MATRIX STRUCTURAL ANALYSIS

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Preface

This book has been written in response to the need for a textbook on matrix structural analysis that places proper emphasis on the methods in use in current practice and that lays the groundwork for allied, more advanced, subject matter such as the finite element method. To retain a wieldy volume oriented toward a one-semester senior or introductory graduate level course, account has been taken of the background that one might reasonably expect in a modern engineering education. Thus, it is assumed that the student is well-acquainted with matrix algebra and has already received instruction in the more traditional methods of structural analysis.

Following the appearance, in the 1950s, of electronic digital computers in design offices, the necessity of courses specifically directed toward computerized structural analysis became apparent. At that time the notions of matrix algebra were unfamiliar to all but a few experts. Structural analysis procedures consisted of a wide variety of uncoordinated "bag of tricks," and here, as well, only a relatively few experts were equipped to perform accurate analyses of complicated structures. Even fewer were acquainted with computer programming procedures. With these considerations in mind, a large number of textbooks appeared, some of them excellent in the context of the era. In addition to the subject matter intrinsic to the theory, these books covered matrix algebra and programming in considerable detail. Many of them included computer programs.

It is well to recognize that this earlier generation of texts also dealt with subject matter that was not fully crystallized. For example, much attention was showered upon the flexibility (or force) method, which today is hardly known in practice. In some cases the stiffness method was cast in the same awkward form as, the flexibility method, rather than in the currently widely practiced, "direct stiffness" format. Many operations, essential to practice, were not perceived. Cases in point include constraint conditions and substructuring. Often a grasp of the even more important relationship to the more general capabilities of finite element analysis was lacking. This book places the subject matter of matrix structural analysis in closer alignment with what we regard as current and future practice. A discussion of the motivation and salient features of the respective chapters best explains the way in which this is done.

Three purposes are served by Chapter 1, the Introduction. First, a concise sketch is drawn of the history of development of the subject. One intent of this history is to emphasize that computerized structural analysis methods are merely one part of a continuing progress that extends back more than 150 years. Second, the role that computerized structural analysis has played in the design of standing structures is outlined. These examples range from designs of modest complexity to monumental structures. Finally, the computer capabilities themselves are tied to the programs written for structural analysis. The awesome rate of development of computer hardware and software concepts limits this discussion to some rather general descriptions of what students might expect when they enter the design office.

Chapters 2 to 5 represent closely allied subject matter. We note above that the direct stiffness method predominates in practice, and it is to this approach that this group of chapters is addressed. Chapter 2 ostensibly serves to define terminology, coordinate systems, and the most fundamental notions of structural behavior. It also presents, however, two developments of great generality. The first is the basic character of elemental relationships in the form of stiffness and flexibility and their transformability from one to another and even to alternative formats. The second is the fundamental idea of direct stiffness analysis, described here by means of the simplest structural element.

A more formal treatment is given to direct stiffness analysis in Chapter 3 and, consequently, it is possible to examine more closely the implications for large-scale, practical computation. The latter include considerations such as the characteristics of the algebraic equations that are to be formed and solved. In Chapters 4 and 5 the remaining tools needed for the linear stiffness analysis of complete frames are established. The stiffness matrix of a rather general space-frame element is formulated and then applied, in illustrative examples, to a wide variety of specific situations.

One important aspect of Chapters 2 to 5 deserves comment. This initial development of the subject is based on consideration of the basic conditions of structural analysis in the form of equilibrium, compatibility, and material mechanical properties. Only slight attention is paid to work and energy concepts. This stems from our view that students entering the study of matrix structural analysis do not have the essential grounding in the latter concepts, and their treatment is best delayed until the student is attuned to the overall approach in computerized structural analysis.

Chapters 6 and 7, taken together, represent all of the study given in this text to the flexibility, or force method of matrix structural analysis. It has already been pointed out that the flexibility method does not enjoy significant utilization in current practice. A major reason for this is the inconvenience of redundant force systems, which must be identified and calculated prior to the actual determination of the unknown forces in statically indeterminate structures. For reasons given below, it is believed that the flexibility method does merit the attention given to it in these two chapters.

Formally, Chapter 6 is concerned with the calculation of internal forces in framework structures by use of equilibrium conditions alone. These procedures are of use in the analysis of statically determinate structures and in the detection of kinematic instability, for example, the case of insufficient support. Most importantly, however, these procedures represent an automatic (programmable) approach to the calculation of redundant force systems. With this in hand it is possible, in Chapter 7, to deal concisely with the flexibility method. Numerous example-solutions are presented to illustrate this approach. It is entirely possible that the availability of this more convenient and organized approach will enable future practical utilization of the flexibility method.

In Chapters 8 and 9 we consider the formulation of matrix structural analysis on the basis of virtual work concepts. Although the appearance of this subject is somewhat late in the text, it is believed that it enables the student to receive a more general and substantive treatment than is usually the case. A treatment of this type is necessary to give maximum scope to various aspects of practical design analysis of frameworks, such as tapered members and distributed loads, and it is the essential basis of later study of the finite element approach.

The theoretical groundwork of the virtual work principle is laid in Chapter 8. Both virtual displacement and virtual force concepts are covered, but far greater attention is given to the former on account of their role in stiffness formulations. Chapter 9 examines the implementation of the virtual work principle in matrix structural analysis.

The development of the stiffness and flexibility procedures, given in Chapters 2 to 7, is necessarily idealistic on account of the introductory intent of that portion of the text. In it, several detailed procedures, essential to the efficient solution of practical design analysis problems, are disregarded. Chapter 10 is devoted to the exposition of certain of these practices, including condensation of analysis equations prior to solution by elimination of specified unknowns, substructuring, the imposition of constraints, procedures for the economical reanalysis of structures when in an iterative design sequence, and the exploitation of symmetry and antisymmetry. Careful attention is given to the detailed circumstances of various types of coordinate systems that are alternative to global coordinates, such as the local coordinate systems that are convenient for sloping supports. Also, the transfer matrix approach to matrix structural analysis is formulated.

Chapter 11 comprises a detailed study of some of the more popular methods of solution of linear algebraic equations. Equation solving is more in the realm of applied mathematics, numerical analysis, or computer science than in structural engineering. Nevertheless, since primary responsibility for the entire analysis generally belongs to structural engineers, they should have more than a superficial knowledge of the important aspects of equation solving. This chapter is an introduction to the subject that highlights the methods found to be most useful in structural analysis. The pitfalls one may encounter are mentioned and illustrated. Guides to more comprehensive literature are included, and the reader is prepared for the study of these more specialized sources.

We noted that one broad objective of the present book is to present the total subject matter in such a way as to lay a proper groundwork for the subsequent study of the finite element method. The latter is distinguished from framework analysis, in this context, by the treatment of two- and three-dimensional continua in contrast with frameworks. Chapter 12 introduces the rudimentary considerations of finite element analysis, with attention limited to planar conditions. It is demonstrated that a relatively small amount of new subject matter—principally in the formulation of relationships between strain and displacement—is needed to extend the work of the prior chapters to the rather general capabilities of finite element analysis.

The contents outlined above, in our experience, represent subject matter on the scale of a 3-credit-hour, one-semester course with some selectivity of coverage available to the instructor. Much additional, relevant subject matter could be identified, for example, dynamic analysis, elastic instability analysis, and inelastic behavior, but it is not likely that it could be adequately covered in a semester time span.

A final word is in order with respect to the computer programs that lie at the heart of this topic. We decided to exclude computer programs. The rapid changes in programming philosophy, alluded to previously, bring about obsolescence of specific codes; an ample supply of such codes is available in other published works. Also, many instructors prefer to employ the widely used structural analysis programs (e.g., STRUDL) in support of the computational side of their courses. An attractive alternative, used by us in courses taught with support of this text, is to require that students develop their own matrix structural analysis program as a term-long project.

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Symbols

In matrix structural analysis many physical quantities and mathematical operations must be represented symbolically. Preparation of the equations of analysis in a form suitable for computer solution requires that all symbols used be defined in a rigid fashion amenable to numerical interpretation. On the other hand, the development of these equations, with stress upon their physical significance, is often best accomplished through the use of simple, less formal symbols, symbols that vary with the principle under discussion and have a clear physical connotation in the case at hand. In the interest of generality and uniformity we shall use some basic symbols to denote certain quantities throughout the text. But the precise interpretation of any of these symbols must be obtained from the local context in which it is used, and in which it will be explained.

In general, we use the letter P to designate applied direct forces and P_m to designate applied moments. R and R_m will be used for direct and moment reactions, respectively. At joints we denote internal direct forces by F and internal moments by M. All of these symbols may carry clarifying subscripts and superscripts to indicate direction, point of application, or member to which the symbol applies. The symbols may appear in either single component or vector form. The symbols u, v, and w will designate translational displacements in the x, y, and z directions and θ_x, θ_y, and θ_z rotational displacements about these axes. Generally, the letter k will refer to a stiffness quantity and d to a flexibility quantity.

Matrices are denoted by a boldface letter within the symbols [ ] (for a rectangular matrix), { } (for a column vector), ⌊ ⌋ (for a row vector), and ⌈ ⌋ for a diagonal matrix.

As a further guide, the following is a list of the principal symbols used in the text. As indicated above, most of these may contain clarifying or modifying subscripts, superscripts, or supplementary marks (overbars ¯, hat symbols ˆ, etc.) that will be defined in context. The same applies to the individual components of matrices and vectors listed below.

A	Area
[**A**]	Kinematics matrix
[**B**]	Statics matrix
[$\mathbf{C}_1$], [$\mathbf{C}_2$]	Matrices derived from the statics matrix
[**d**]	Element flexibility matrix
[**D**]	Global flexibility matrix
E, [**E**]	Elastic modulus, matrix of elastic constants
{**F**}	Vector of element nodal forces
G	Shear modulus

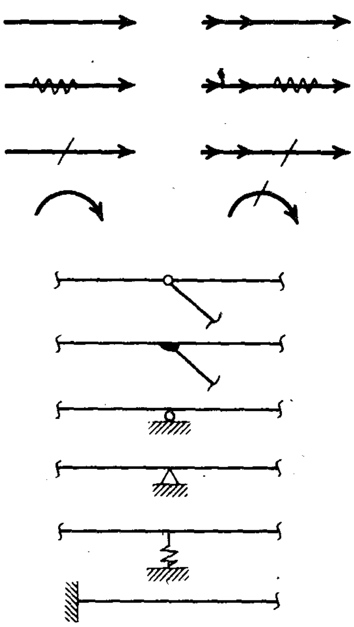
I	Moment of inertia
$[\mathbf{I}]$	Identity matrix
J	St. Venant torsion constant
$[\mathbf{k}]$	Element stiffness matrix
$[\mathbf{K}]$	Global stiffness matrix
L	Length
M	Element nodal moment
$\mathcal{M}$	Internal bending or twisting moment
$\lfloor \mathbf{N} \rfloor$	Vector of element shape functions
n	Number of degrees of freedom, number of nodes
$[\mathbf{0}]$, $\{\mathbf{0}\}$	Null matrix and vector
$\{\mathbf{P}\}$	Vector of global nodal forces
p	Number of elements
q	Distributed load intensity
$\{\mathbf{R}\}$	Vector of reaction forces
T	Temperature change above stress-free state
t	Plate thickness
U, U^*	Strain energy and complementary strain energy
u, v, w	Displacement components
Vol	Volume
W	Work
x, y, z	Cartesian coordinates

Greek Symbols

α	Coefficient of thermal expansion
α, β, δ	Direction angles
β	Angle of twist per unit of length
$[\mathbf{\Gamma}]$, $[\boldsymbol{\gamma}]$	Transformation matrix
γ	Shear strain
$\{\mathbf{\Delta}\}$	Vector of nodal point displacements
Δ	Displacement
δ	Relative axial displacement
ε	Normal strain
θ	Angular displacement
κ	Condition number, curvature
λ	Eigenvalue
λ, μ, ν	Direction cosines
ν	Poisson's ratio

Π	Energy
π	3.1416 . . .
ρ	Radius of curvature
σ	Normal stress
τ	Shear stress
$\{\Upsilon\}$	Vector of element relative displacements
$[\Phi]$	Static equilibrium matrix
ϕ	Angle of measure
$[\Omega]$	Mixed format force-displacement matrix

In addition to the above literal and matrix symbolism, we shall use the following graphic symbols wherever it is desired to indicate or to stress some particular characteristic of a force or structure.



Direct force and moment components

Direct force and moment resultants

Reactive force and moment components

Alternative representations of moments

Pinned joint

Rigid joint

Roller support

Pinned support

Elastic support

Fixed support

Contents

Ch.1 INTRODUCTION

One of the responsibilities of the structural design engineer is to devise arrangements and proportions of members that can withstand, economically and efficiently, the conditions anticipated during the lifetime of a structure. A central aspect of this function is the calculation of the distribution of forces within the structure and the displaced state of the system. Our objective is to describe modern methods for performing these calculations in the particular case of framed structures—trusses, planar frames, and space frames—under conditions of linear elastic behavior.

The number of structures that are actually frameworks represents only a part of those whose idealization in the form of a framework is acceptable for the purposes of analysis. Buildings of various types, portions of aerospace and ship structures, and radio telescopes can often be idealized as frameworks. The philosophy of design is predominantly one requiring elastic behavior under working loads. Although ultimate load concepts continue to gain acceptance, they, too, generally depend on linear analysis in the calculation of internal load distributions.

Fundamentally, the behavior of all types of structures—frameworks, plates, shells, or solids—is described by means of differential equations. In practice, the writing of differential equations for framed structures is rarely necessary. It has been long established that such structures may be treated as assemblages of one-dimensional members. Exact or approximate solutions to the ordinary differential equations for each member are well-known. These solutions can be cast in the form of relationships between the forces and the displacements at the ends of the member. Proper combinations of these relationships with the equations of equilibrium and compatibility at the joints and supports yields a system of algebraic equations that describes the behavior of the structure.

Structures consisting of two- or three-dimensional components—plates, membranes, shells, solids—are more complicated in that rarely do exact solutions exist for the applicable partial differential equations. One approach to obtaining practical, numerical solutions is the *finite element method*. The basic concept of the method is that a *continuum* (the total structure) can be modeled analytically by its subdivision into regions (the *finite elements*), in each of which the behavior is described by a set of assumed functions representing the stresses or displacements in that region. This permits the problem formulation to be altered to one of the establishment of a system of algebraic equations.

The practical, numerical solution of problems in structural analysis thus is seen to involve the formation and solution of systems—sometimes very large systems—of algebraic equations. Also it should be fairly clear that a member of a framed structure is simply one example of a more broadly defined family of finite elements.

Viewed in this way, structural analysis may be broken down into five parts:

1. *Basic mechanics.* The fundamental relationships of stress and strain, compatibility, and equilibrium.
2. *Finite element mechanics.* The exact or approximate solution of the differential equations of the element.
3. *Equation formulation.* The establishment of the governing algebraic equations of the system.
4. *Equation solution.* Computational methods and algorithms.
5. *Solution interpretation.* The presentation of results in a form useful in design.

This book deals chiefly with parts 3 to 5 of the above process. Specifically, it is on *matrix structural analysis.* This is the approach to these parts that currently seems to be most suitable for automation of the equation-formulation process and for taking advantage of the powerful capabilities of the electronic digital computer in solving large-order systems of equations. An understanding of basic structural mechanics and basic matrix algebra is presumed. Except for a brief introduction to the concepts of finite elements in Chapter 12, only that segment of the finite element method relating to framed structures will be included, other aspects being left to texts specializing in the fundamentals of finite element mechanics (e.g., Ref. 1.1). Computational methods and algorithms will be discussed in Chapter 11, but more comprehensive coverage can be found in books on numerical analysis (e.g., Ref. 1.2).

An appreciation of the approach to structural analysis we are taking requires some understanding of the history of this and related subjects. The following brief review may help.

1.1 A Brief History of Structural Analysis

Although it was immediately preceded by the great accomplishments of the school of French elasticians, such as Navier and St. Venant, the period from 1850 to 1875 is a logical starting point for our review. The concept of framework analysis emerged during this period, through the efforts of Maxwell (Ref. 1.3), Castigliano (Ref. 1.4), and Mohr (Ref. 1.5), among others. At the same time, the concepts of matrices were being introduced and defined by Sylvester, Hamilton, and Cayley (Ref. 1.6). These concepts are the foundations of matrix structural analysis, which did not take form until nearly 80 years later.

An excellent chronicle of developments in structural mechanics in the period 1875 to 1920 is found in Timoshenko's *History of Strength of Mate-*