

国外数学名著系列(续一)

(影印版) 38

Tony F. Chan Jianhong Shen

# Image Processing and Analysis

Variational, PDE, Wavelet and  
Stochastic Methods

## 图像处理与分析

变分, PDE, 小波及随机方法



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## 《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来,需要数学家淡泊名利并付出更艰苦地努力。另一方面,我们也要从客观上为数学家创造更有利的发展数学事业的外部环境,这主要是加强对数学事业的支持与投资力度,使数学家有较好的工作与生活条件,其中也包括改善与加强数学的出版工作。

从出版方面来讲,除了较好较快地出版我们自己的成果外,引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说,施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书,使我国广大数学家能以较低的价格购买,特别是在边远地区工作的数学家能普遍见到这些书,无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权,一次影印了 23 本施普林格出版社出版的数学书,就是一件好事,也是值得继续做下去的事情。大体上分一下,这 23 本书中,包括基础数学书 5 本,应用数学书 6 本与计算数学书 12 本,其中有些书也具有交叉性质。这些书都是很新的,2000 年以后出版的占绝大部分,共计 16 本,其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿,例如基础数学中的数论、代数与拓扑三本,都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点,基础数学类的书以“经典”为主,应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家,例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士,曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然,23 本书只能涵盖数学的一部分,所以,这项工作还应该继续做下去。更进一步,有些读者面较广的好书还应该翻译成中文出版,使之有更大的读者群。

总之,我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持,并盼望这一工作取得更大的成绩。

王 元

2005 年 12 月 3 日

*To Monica, Michelle, Ryan, and Claudia*



# Preface

No time in human history has ever witnessed such explosive influence and impact of image processing on modern society, sciences, and technologies. From nanotechnologies, astronomy, medicine, vision psychology, remote sensing, security screening, and the entertainment industry to the digital communication technologies, images have helped mankind to see objects in various environments and scales, to sense and communicate distinct spatial or temporal patterns of the physical world, as well as to make optimal decisions and take right actions. Image processing and understanding are therefore turning into a critical component in contemporary sciences and technologies with many important applications.

As a branch of signal processing, image processing has traditionally been built upon the machinery of Fourier and spectral analysis. In the past few decades, there have emerged numerous novel competing methods and tools for successful image processing. They include, for example, stochastic approaches based upon Gibbs/Markov random fields and Bayesian inference theory, variational methods incorporating various geometric regularities, linear or nonlinear partial differential equations, as well as applied harmonic analysis centered around wavelets.

These diversified approaches are apparently distinct but in fact intrinsically connected. Each method excels from certain interesting angles or levels of approximations but is also inevitably subject to its limitations and applicabilities. On the other hand, at some deeper levels, they share common grounds and roots, from which more efficient *hybrid* tools or methods can be developed. This highlights the necessity of *integrating* this diversity of approaches.

The present book takes a concerted step towards this integration goal by synergistically covering all the aforementioned modern image processing approaches. We strive to reveal the few key common threads connecting all these major methodologies in contemporary image processing and analysis, as well as to highlight some emergent integration efforts that have been proven very successful and enlightening. However, we emphasize that we have made no attempt to be comprehensive in covering each subarea. In addition to the efforts of organizing the vast contemporary literature into a coherent logical structure, the present book also provides some in-depth analysis for those relatively newer areas. Since very few books have attempted this integrative approach, we hope ours will fill a need in the field.

Let  $u$  denote an observed image function, and  $T$  an image processor, which can be either deterministic or stochastic, as well as linear or nonlinear. Then a typical image

processing problem can be expressed by the flow chart

$$(\text{input})\ u \longrightarrow (\text{processor})\ T \longrightarrow (\text{output})\ F = T[u],$$

where  $F$  represents important image or visual features of interest. In the present book, we explore all three key aspects of image processing and analysis.

- **Modeling:** What are the suitable mathematical models for  $u$  and  $T$ ? What are the fundamental principles governing the constructions of such models? What are the key features that have to be properly respected and incorporated?
- **Model Analysis:** Are the two models for  $u$  and  $T$  compatible? Is  $T$  stable and robust to noises or general perturbations? Does  $F = T[u]$  exist, and if so, is it unique? What are the fine properties or structures of the solutions? In many applications, image processors are often formulated as inverse problem solvers, and as a result, issues like stability, existence, and uniqueness become very important.
- **Computation and Simulation:** How can the models be efficiently computed or simulated? Which numerical regularization techniques should be introduced to ensure stability and convergence? And how should the targeted entities be properly represented?

This view governs the structure and organization of the entire book. The first chapter briefly summarizes the emerging novel field of imaging science, as well as outlines the main tasks and topics of the book. In the next two chapters, we introduce and analyze several universal modern ways for image modeling and representation (for  $u$ ), which include wavelets, random fields, level sets, etc. Based on this foundation, we then in the subsequent four chapters develop and analyze four specific and significant processing models (for  $T$ ) including image denoising, image deblurring, inpainting or image interpolation, and image segmentation. Embedded within various image processing models are their computational algorithms, numerical examples, or typical applications.

As the whole spectra of image processing spread so vastly, in this book we can only focus on several most representative problems which emerge frequently from applications. In terms of computer vision and artificial intelligence, these are often loosely categorized as *low-level* vision problems. We do not intend to cover *high-level* vision problems which often involve pattern learning, identification, and representation.

We are enormously grateful to Linda Thiel, Alexa Epstein, Kathleen LeBlanc, Michelle Montgomery, David Riegelhaupt, and Sara Murphy of the SIAM Publisher for their constant encouragement and care throughout the project. It has been such a wonderful experience of planning, communication, and envisaging.

We also owe profound gratitude to the following colleagues whose published works and personal discussions have greatly influenced and shaped the contents and structures of the current book (in alphabetical order): Antonin Chambolle, Ron Coifman, Ingrid Daubechies, Rachid Deriché, Ron DeVore, David Donoho, Stu Geman, Brad Lucier, Jitendra Malik, Yves Meyer, Jean-Michel Morel, David Mumford, Stan Osher, Pietro Perona, Guillermo Sapiro, Jayant Shah, James Sethian, Harry Shum, Steve Smale, Gilbert Strang, Curt Vogel, Yingnian Wu, Alan Yuille, and Song-Chun Zhu, and the list further expands.

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We also acknowledge the tremendous benefits from the participation of numerous imaging sciences related workshops at the Institute of Pure and Applied Mathematics (IPAM) at UCLA, the Institute of Mathematics and Its Applications (IMA) at the University of Minnesota, the Institute of Mathematical Sciences (IMS) at the National University of Singapore, the Mathematical Sciences Research Institute (MSRI) at Berkeley, as well as the Center of Mathematical Sciences (CMS) at Zhejiang University, China.

Finally, this book project, like all the others in our life, is an intellectual product under numerous constraints, including our busy working schedules and many other scholastic duties. Its contents and structures as presented herein are therefore only optimal subject to such inevitable conditions. All errata and suggestions for improvements will be received gratefully by the authors.

This book is absolutely impossible without the pioneering works of numerous insightful mathematicians and computer scientists and engineers. It would be our great pleasure to see that the book can faithfully reflect many major aspects of contemporary image analysis and processing. But unintentional biases are inevitable due to the limited views and experiences of the authors, and we are happy to hear any criticisms from our dear readers.

# Contents

<b>List of Figures</b>	<b>xiii</b>
<b>Preface</b>	<b>xix</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Dawning of the Era of Imaging Sciences . . . . .	1
1.1.1 Image Acquisition . . . . .	1
1.1.2 Image Processing . . . . .	5
1.1.3 Image Interpretation and Visual Intelligence . . . . .	6
1.2 Image Processing by Examples . . . . .	6
1.2.1 Image Contrast Enhancement . . . . .	6
1.2.2 Image Denoising . . . . .	8
1.2.3 Image Deblurring . . . . .	9
1.2.4 Image Inpainting . . . . .	9
1.2.5 Image Segmentation . . . . .	11
1.3 An Overview of Methodologies in Image Processing . . . . .	12
1.3.1 Morphological Approach . . . . .	12
1.3.2 Fourier and Spectral Analysis . . . . .	14
1.3.3 Wavelet and Space-Scale Analysis . . . . .	15
1.3.4 Stochastic Modeling . . . . .	16
1.3.5 Variational Methods . . . . .	17
1.3.6 Partial Differential Equations (PDEs) . . . . .	19
1.3.7 Different Approaches Are Intrinsically Interconnected . . . . .	21
1.4 Organization of the Book . . . . .	24
1.5 How to Read the Book . . . . .	26
<b>2 Some Modern Image Analysis Tools</b>	<b>31</b>
2.1 Geometry of Curves and Surfaces . . . . .	31
2.1.1 Geometry of Curves . . . . .	31
2.1.2 Geometry of Surfaces in Three Dimensions . . . . .	36
2.1.3 Hausdorff Measures and Dimensions . . . . .	44
2.2 Functions with Bounded Variations . . . . .	45
2.2.1 Total Variation as a Radon Measure . . . . .	46
2.2.2 Basic Properties of BV Functions . . . . .	49

2.2.3	The Co-Area Formula . . . . .	52
2.3	Elements of Thermodynamics and Statistical Mechanics . . . . .	54
2.3.1	Essentials of Thermodynamics . . . . .	55
2.3.2	Entropy and Potentials . . . . .	56
2.3.3	Statistical Mechanics of Ensembles . . . . .	58
2.4	Bayesian Statistical Inference . . . . .	61
2.4.1	Image Processing or Visual Perception as Inference . . . . .	61
2.4.2	Bayesian Inference: Bias Due to Prior Knowledge . . . . .	62
2.4.3	Bayesian Method in Image Processing . . . . .	64
2.5	Linear and Nonlinear Filtering and Diffusion . . . . .	65
2.5.1	Point Spreading and Markov Transition . . . . .	65
2.5.2	Linear Filtering and Diffusion . . . . .	67
2.5.3	Nonlinear Filtering and Diffusion . . . . .	70
2.6	Wavelets and Multiresolution Analysis . . . . .	73
2.6.1	Quest for New Image Analysis Tools . . . . .	73
2.6.2	Early Edge Theory and Marr's Wavelets . . . . .	75
2.6.3	Windowed Frequency Analysis and Gabor Wavelets . . . . .	76
2.6.4	Frequency-Window Coupling: Malvar-Wilson Wavelets . . . . .	77
2.6.5	The Framework of Multiresolution Analysis (MRA) . . . . .	80
2.6.6	Fast Image Analysis and Synthesis via Filter Banks . . . . .	86
<b>3</b>	<b>Image Modeling and Representation</b> . . . . .	<b>91</b>
3.1	Modeling and Representation: What, Why, and How . . . . .	91
3.2	Deterministic Image Models . . . . .	93
3.2.1	Images as Distributions (Generalized Functions) . . . . .	93
3.2.2	$L^p$ Images . . . . .	96
3.2.3	Sobolev Images $H^n(\Omega)$ . . . . .	98
3.2.4	BV Images . . . . .	98
3.3	Wavelets and Multiscale Representation . . . . .	99
3.3.1	Construction of 2-D Wavelets . . . . .	99
3.3.2	Wavelet Responses to Typical Image Features . . . . .	104
3.3.3	Besov Images and Sparse Wavelet Representation . . . . .	107
3.4	Lattice and Random Field Representation . . . . .	115
3.4.1	Natural Images of Mother Nature . . . . .	115
3.4.2	Images as Ensembles and Distributions . . . . .	116
3.4.3	Images as Gibbs' Ensembles . . . . .	117
3.4.4	Images as Markov Random Fields . . . . .	119
3.4.5	Visual Filters and Filter Banks . . . . .	122
3.4.6	Entropy-Based Learning of Image Patterns . . . . .	124
3.5	Level-Set Representation . . . . .	126
3.5.1	Classical Level Sets . . . . .	127
3.5.2	Cumulative Level Sets . . . . .	127
3.5.3	Level-Set Synthesis . . . . .	129
3.5.4	An Example: Level Sets of Piecewise Constant Images . . . . .	129
3.5.5	High Order Regularity of Level Sets . . . . .	130
3.5.6	Statistics of Level Sets of Natural Images . . . . .	131

3.6	The Mumford–Shah Free Boundary Image Model . . . . .	132
3.6.1	Piecewise Constant 1-D Images: Analysis and Synthesis . . . . .	132
3.6.2	Piecewise Smooth 1-D Images: First Order Representation . . . . .	134
3.6.3	Piecewise Smooth 1-D Images: Poisson Representation . . . . .	135
3.6.4	Piecewise Smooth 2-D Images . . . . .	136
3.6.5	The Mumford–Shah Model . . . . .	138
3.6.6	The Role of Special BV Images . . . . .	140
<b>4</b>	<b>Image Denoising</b> . . . . .	<b>145</b>
4.1	Noise: Origins, Physics, and Models . . . . .	145
4.1.1	Origins and Physics of Noise . . . . .	145
4.1.2	A Brief Overview of 1-D Stochastic Signals . . . . .	147
4.1.3	Stochastic Models of Noises . . . . .	150
4.1.4	Analog White Noises as Random Generalized Functions . . . . .	151
4.1.5	Random Signals from Stochastic Differential Equations . . . . .	153
4.1.6	2-D Stochastic Spatial Signals: Random Fields . . . . .	155
4.2	Linear Denoising: Lowpass Filtering . . . . .	156
4.2.1	Signal vs. Noise . . . . .	156
4.2.2	Denoising via Linear Filters and Diffusion . . . . .	157
4.3	Data-Driven Optimal Filtering: Wiener Filters . . . . .	159
4.4	Wavelet Shrinkage Denoising . . . . .	160
4.4.1	Shrinkage: Quasi-statistical Estimation of Singletons . . . . .	160
4.4.2	Shrinkage: Variational Estimation of Singletons . . . . .	163
4.4.3	Denoising via Shrinking Noisy Wavelet Components . . . . .	165
4.4.4	Variational Denoising of Noisy Besov Images . . . . .	171
4.5	Variational Denoising Based on BV Image Model . . . . .	174
4.5.1	TV, Robust Statistics, and Median . . . . .	174
4.5.2	The Role of TV and BV Image Model . . . . .	175
4.5.3	Biased Iterated Median Filtering . . . . .	175
4.5.4	Rudin, Osher, and Fatemi’s TV Denoising Model . . . . .	177
4.5.5	Computational Approaches to TV Denoising . . . . .	178
4.5.6	Duality for the TV Denoising Model . . . . .	183
4.5.7	Solution Structures of the TV Denoising Model . . . . .	185
4.6	Denoising via Nonlinear Diffusion and Scale-Space Theory . . . . .	191
4.6.1	Perona and Malik’s Nonlinear Diffusion Model . . . . .	191
4.6.2	Axiomatic Scale-Space Theory . . . . .	194
4.7	Denoising Salt-and-Pepper Noise . . . . .	198
4.8	Multichannel TV Denoising . . . . .	203
4.8.1	Variational TV Denoising of Multichannel Images . . . . .	203
4.8.2	Three Versions of TV[ $u$ ] . . . . .	204
<b>5</b>	<b>Image Deblurring</b> . . . . .	<b>207</b>
5.1	Blur: Physical Origins and Mathematical Models . . . . .	207
5.1.1	Physical Origins . . . . .	207
5.1.2	Mathematical Models of Blurs . . . . .	208
5.1.3	Linear vs. Nonlinear Blurs . . . . .	214

5.2	Ill-posedness and Regularization . . . . .	216
5.3	Deblurring with Wiener Filters . . . . .	217
5.3.1	Intuition on Filter-Based Deblurring . . . . .	217
5.3.2	Wiener Filtering . . . . .	218
5.4	Deblurring of BV Images with Known PSF . . . . .	220
5.4.1	The Variational Model . . . . .	220
5.4.2	Existence and Uniqueness . . . . .	222
5.4.3	Computation . . . . .	224
5.5	Variational Blind Deblurring with Unknown PSF . . . . .	226
5.5.1	Parametric Blind Deblurring . . . . .	226
5.5.2	Parametric-Field-Based Blind Deblurring . . . . .	230
5.5.3	Nonparametric Blind Deblurring . . . . .	233
<b>6</b>	<b>Image Inpainting</b> . . . . .	<b>245</b>
6.1	A Brief Review on Classical Interpolation Schemes . . . . .	246
6.1.1	Polynomial Interpolation . . . . .	246
6.1.2	Trigonometric Polynomial Interpolation . . . . .	248
6.1.3	Spline Interpolation . . . . .	249
6.1.4	Shannon's Sampling Theorem . . . . .	251
6.1.5	Radial Basis Functions and Thin-Plate Splines . . . . .	253
6.2	Challenges and Guidelines for 2-D Image Inpainting . . . . .	256
6.2.1	Main Challenges for Image Inpainting . . . . .	256
6.2.2	General Guidelines for Image Inpainting . . . . .	257
6.3	Inpainting of Sobolev Images: Green's Formulae . . . . .	258
6.4	Geometric Modeling of Curves and Images . . . . .	263
6.4.1	Geometric Curve Models . . . . .	264
6.4.2	2-, 3-Point Accumulative Energies, Length, and Curvature . . . . .	265
6.4.3	Image Models via Functionalizing Curve Models . . . . .	268
6.4.4	Image Models with Embedded Edge Models . . . . .	269
6.5	Inpainting BV Images (via the TV Radon Measure) . . . . .	270
6.5.1	Formulation of the TV Inpainting Model . . . . .	270
6.5.2	Justification of TV Inpainting by Visual Perception . . . . .	272
6.5.3	Computation of TV Inpainting . . . . .	274
6.5.4	Digital Zooming Based on TV Inpainting . . . . .	274
6.5.5	Edge-Based Image Coding via Inpainting . . . . .	276
6.5.6	More Examples and Applications of TV Inpainting . . . . .	277
6.6	Error Analysis for Image Inpainting . . . . .	279
6.7	Inpainting Piecewise Smooth Images via Mumford and Shah . . . . .	282
6.8	Image Inpainting via Euler's Elastica and Curvatures . . . . .	284
6.8.1	Inpainting Based on the Elastica Image Model . . . . .	284
6.8.2	Inpainting via Mumford-Shah-Euler Image Model . . . . .	287
6.9	Inpainting of Meyer's Texture . . . . .	289
6.10	Image Inpainting with Missing Wavelet Coefficients . . . . .	291
6.11	PDE Inpainting: Transport, Diffusion, and Navier-Stokes . . . . .	295
6.11.1	Second Order Interpolation Models . . . . .	295
6.11.2	A Third Order PDE Inpainting Model and Navier-Stokes . . . . .	299

6.11.3	TV Inpainting Revisited: Anisotropic Diffusion . . . . .	301
6.11.4	CDD Inpainting: Curvature Driven Diffusion . . . . .	302
6.11.5	A Quasi-axiomatic Approach to Third Order Inpainting . . . . .	303
6.12	Inpainting of Gibbs/Markov Random Fields . . . . .	307
<b>7</b>	<b>Image Segmentation</b>	<b>309</b>
7.1	Synthetic Images: Monoids of Occlusive Preimages . . . . .	309
7.1.1	Introduction and Motivation . . . . .	309
7.1.2	Monoids of Occlusive Preimages . . . . .	310
7.1.3	Miminal and Prime (or Atomic) Generators . . . . .	315
7.2	Edges and Active Contours . . . . .	318
7.2.1	Pixelwise Characterization of Edges: David Marr's Edges . . . . .	318
7.2.2	Edge-Regulated Data Models for Image Gray Values . . . . .	320
7.2.3	Geometry-Regulated Prior Models for Edges . . . . .	322
7.2.4	Active Contours: Combining Both Prior and Data Models . . . . .	325
7.2.5	Curve Evolutions via Gradient Descent . . . . .	327
7.2.6	$\Gamma$ -Convergence Approximation of Active Contours . . . . .	329
7.2.7	Region-Based Active Contours Driven by Gradients . . . . .	331
7.2.8	Region-Based Active Contours Driven by Stochastic Features . . . . .	332
7.3	Geman and Geman's Intensity-Edge Mixture Model . . . . .	338
7.3.1	Topological Pixel Domains, Graphs, and Cliques . . . . .	338
7.3.2	Edges as Hidden Markov Random Fields . . . . .	339
7.3.3	Intensities as Edge-Regulated Markov Random Fields . . . . .	342
7.3.4	Gibbs' Fields for Joint Bayesian Estimation of $u$ and $\Gamma$ . . . . .	343
7.4	The Mumford–Shah Free-Boundary Segmentation Model . . . . .	344
7.4.1	The Mumford–Shah Segmentation Model . . . . .	344
7.4.2	Asymptotic M.–S. Model I: Sobolev Smoothing . . . . .	345
7.4.3	Asymptotic M.–S. Model II: Piecewise Constant . . . . .	347
7.4.4	Asymptotic M.–S. Model III: Geodesic Active Contours . . . . .	351
7.4.5	Nonuniqueness of M.–S. Segmentation: A 1-D Example . . . . .	355
7.4.6	Existence of M.–S. Segmentation . . . . .	355
7.4.7	How to Segment Sierpinski Islands . . . . .	359
7.4.8	Hidden Symmetries of M.–S. Segmentation . . . . .	362
7.4.9	Computational Method I: $\Gamma$ -Convergence Approximation . . . . .	364
7.4.10	Computational Method II: Level-Set Method . . . . .	366
7.5	Multichannel Logical Segmentation . . . . .	369
	<b>Bibliography</b>	<b>373</b>
	<b>Index</b>	<b>393</b>

# List of Figures

1.1	An ideal image: noiseless, complete, and in good contrast. . . . .	7
1.2	A low-contrast version of the ideal image. . . . .	7
1.3	Degradation by additive Gaussian noise (Chapter 4). . . . .	8
1.4	Salt-and-pepper noise with 10% spatial density (Chapter 4). . . . .	9
1.5	Degradation by out-of-focus blur: the digital camera focuses on a finger- tip about 1 inch away while the target scene is about 1 foot away (Chapter 5). . . . .	10
1.6	Degradation by motion blur due to horizontal hand jittering during a single exposure (Chapter 5). . . . .	10
1.7	150 8-by-8 packets are randomly lost during transmission (Chapter 6). The goal of error concealment (or more generally, inpainting) is to develop models and algorithms that can automatically fill in the blanks [24, 67, 166].	11
1.8	Such cartoonish segmentation seems trivial to human vision but still re- mains the most fundamental and challenging problem in image processing and low-level computer vision (Chapter 7). (The background segment $\Omega_0$ is not shown explicitly here.) . . . . .	11
1.9	A binary set $A$ and a structure element $S$ (for dilation and erosion). . . .	13
1.10	Dilation $D_S(A)$ (left) and erosion $E_S(A)$ (right): dilation closes up small holes or gaps, while erosion opens them up. . . . .	14
1.11	A digital image and its discrete Fourier transform. The example reveals a couple of salient features of Fourier image analysis: (1) most high- amplitude coefficients concentrate on the low-frequency band; (2) dom- inant directional information in the original image is easily recognizable in the Fourier domain; and (3) the coefficients, however, decay slowly for Heaviside-type directional edges (i.e., a jump line with distinct constant values along its two shoulders). . . . .	15
1.12	An example of a (mother) wavelet by Daubechies' design [96]. Local- ization and oscillation are characteristic to all wavelets. . . . .	16
1.13	A noisy 1-D signal and its optimal restoration according to (1.7). . . . .	19
1.14	A trefoil evolves under mean-curvature motion (1.11) or (1.12). . . . .	21
1.15	Different approaches are intrinsically connected: an example. . . . .	22
1.16	Organization of the book. . . . .	24

2.1	A planar curve: tangent $\mathbf{t}$ , normal $\mathbf{n}$ , and curvature $\kappa$ . . . . .	32
2.2	Second order local geometry: surface normal $N$ and two perpendicular principle curvature circles. . . . .	38
2.3	A surface patch $z = h(u, v) = \cos(u) \cos(v)$ (top), its mean curvature field $H$ (lower left), and Gauss curvature field $K$ (lower right). . . . .	40
2.4	A set $E$ (left) and a covering $\mathcal{A}$ (right). Covering elements like the huge box with dashed borders (right) fail to faithfully capture small-scale details. The definition of Hausdorff measures forces covering scales to tend to zero. . . . .	45
2.5	The Hausdorff dimension $\dim_H(E)$ of a set $E$ is the critical $d$ , above which $E$ appears too “thin,” while below it appears too “fat.” . . . .	45
2.6	Three 1-D images with $\text{TV}[f] = \text{TV}[g] = \text{TV}[h] = 2$ . . . . .	47
2.7	An example of $L^1$ -lower semicontinuity. The sequence $(u_n)$ of 1-D images on $[0, 1]$ converges to $u = 0$ in $L^1$ since $\ u_{n+1} - u\ _{L^1} \leq 2^{-n}$ . Notice that $\text{TV}(u) = 0$ while $\text{TV}(u_n) \equiv 2$ , which is consistent with the property of lower semicontinuity: $\text{TV}(u) \leq \liminf_n \text{TV}(u_n)$ . In particular, strict inequality is indeed realizable. . . . .	49
2.8	Geometric meaning of TV: the co-area formula. For smooth images, $\text{TV}[u]$ is to sum up the lengths of all the level curves, weighted by the Lebesgue element $d\lambda$ . Plotted here is a discrete approximation: $\text{TV}[u] \simeq \sum_n \text{length}(u \equiv \lambda_n) \Delta\lambda$ . . . . .	54
2.9	Gibbs’ CE: higher temperature $T$ corresponds to smaller $\beta$ and more uniform distribution; on the other hand, when $T \simeq 0$ , the system exclusively remains at the ground state (leading to superfluids or superconductance in physics). . . . .	59
2.10	Gibbs’ entropies for two imaginary 4-state systems (with $\kappa$ set to 1). Entropy generally measures the degrees of freedom of a system. A lower entropy therefore means that the target system is more restrained. (This observation led Shannon [272] to define negative entropies as information metrics, since less randomness implies more information.) . . . . .	60
2.11	An example of linear anisotropic diffusion by (2.38) with diagonal diffusivity matrix $D = \text{diag}(D_x, D_y)$ and $D_y : D_x = 10 : 1$ . The image thus diffuses much faster along the vertical $y$ -direction. . . . .	68
2.12	The median of a 5-component segment from an imaginary 1-D signal $(x_n)$ . . . . .	70
2.13	An example of using the median filter (with $7 \times 7$ centered square window) to denoise an image with severe salt-and-pepper noise. Notice the outstanding feature of median filtering: the edges in the restored image are not blurry. . . . .	71
2.14	Two examples of Marr’s wavelets (“Mexican hats”) as in (2.50). . . . .	75
2.15	A generic approach to designing the window template $w(x)$ via symmetrically constructing the profile of its square $w^2(x)$ . . . . .	79
2.16	MRA as (Hilbert) space decompositions: the finer resolution space $V_2$ is decomposed to the detail (or wavelet) space $W_1$ and coarser resolution space $V_1$ . The same process applies to $V_1$ and all other $V_j$ ’s [290]. . . . .	84
2.17	A pair of compactly supported scaling function $\phi(x)$ and mother wavelet $\psi(x)$ by Daubechies’ design [96]. . . . .	87



2.18	Fast wavelet transform via filter banks: the two-channel analysis (or decomposition) and synthesis (or reconstruction) banks. . . . .	88
3.1	Images as distributions or generalized functions. Test functions then model various biological or digital sensors, such as retinal photoreceptors of human vision or coupled charge devices in CCD cameras. . . . .	94
3.2	Three Haar mother wavelets in two dimensions via tensor products. . . .	103
3.3	Besov norms in $B_q^\alpha(L^p)$ 's measure the strength of signals in the space-scale plane: $L^p$ for intrascale variations, $L^q$ for inter- or cross-scale variations (in terms of $d\lambda = dh/h$ ), while $h^{-\alpha}$ for comparison with Hölder continuity. . . . .	108
3.4	Ingredients of Markov random fields by examples: a neighborhood $N_\alpha$ (left), two doubleton cliques $C \in \mathcal{C}$ (middle), and locality of conditional inference $p(u_\alpha   u_{\Omega \setminus \alpha}) = p(u_\alpha   u_{N_\alpha})$ (right). . . . .	120
3.5	An example of branches $L_{\lambda, \Delta\lambda}$ 's with $\Delta\lambda = h$ and $\lambda = nh$ . The branch $L_{3h, h}$ contains two leaflets. . . . .	132
3.6	The $(x_n, b_n, g_n)$ -representation of a (compactly supported) piecewise smooth signal $u$ . On smooth regions where the signal $u$ varies slowly, $g_n$ is often small and only a few bits suffice to code them. . . . .	135
3.7	Poisson representation of a piecewise smooth signal $u$ . Shown here is only a single representative interval. The signal $u$ is represented by the two boundary values $u_n^+$ and $u_{n+1}^-$ and its second order derivative $f = u''$ (corresponding to the <i>source</i> distribution in electromagnetism). Reconstructing $u$ on the interval then amounts to solving the Poisson equation (3.62). The advantage of such representation is that $f$ is often small (just like wavelet coefficients) for smooth signals and demands fewer bits compared with $u$ . . . . .	136
4.1	A sample 2-D Brownian path $W(t)$ with $0 < t < 4$ , and the reference circle with radius $2 = \sqrt{4}$ , the standard deviation of $W(4)$ . . . . .	152
4.2	A sample random signal generated by the SDE $dX = 2Xd t + XdW$ with $X(0) = 1$ . The smooth dashed curve denotes the mean curve $x(t) = EX(t) = e^{2t}$ . The random signal is clearly not stationary since both the means and variances of $X(t)$ evolve. . . . .	155
4.3	Hard thresholding $T_\lambda(t)$ and soft shrinkage $S_\lambda(t)$ . . . . .	161
4.4	The shrinkage operator $\hat{a}_* = S_\sigma(a_0)$ achieves minimum shrinkage (Theorem 4.4) among all the estimators $\hat{a}$ that satisfy the <i>uniform shrinkage condition</i> (4.25). . . . .	163
4.5	Two singleton error functions $e_1(t a_0)$ with $\lambda = \mu = 1$ and $a_0 = 1, 2$ . Notice that the optimal estimators (i.e., the valleys) $\hat{a} = 0, 1$ are precisely the shrinkages $S_\sigma(a_0)$ with $\sigma = \mu/\lambda = 1$ . . . . .	165