Numerical Methods in Fluid Dynamics

Edited by C. A. Brebbia and J. J. Connor

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Preface

Fluid dynamics problems have increasingly demanded the application of computer techniques thus promoting considerable research in this field which has mainly been based on finite difference methods. However the use of finite element techniques is growing although there is still much work to be done in stability, accuracy, etc. of the solutions. This growth is mainly due to its versatility but finite differences have the advantages of sounder theoretical foundations. This Conference was organised to bring together researchers in both fields and to foster what proved to be a profitable exchange.

In particular, the Conference brought together aerodynamicists and hydrodynamicists. Aerodynamics has always been a discipline which attracted mathematicians, whereas hydrodynamics appeals more to engineers. These constitute two different approaches in research, engineers tend to emphasise the physical model and mathematicians the formulation. Each of them has a lot to learn from the other and it is hoped that the Conference created a greater rapport between them.

It is clear from the Proceedings that variational techniques, such as finite elements will become more important in fluid dynamics applications. This however should not imply the abandonment of all other methods but rather their reappraisal. For instance, finite elements are obviously inadequate to represent infinite domains but a combination of them with some integral techniques could produce a method having the advantages of both. Similarly, ways need to be found of freeing the finite difference method from the severe restriction of regular grids and fulfilment of all—natural and essential—boundary conditions. These are only two suggestions indicating the vast area still open to research.

Several papers on environmental topics such as pollution were also presented, many of them using finite elements. (This led to a colleague pointing out the dangers of finite element pollution?) No doubt a great deal more research will be conducted in this field.

The main aim of this Conference was to foster an exchange of ideas between researchers in different fields, in the hope of approaching, in a modest way, that level of generality and simplicity which is the hallmark of true science.

C. A. Brebbia
J. J. Connor
Southampton, March. 1974

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DYNAMIC RELAXATION SOLUTION OF THREE DIMENSIONAL SUBSONIC COMPRESSIBLE OSCILLATORY FLOW

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S U M M A R Y

When the equations of subsonic compressible oscillatory flow are written in terms of the real and imaginary parts of a velocity potential, two 'Laplace type' differential equations in three space dimensions are obtained interconnected by involved boundary conditions. Solutions are obtained using the dynamic relaxation method which is an iterative finite difference technique. The advantages of using the dynamic relaxation method for this type of problem are discussed.

1. Introduction

The Laplace equation is the governing equation in a large number of problems. One example which presents several computational difficulties is the analysis of interference effects of a small wing oscillating in subsonic compressible flow in a wind tunnel. This problem can be reduced to the solution of Laplace type equations in terms of the real and imaginary parts of a potential function with conditions, in terms of both the real and imaginary parts of the function, representing the behaviour at the boundaries of the tunnel.

Many successful solutions of this type of problem have been obtained using finite difference methods. The main difficulty, however, arises from the large number of simultaneous equations to be solved with the additional complication of the interconnection of the real and imaginary sets of simultaneous equations at the boundaries. Clearly an iterative method is advantageous; the method chosen in this instance was the dynamic relaxation method.

Dynamic relaxation was introduced by Otter (1965) and Day (1965) and has been applied mainly to structural problems, though it is also suitable for fluid flow problems. It is simply an iterative method which has the advantage that it is based on a physical analogy which greatly assists the choice of the convergence parameters.

The aim of this paper is to show how the dynamic relaxation method is used for the particular problem of flow in a three-dimensional wind tunnel. Emphasis will be placed on the method of deriving the dynamic relaxation equations, the choice of the finite difference approximation to both the governing equations and the boundary conditions and the preparation of the computer programme. Further details of the aerodynamic significance can be found in Rushton and Tomlinson (1972).

2. Formulation of the Problem

2.1 Particular problem under consideration

The particular problem to be considered in this paper relates to a rectangular wind tunnel or breadth, b, and height, h, containing a small oscillating wing positioned at the centre of the tunnel, Figure 1. Due to symmetry, the analysis will be restricted to one quarter of the cross section of the tunnel. The flow in the tunnel is compressible and oscillatory. The sidewalls of the tunnel are closed but the roof and floor contain perforations.

2.2 Governing equations

Using linearised theory which ignores viscous effects, the governing equation for oscillatory compressible flow is

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{M^2}{U^2}\left(U^2\frac{\partial^2\Phi}{\partial x^2} + 2U\frac{\partial^2\Phi}{\partial x\partial t} + \frac{\partial^2\Phi}{\partial t^2}\right) , \qquad (1)$$

where $\Phi\,(x,y,z,t)$ is the perturbation velocity potential and U is the velocity of the undisturbed stream.

Since the flow is socillating with an angular frequency, ω , it is advantageous to write

$$\Phi = \text{Real part } \bar{\phi}\,(x,y,z)e^{i\omega t} \qquad (2)$$

which, when substituted in (1) leads to

$$\beta^2\frac{\partial^2\bar{\phi}}{\partial x^2} + \frac{\partial^2\bar{\phi}}{\partial y^2} + \frac{\partial^2\bar{\phi}}{\partial z^2} - \frac{2i\omega M^2}{U}\frac{\partial\bar{\phi}}{\partial x} + \frac{\omega^2M^2}{U^2}\bar{\phi} = 0 , \qquad (3)$$

where $\beta^2 = 1 - M^2$.

It is convenient to introduce a modified potential function

$$\bar{\psi} = \bar{\phi} \exp\left(-\frac{i\omega M^2 x}{\beta^2 U}\right) \qquad (4)$$

This modified potential can be separated into real and imaginary parts,

$$\bar{\psi} = \psi_R + i\,\psi_I . \qquad (5)$$

Using (4) and (5), equation (3) becomes

$$\beta^2 \frac{\partial^2 \psi_R}{\partial x^2} + \frac{\partial^2 \psi_R}{\partial y^2} + \frac{\partial^2 \psi_R}{\partial z^2} + \frac{\omega^2 M^2 \psi_R}{\beta^2 U^2} = 0 \quad , \qquad (6a)$$

$$\beta^2 \frac{\partial^2 \psi_I}{\partial x^2} + \frac{\partial^2 \psi_I}{\partial y^2} + \frac{\partial^2 \psi_I}{\partial z^2} + \frac{\omega^2 M^2 \psi_I}{\beta^2 U^2} = 0 \quad , \qquad (6b)$$

2.3 Small oscillating wing

The disturbance in the wind tunnel is caused by a small wing with oscillatory lift; the undisturbed velocity potential for such a wing is

$$\bar{\psi}_m = \frac{U S \bar{C}_L \beta^2 z}{8\pi} \int_0^{\infty} \left(1 + \frac{i\omega M r}{\beta^2 U}\right) \exp\left(-\frac{i\omega}{\beta^2 U}(x' + Mr)\right) \frac{dx'}{r^3} , \qquad (7)$$

where $r^2 = (x-x')^2 + \beta^2(y^2+z^2)$. The real and imaginary parts of this function are given in Table 1.

Since r occurs in the denominator of equation (7), the value of the velocity potential cannot be evaluated at the centre of the tunnel, where r is zero. Instead it is evaluated at nodel points surrounding the tunnel centre by means of numerical integration.

2.4 Boundary Conditions

In wind tunnels perforations are often provided in the roof and floor of the tunnel. According to Garner et al (1966) the condition on these boundaries is

$$\left(\frac{\partial}{\partial x} + \frac{i\omega}{U}\right)\left(\bar{\phi} + \bar{K}\frac{\partial\bar{\phi}}{\partial n}\right) + \frac{1}{P}\frac{\partial\bar{\phi}}{\partial n} = 0 \quad ,$$

where $\bar{K}$ and P are parameters describing a particular boundary.

When written in terms of the real and imaginary parts of the modified potential function (equations (4) and (5)) this becomes, on a plane z = constant,

$$\frac{\partial\psi_R}{\partial x} - \frac{\omega\psi_I}{\beta^2 U} + \bar{K}\frac{\partial^2\psi_R}{\partial x\partial z} - \frac{\omega}{\beta^2 U}\frac{\partial\psi_I}{\partial z} + \frac{1}{P}\frac{\partial\psi_R}{\partial z} = 0 \qquad (8a)$$

$$\frac{\partial\psi_I}{\partial x} + \frac{\omega\psi_R}{\beta^2 U} + \bar{K}\frac{\partial^2\psi_I}{\partial x\partial z} + \frac{\omega}{\beta^2 U}\frac{\partial\psi_R}{\partial z} + \frac{1}{P}\frac{\partial\psi_I}{\partial z} = 0 \;. \qquad (8b)$$

For the particular case of closed walls ($\bar{K}$ and 1/P equal infinity) this reduces to,

$$\frac{\partial\psi_R}{\partial z} = 0 \quad \text{and} \quad \frac{\partial\psi_I}{\partial z} = 0 \qquad (9)$$

At distances far upstream there is no disturbance and therefore

$$\frac{\partial\psi_R}{\partial x} = 0 \quad \text{and} \quad \frac{\partial\psi_I}{\partial x} = 0 \quad .$$

However, far downstream the velocity potential is oscillatory and there the tunnel is extended to $x = X_R$ where $\psi_R = 0$, and $x = X_I$ where $\psi_I = 0$.

2.5 Summary of equations

All the relevant equations for this problem are summarised in Table 1. Note that the equations for the real and imaginary parts are independent apart from those on the perforated boundary.

3. Numerical Solution

Numerical solutions to the problem formulated in Section 2. can be obtained using the dynamic relaxation method. The primary reason for choosing this method is that it is an iterative finite difference method in which values of the real and imaginary parts are calculated simultaneously, and the interactive boundary conditions on the tunnel roof and floor are easily included in the iterative procedure.

3.1 Dynamic relaxation equations

In the dynamic relaxation method auxiliary variables **U, V and M** are introduced such that

$$\begin{aligned} \partial\psi_R/\partial x &= DU + \partial U/\partial t \ , \\ \partial\psi_R/\partial y &= DV + \partial V/\partial t \ , \\ \partial\psi_R/\partial z &= DW + \partial W/\partial t \ , \end{aligned} \qquad (10)$$

where D is a damping factor. When these equations are substituted into the following equation,

$$\beta^2 \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} + \frac{\omega^2 M^2 \psi_R}{D\beta^2 U^2} = \frac{\partial \psi_R}{\partial t} \ , \qquad (11)$$

then the resultant equation,

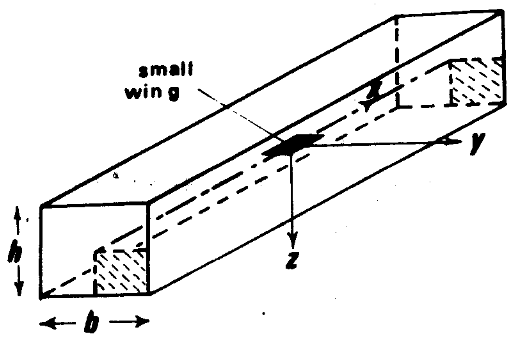

Figure 1. Arrangement of tunnel. Shaded sections show area considered in analysis.

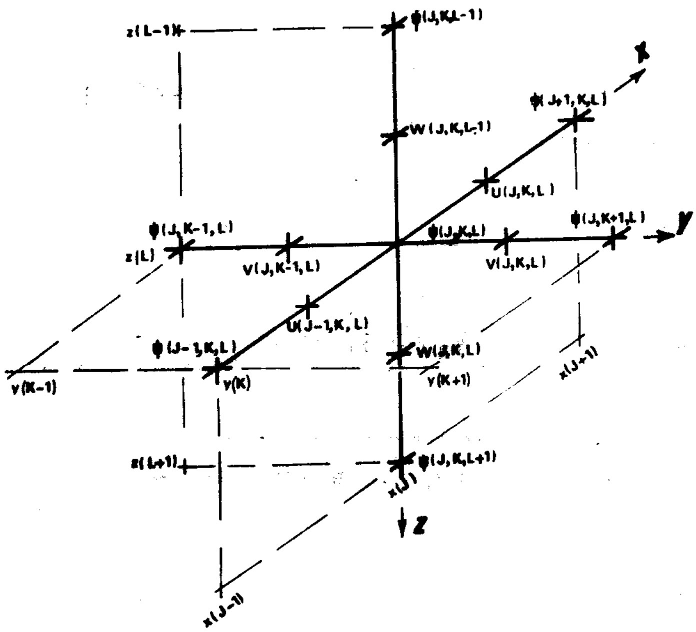

Figure 2. Interlacing mesh.

$$\beta^2 \frac{\partial^2 \psi_R}{\partial x^2} + \frac{\partial^2 \psi_R}{\partial y^2} + \frac{\partial^2 \psi_R}{\partial z^2} + \frac{\omega^2 M^2 \psi_R}{\beta^2 U} = \left(D - \frac{\omega^2 M^2}{D \beta^2 U^2}\right) \frac{\partial \psi_R}{\partial t} + \frac{\partial^2 \psi_R}{\partial t^2} \tag{12}$$

has the same left hand side as the governing equation (6a). The right hand side consists of dynamic terms which, with suitable damping, quickly become small. When the dynamic terms are negligibly small, equation (12) becomes identical to equation (6a).

Efficient reduction of the oscillatory terms to a negligible small value depends on the damping coefficient $D-\omega^2 M^2/D\beta^2 U^2$ (the second term is small compared with D for the present problems). The method of choosing the optimum value of D is discussed later.

A similar set of equations is used for the calculation of the imaginary part (6b).

A solution to these equations can readily be obtained using an explicit finite difference technique.

3.2 Finite difference approximations

It is convenient to use an interlacing finite difference net, Figure 2, with the functions ψ_R and ψ_I defined on the main net, and the auxiliary variables U, V and W defined at intermediate points. In a similar fashion, interlacing times are used with ψ_R and ψ_I defined at times n, n+1, n+2, ... and U, V and W defined at times $n-\frac{1}{2}$, $n+\frac{1}{2}$, $n+1\frac{1}{2}$, $n+2\frac{1}{2}$, ...

The explicit finite difference form of the first of equations (10) centred at time n is

$$(\psi_R(J+1,K,L)_n - \psi_R(J,K,L)_n)/\Delta x = 0.5D(U(J,K,L)_{n+\frac{1}{2}} + U(J,K,L)_{n-\frac{1}{2}})$$

$$+ (U(J,K,L)_{n+\frac{1}{2}} - U(J,K,L)_{n-\frac{1}{2}})/\Delta t. \qquad (13)$$

This can be re-arranged to give a simple substitution formula for the unknown function $U(J,K,L)_{n+\frac{1}{2}}$ which is included as the first equation of Table 2.

Equation (11) centred at time $n+\frac{1}{2}$ becomes

$$\beta^2(U,(J,K,L) - U(J-1,K,L))/\Delta x + (V(J,K,L) - V(J,K-1,L))/\Delta y +$$

$$(W(J,K,L) - W(J,K,L-1))/\Delta z_{n+\frac{1}{2}} + (\psi_{R,n}+\psi_{R,n+1})\omega^2 M^2/2D\beta^2 U^2 =$$

$$(\psi_{R,n+1} - \psi_{R,n})/\Delta t. \qquad (14)$$

Re-arranging leads to an equation for the unknown $\psi_{R,n+1}$ which is the second equation of Table 2.

Finite difference forms of certain of the boundary conditions are also included in Table 2. Boundary conditions in terms of differentials are enforced by means of fictitious nodes positioned outside the true boundaries, Figure 3.

The perforated boundary requires particular comment. The relevant equations (e) and (ℓ) of Table 1, can be written with the terms $\partial\psi_R/\partial z$ and $\partial\psi_I/\partial z$ on the left hand sides. All of the other terms can be written in terms of the known values of the functions on the boundary and at internal nodal points if an extrapolation formula is used for expressions of the form $\partial/\partial z$, and a three point formula for $\partial/\partial x$. The full finite difference equations are written in Table 2.

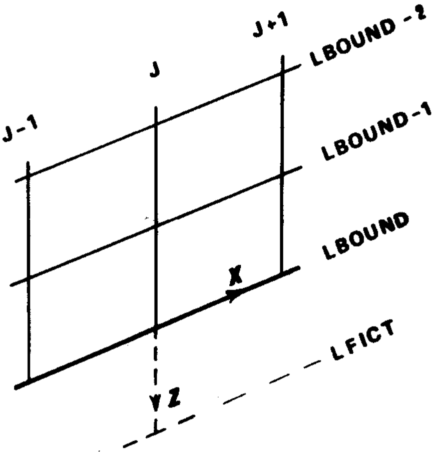

Figure 3. Mesh numbering at boundary.

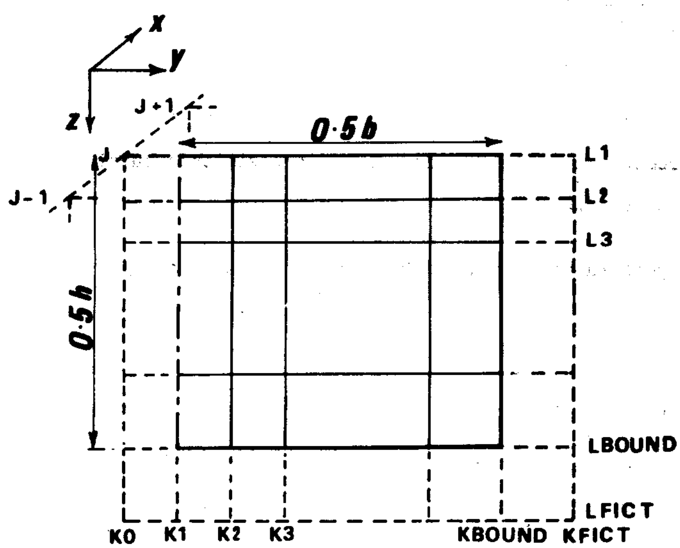

Figure 4. Mesh numbering on tunnel section.

4. Numerical Solution

In obtaining a numerical solution of the set of equations in Table 2, the following procedure is adopted with the nodal points numbered as shown in Figure 4.

(i) Initially the values of U,V,W and ψ are set to zero apart from those values of ψ at the nodes around the small wing which are determined by the integrals (b) and (i) of Table 1.

(ii) Values of U,V,W and ψ_R, U,V,W and ψ_I are calculated using equations (a) of Table 2 and the equivalent imaginary equations at all internal and boundary nodes apart from those surrounding the wing.

(iii) The boundary conditions are applied, equations (c) to (e) of ψ_R and ψ_I in finite difference approximations of the governing equations (6a) and (6b). If every residual is less than a specified value then the calculation is terminated. Otherwise return to step (ii) and repeat the calculation.

The outline computer programme, Table 3, follows the above order. Values of the functions are stored in three-dimensional arrays, with eight functions for each nodal point. Since the explicit formulation leads to simple substitution formulae, the value for the previous time step can be overwritten by the new value. The order in which the equations are written in the computer programme represents the interlacing time steps.

5. Details of Numerical Solution for a Wind Tunnel

Details are given below of the solution for a wind tunnel of square cross section (b = h) with a small wing oscillating at a fre-