every member in $\mathscr{E}$ is an *edge*, every member in Ω is a *vertex*, and $\mathscr{E} \cup \Omega$† is a *linear graph*.

With this definition, the endpoints, oriented edges, and nonoriented edges are defined abstractly as follows.

Definition 1-1-2. Let e be an edge corresponding to a pair (v, v') of vertices. Then the two vertices v and v' are called the endpoints of edge e. If the pair (v, v') is ordered, then e is said to be ***oriented*** or called an ***oriented edge***. Otherwise, e is said to be ***nonoriented*** or is called a ***nonoriented edge***.

Furthermore, we define oriented and nonoriented (linear) graphs as follows.

Definition 1-1-3. If all edges in a linear graph are oriented, then the linear graph is said to be oriented or called an oriented (linear) graph. If all edges are nonoriented, a linear graph is said to be nonoriented or called a nonoriented (linear) graph.

* The symbol $\in$ means "belong to" or "in". $e \in \mathscr{E}$ means e in $\mathscr{E}$.
† $\mathscr{E} \cup \Omega$ is a set of all members in either $\mathscr{E}$ or Ω or both.

Example 1-1-1. We use the symbol $\alpha \to \beta$ for indicating that α corresponds to β. Consider two sets $\mathscr{E} = (a, b, c, d, e, f, g)$ and $\Omega = (1, 2, 3, 4, 5)$ where

$$\begin{aligned} a &\to (1, 2) \\ b &\to (1, 3) \\ c &\to (2, 3) \\ d &\to (1, 4) \\ e &\to (1, 4) \\ f &\to (4, 2) \end{aligned}$$

and

$$g \to (2, 4)$$

Since each member in $\mathscr{E}$ corresponds to exactly one pair of vertices in Ω, a, b, c, d, e, f, and g are edges, 1, 2, 3, 4, and 5 are vertices, and $\mathscr{E} \cup \Omega$ is a linear graph by Definition 1-1-1. Vertices 1 and 2 are the endpoints of edge a, vertices 1 and 3 are the endpoints of edge b, and so on (Definition 1-1-2).

Instead of using the symbol $\alpha \to \beta$ as in the foregoing example, we use a drawing to indicate edges and the corresponding pairs of vertices. For this, we make the following agreements.

1. A vertex will be indicated by a small circle. When the name of a vertex must be indicated, it will be written either at a place near a circle or inside the circle as shown in Fig. 1-1-3*a* and *b*.

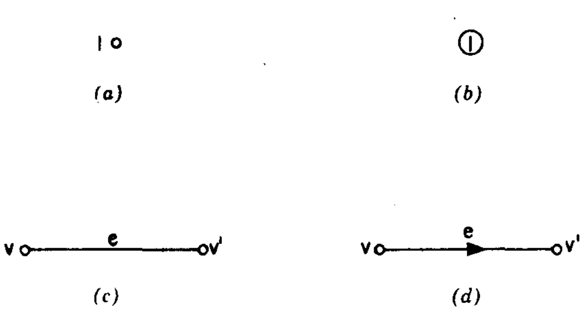

Fig. 1-1-3. Representation of vertices and edges. (*a*) Representation of a vertex 1; (*b*) representation of a vertex 1; (*c*) nonoriented edge e (v, v'); (*d*) oriented edge e (v, v').

2. When an edge is nonoriented, the edge is represented by a line between two vertices which are the endpoints of the edge. The name of an edge will be given at a place near the line if needed. As an example, a nonoriented edge $e \to (v, v')$ will be represented by a line shown in Fig. 1-1-3*c*.
3. When an edge is oriented, the edge will be represented by a line with an

arrow to indicate its orientation. As an example, the representation of an oriented edge $e \rightarrow (v, v')$ is shown in Fig. 1-1-3*d*.

Note that when v and v' of $e \rightarrow (v, v')$ are the same, then edge e is a self-loop and the line representing edge e is a loop starting from vertex v (a small circle representing vertex v) and terminated at the same vertex.

Since we need know only a pair of vertices for each edge, a shape of line representing an edge is immaterial. For example, an edge in Fig. 1-1-3*c* and those in Fig. 1-1-4*a* and *b* represent the same edge.

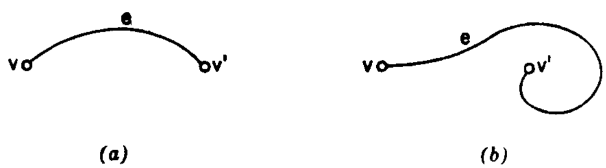

Fig. 1-1-4. Representation of edge e (v, v'). (*a*) Edge e, (*b*) edge e.

In a drawing, crossing points of edges other than those represented by small circles are also immaterial. For example, even though edges a and b in Fig. 1-1-5*a* are crossing each other, this drawing indicates only that non-

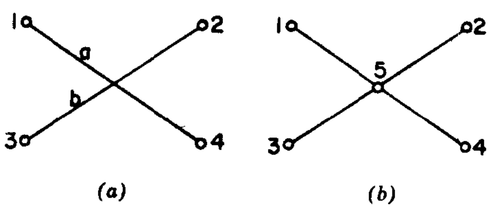

Fig. 1-1-5. Crossing of edges. (*a*) Edges a and b; (*b*) four edges

oriented edge a corresponds to pair (1, 4) and nonoriented edge b corresponds to pair (2, 3). Note the difference between those in Fig. 1-1-5*a* and *b*.

With these agreements, we can specify a linear graph by a drawing. As an example, the linear graph $\mathscr{E} \cup \Omega$ in Example 1-1-1, where $\mathscr{E} = (a, b, c, d, e, f, g)$ and $\Omega = (1, 2, 3, 4, 5)$ with each pair of vertices being considered as a non-ordered pair, can be represented by Fig. 1-1-6. Note that vertex 5 is not in any pair in this example. Hence there will be no lines connected to vertex 5.

Since a drawing gives a clear and simple representation of a linear graph, we use such drawings for specifying linear graphs in this book.

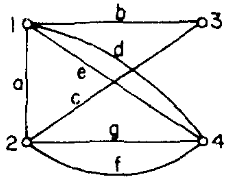

Fig. 1-1-6. A linear graph.

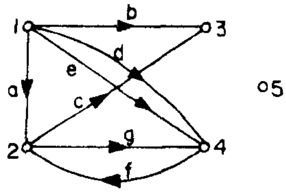

Fig. 1-1-7. An oriented graph.

With Definitions 1-1-2 and 1-1-3, we can see that a linear graph in Fig. 1-1-6 is nonoriented and that in Fig. 1-1-7 is an oriented (linear) graph. Note that Fig. 1-1-7 is the linear graph when all pairs of vertices in Example 1-1-1 are ordered pairs.

Until the later chapters, we will consider only nonoriented linear graphs. In other words, in the next few chapters, a *linear graph means a nonoriented linear graph.*

1-2 PATHS AND CIRCUITS

An edge is said to be *incident* or *connected* at a vertex if the vertex is one of the two endpoints of the edge. For example, edges a, b, and c in the linear graph Fig. 1-2-1 are incident (or connected) at vertex A. Edges a, d, e, and f are incident at vertex B.

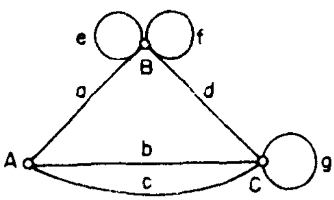

Fig. 1-2-1. A linear graph with self-loops.

The number of edges incident at each vertex is very important to characterize linear graphs such as paths, circuits, and Euler graphs. So we define the degree of a vertex as follows.

Definition 1-2-1. *The degree of a vertex v*, symbolized by $d(v)$, is defined as

$$d(v) = 2n_s + n_n \qquad (1\text{-}2\text{-}1)$$

where n_s is the number of self-loops incident at vertex v and n_n is the number of edges other than self-loops incident at vertex v.

For example, the degree of vertex A of a linear graph in Fig. 1-2-1 is $d(A) = 3$ because $n_s = 0$ and $n_n = 3$ at this vertex. The degree of vertex B is $d(B) = 2(2) + 2 = 6$ where $n_s = 2$ and $n_n = 2$ at vertex B. The degree of vertex C is $d(C) = 5$.

Suppose edge e is connected between vertices p and q (i.e., the two endpoints of edge e are p and q). Then we count edge e as 1 for both $d(p)$ and $d(q)$ for $p \neq q$. When $p = q$, edge e (which is a self-loop) will be counted as 2 for $d(p)$. This is true for every edge in a linear graph. Hence the summation of the degrees of all vertices is equal to twice of the number of edges in a linear graph. That is,

$$\sum_{v \in G} d(v) = 2n_e \qquad (1\text{-}2\text{-}2)$$

where $\sum_{v \in G}$ means the summation for all vertices in linear graph G and n_e is the number of edges in G. For example, in Fig. 1-2-2, $d(A) = 3$, $d(B) = 2$, $d(C) = 3$, $d(D) = 2$, and $d(E) = 4$. Thus $\sum_{i \in G} d(i) = 14$. The number of edges in the linear graph is 7.

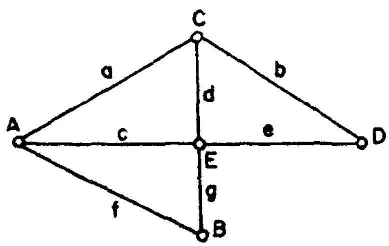

Fig. 1-2-2. A linear graph.

Consider the linear graph in Fig. 1-2-2 as a map in which vertices indicate cities and edges indicate highways. We can see that there are several highways going from one city to another. Suppose we are planning to travel from a city A to a city D. If we list highways according to the order by which we are going to travel from city A to city D, we will have a sequence of edges which specifies a particular route from A to D. The vertex corresponding to the origin is called the initial vertex and the vertex corresponding to the destination is called the final vertex. As an example, with initial vertex A and final vertex D, some sequences of edges of a linear graph in Fig. 1-2-2 are (c, e), (a, d, c, f, g, e), and (c, d, b). It must be noted that each edge in the sequence discussed has one vertex in common with the preceding edge and the other vertex in common with the succeeding edge. For example, in sequence (c, d, b), vertex E of edge d is an endpoint of preceding edge c and vertex C