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Random Sets

Theory and Applications

With 39 Illustrations

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FOREWORD

This IMA Volume in Mathematics and its Applications

RANDOM SETS: THEORY AND APPLICATIONS

is based on the proceedings of a very successful 1996 three-day Summer Program on "Application and Theory of Random Sets." We would like to thank the scientific organizers: John Goutsias (Johns Hopkins University), Ronald P.S. Mahler (Lockheed Martin), and Hung T. Nguyen (New Mexico State University) for their excellent work as organizers of the meeting and for editing the proceedings. We also take this opportunity to thank the Army Research Office (ARO), the Office of Naval Research (ONR), and the Eagan, Minnesota Engineering Center of Lockheed Martin Tactical Defense Systems, whose financial support made the summer program possible.

Avner Friedman

Robert Gulliver

PREFACE

"Later generations will regard set theory as a disease from which one has recovered."

– Henri Poincaré

Random set theory was independently conceived by D.G. Kendall and G. Matheron in connection with stochastic geometry. It was however G. Choquet with his work on capacities and later G. Matheron with his influential book on *Random Sets and Integral Geometry* (John Wiley, 1975), who laid down the theoretical foundations of what is now known as the theory of *random closed sets.* This theory is based on studying probability measures on the space of *closed subsets* of a locally compact, Hausdorff, and separable base space, endowed with a special topology, known as the *hit-or-miss topology.* Random closed sets are just random elements on these spaces of closed subsets. The mathematical foundation of random closed sets is essentially based on *Choquet's capacity theorem*, which characterizes distribution of these set-valued random elements as nonadditive set functions or "nonadditive measures." In theoretical statistics and stochastic geometry such nonadditive measures are known as *infinitely monotone*, *alternating capacities of infinite order*, or *Choquet capacities*, whereas in expert systems theory they are more commonly known as *belief measures*, *plausibility measures*, *possibility measures*, etc. The study of random sets is, consequently, inseparable from the study of nonadditive measures.

Random set theory, to the extent that is familiar to the broader technical community at all, is often regarded as an obscure and rather exotic branch of pure mathematics. In recent years, however, various aspects of the theory have emerged as promising new theoretical paradigms for several areas of academic, industrial, and defense-related R&D. These areas include stochastic geometry, stereology, and image processing and analysis; expert systems theory; an emerging military technology known as "information fusion;" and theoretical statistics.

Random set theory provides a solid theoretical foundation for certain image processing and analysis problems. As a simple example, Fig. 1 illustrates an image of an object (a cube), corrupted by various noise processes, such as clutter and occlusions. Images, as well as noise processes, can be modeled as random sets. Nonlinear algorithms, known collectively as *morphological operators*, may be used here in order to provide a means of recovering the object from noise and clutter. Random set theory, in conjunction with *mathematical morphology*, provides a rigorous statistical foundation for nonlinear image processing and analysis problems that is analogous to that of conventional linear statistical signal processing. For example, it allows one to demonstrate that there exist optimal algorithms that recover images from certain types of noise processes.

In *expert systems theory*, random sets provide a means of modeling and

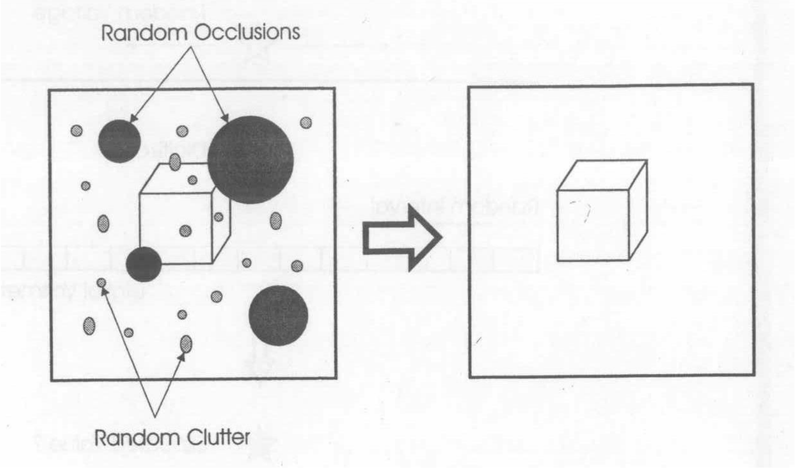

FIG. 1. *Random sets and image processing.*

manipulating evidence that is imprecise (e.g., poorly characterized sensor signatures), vague or fuzzy (e.g., natural language statements), or contingent (e.g., rules). In Fig. 2, for example, we see an illustration of a natural-language statement such as "*Gustav is NEAR the tower.*" Each of the four (closed) ellipses represents a plausible interpretation of the concept "*NEAR the tower*," and the numbers p_1, p_2, p_3, p_4 represent the respective beliefs that these interpretations of the concept are valid. A discrete random variable that takes the four ellipses as its values, and which has respective probabilities p_1, p_2, p_3, p_4 of attaining those values, is a random set representative of the concept.

Random sets provide also a convenient mathematical foundation for a statistical theory that supports *multisensor, multitarget information fusion.* In Fig. 3, for example, an unknown number of unknown targets are being interrogated by several sensors whose respective observations can be of very diverse type, ranging from statistical measurements generated by radars to English-language statements supplied by human observers. If the sensor suite is interpreted as a single sensor, if the target set is interpreted as a single target, and if the observations are interpreted as a single *finite-set* observation, then it turns out that problems of this kind can be attacked using direct generalizations of standard statistical techniques by means of the theory of random sets.

Finally, random set theory is playing an increasingly important role in *theoretical statistics.* For example, suppose that a continuous but random voltage is being measured using a digital voltmeter and that, on the basis of the measured data, we wish to derive bounds on the expected value of the original random variable, see Fig. 4. The observed quantity is a random subset (specifically, a random interval) and the bounds can be expressed in terms of certain nonlinear integrals, called *Choquet integrals*, computed

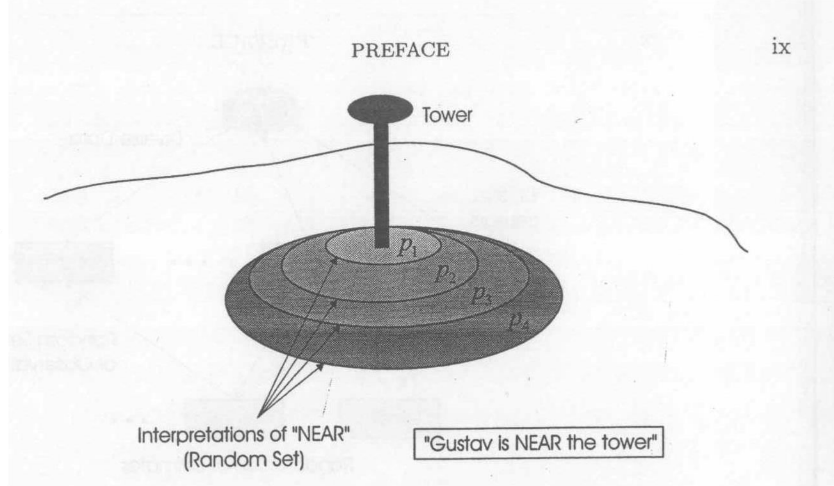

FIG. 2. *Random sets and expert systems.*

with respect to nonadditive measures associated with that random subset.

On August 22–24, 1996, an international group of researchers convened under the auspices of the *Institute for Mathematics and Its Applications* (IMA), in Minneapolis, Minnesota, for a scientific workshop on the "Applications and Theory of Random Sets." To the best of our knowledge this was the first scientific gathering in the United States, devoted primarily to the subject of random sets and allied concepts. The immediate purpose of the workshop was to bring together researchers and other parties from academia, industry, and the U.S. Government who were interested in the potential application of random set theory to practical problems of both industrial and government interest. The long-term purpose of the workshop was expected to be the enhancement of imaging, information fusion, and expert system technologies and the more efficient dissemination of these technologies to industry, the U.S. Government, and academia.

To accomplish these two purposes we tried to bring together, and encourage creative interdisciplinary cross-fertilization between, three communities of random-set researchers which seem to have been largely unaware of each other: theoretical statisticians, those involved in imaging applications, and those involved in information fusion and expert system applications. Rather than "rounding up the usual suspects"–a common, if incestuous, practice in organizing scientific workshops–we attempted to mix experienced researchers and practitioners having complementary interests but who, up until that time, did not have the opportunity for scientific interchange.

The result was, at least for a scientific workshop, an unusually diverse group of researchers: theoretical statisticians; academics involved in applied research; personnel from government organizations and laboratories, such as the National Institutes of Health, Naval Research and Development, U.S. Army Research Office, and USAF Wright Labs, as well as industrial R&D engineers from large and small companies, such as Applied Biomath-

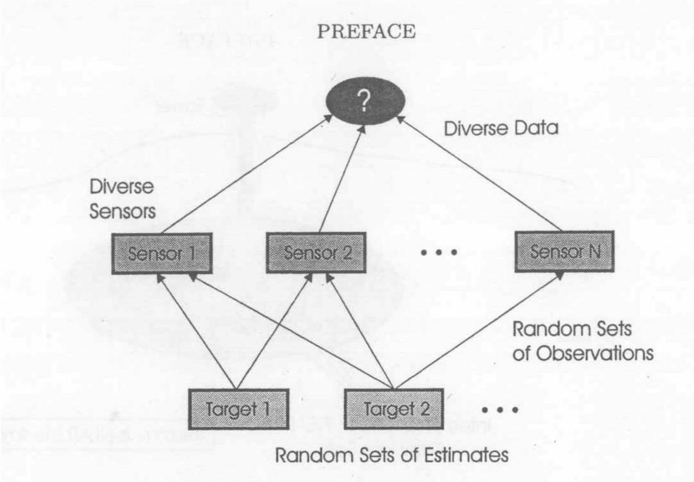

FIG. 3. *Random sets and information fusion.*

ematics, Data Fusion Corporation, Lockheed Martin, Neptune and Company, Oasis Research Center, Raytheon, Texas Instruments, and Xerox. The papers in this volume reflect this diversity. A few papers are tutorial in nature, some are detailed mathematical treatises, some are summary overviews of an entire subject, and still others are investigations rooted in practical engineering intuition.

The workshop was structured into three sessions, devoted respectively to the following topic areas, each organized and chaired by one of the editors:

- *Image Modeling and Analysis* (J. Goutsias).
- *Information/Data Fusion and Expert Systems* (R.P.S. Mahler).
- *Theoretical Statistics and Expert Systems* (H.T. Nguyen).

Each session was preceded by a plenary presentation given by a researcher of world standing:

- Ilya Molchanov, University of Glasgow, Scotland.
- Jean-Yves Jaffray, University of Paris VI, France.
- Ulrich Höhle, Bergische Universität, Germany.

The following institutions kindly extended their support to this workshop:

- *U.S. Office of Naval Research*, Mathematical, Computer, and Information Sciences Division.
- *U.S. Army Research Office*, Electronics Division.
- *Lockheed Martin*, Eagan, Minnesota Engineering Center.

The editors wish to express their appreciation for the generosity of these sponsors. They also extend their special gratitude to the following individuals for their help in ensuring success of the workshop: Avner Friedman,

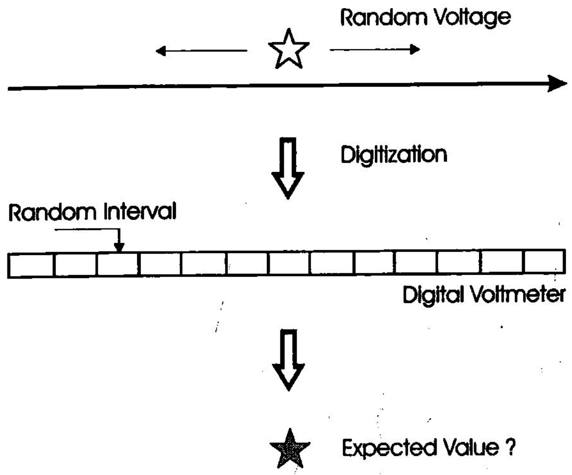

FIG. 4. *Random sets and theoretical statistics.*

IMA, Director; Julia Abrahams, Office of Naval Research; William Sander, Army Research Office; Wesley Snyder, North Carolina State University; Marjorie Hahn, Tufts University; Larry Wasserman, Carnegie-Mellon University; Charles Mills, Lockheed Martin, Director of Engineering; Amy Cavanaugh, IMA, Workshop Coordinator; and John Schepers, IMA, Workshop Financial Coordinator.

In committing these proceedings to the attention of the larger scientific and engineering community, the editors hope that the workshop will have thereby contributed to one of the primary goals of IMA: facilitating creative interchange between statisticians, scientists, and academic and industrial engineers in technical domains of potential practical significance.

John Goutsias
Ronald P.S. Mahler
Hung T. Nguyen

Workshop on
Applications and Theory of Random Sets
Institute for Mathematics and its Applications (IMA)
University of Minnesota
Minneapolis, Minnesota
August 22–24, 1996

Participants
(From left-to-right)

Lower Row: Scott Ferson, Wesley Snyder, Yidong Chen, Bert Fristedt, John Handley, Sinan Batman, Edward Dougherty, Nikolaos Sidiropoulos, Dan Schonfeld, I.R. Goodman, Wolfgang Kober, Stan Music, Ronald Mahler, Jean-Yves Jaffray, Elbert Walker, Carol Walker, Hung Nguyen, John Goutsias

Upper Row: Robert Launer, Paul Black, Tonghui Wang, Shozo Mori, Robert Taylor, Ulrich Hőehle, Ilya Molchanov, Michael Stein, Krishnamoorthy Sivakumar, Fred Daum, Teddy Seidenfeld

CONTENTS

PART I
Image Modeling and Analysis

MORPHOLOGICAL ANALYSIS OF RANDOM SETS AN INTRODUCTION

JOHN GOUTSIAS*

Abstract. This paper provides a brief introduction to the problem of processing random shapes by means of mathematical morphology. Compatibility issues with mathematical morphology suggest that shapes should be modeled as random closed sets. This approach however is limited by theoretical and practical difficulties. Morphological sampling is used to transform a random closed set into a much simpler discrete random set. It is argued that morphological sampling of a random closed set is a sensible thing to do in practical situations. The paper concludes by reviewing three useful random set models.

Key words. Capacity Functional, Discretization, Mathematical Morphology, Random Sets, Shape Processing and Analysis.

AMS(MOS) subject classifications. 60D05, 60K35, 68U10

1. Introduction. Development of stochastic techniques for image processing and analysis is an important area of investigation. Consider, for example, the problem of analyzing microscopic images of cells, like the ones depicted in the first row of Fig. 1. Image analysis consists of obtaining measurements characteristic to the images under consideration. When we are only interested in geometric measurements (e.g., object location, orientation, area, perimeter length, etc.), and in order to simplify our problem, we may decide to reduce gray-scale images into binary images by means of thresholding, thus obtaining shapes, like the ones depicted in the second row of Fig. 1. Since shape information is frequently random, as is clear from Fig. 1, binary microscopic images of cells may be conceived as realizations of a two-dimensional *random set* model. In this case, measurements are considered to be estimates of random variables, and statistical analysis of such random variables may lead to successful shape analysis.

There are other reasons why stochastic techniques are important for shape processing and analysis. In many instances, shape information is not directly observed. For example, it is quite common that a three-dimensional object (e.g., a metal or a mineral) is partially observed through an imaging system that is only capable of producing two-dimensional pictures of cross sections. The problem here is to infer geometric properties of the three-dimensional object under consideration by means of measurements obtained from the two-dimensional cross sections (this is the main theme in *stereology* [1], an important branch of *stochastic geometry* [2]). Another example is the problem of restoring shape information corrupted by sensor

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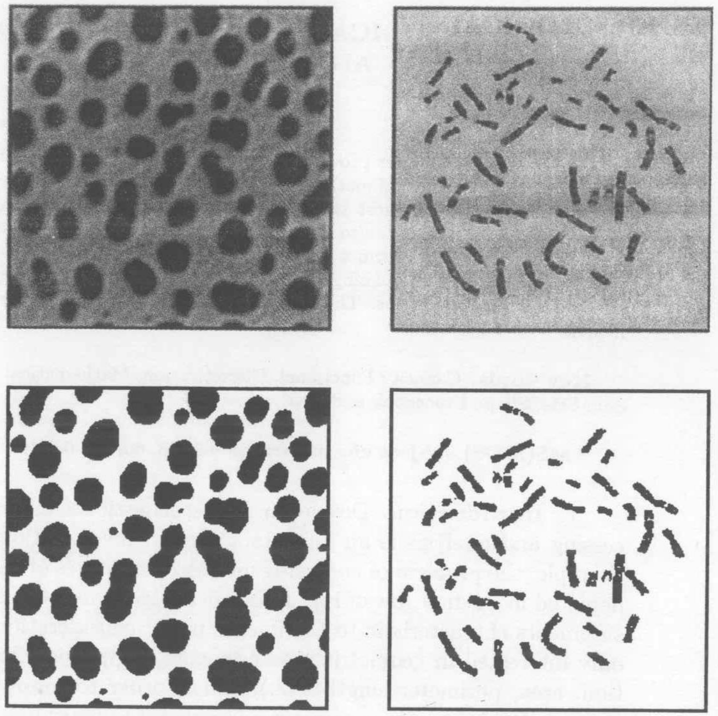

FIG. 1. *Gray-scale (first row) microscopic images of cells and their binary counterparts (second row) obtained after thresholding. Geometric features of interest (e.g., location, orientation, area, perimeter length, etc.) are usually preserved after thresholding.*

noise and clutter. This task is very important in military target detection problems, where targets are frequently imaged through hostile environments by means of imperfect imaging sensors. Both problems consist of recovering shape information from imperfectly or partially observed data and are clearly *ill-posed inverse problems* that need proper *regularization.*

A popular approach to regularizing inverse problems is by means of *stochastic regularization* techniques. A random model is assumed for the images under consideration and statistical techniques are then employed for recovering lost information from available measurements. This approach frequently leads to robust and highly effective algorithms for shape recovery. To be more precise, let us consider the problem of restoring shape information from degraded data. Shapes are usually combined by means of set union or intersection (or set difference, since $A \setminus B = A \cap B^c$). It is therefore reasonable to model shapes as sets (and more precisely as random sets) and assume that data $\mathbf{Y}$ are described by means of a *degradation equation* of the form:

$$\mathbf{Y} = (\mathbf{X} \setminus \mathbf{N}_1) \cup \mathbf{N}_2 \,, \tag{1.1}$$

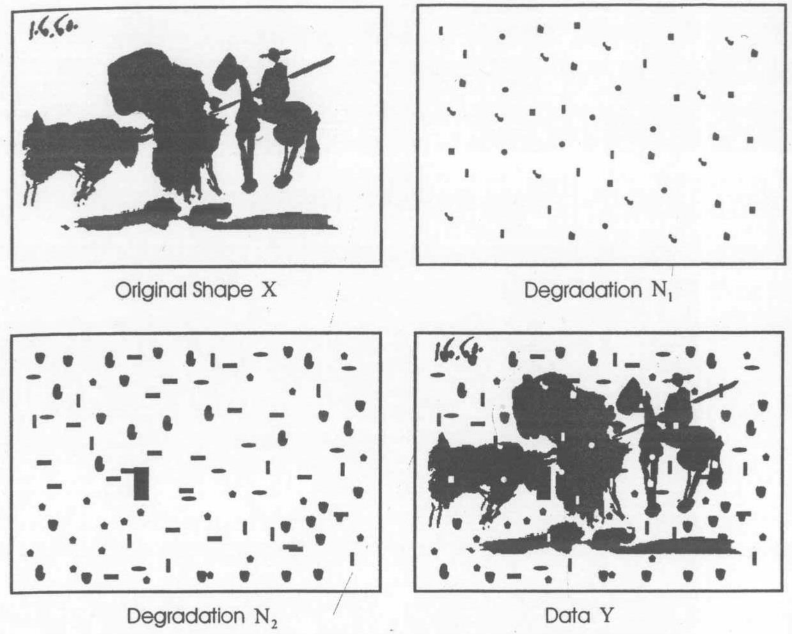

FIG. 2. *A binary Picasso image* $\mathbf{X}$*, the degradation noises* $\mathbf{N}_1$, $\mathbf{N}_2$*, and data* $\mathbf{Y}$*, obtained by means of (1.1).*

where $\mathbf{X}$ is a random set that models the shape under consideration and $\mathbf{N}_1$, $\mathbf{N}_2$ are two random sets that model degradation. In particular, $\mathbf{N}_1$ may model incomplete data collection, whereas, $\mathbf{N}_2$ may model degradation due to sensor noise and clutter. Figure 2 depicts the effect of degradations $\mathbf{N}_1$, $\mathbf{N}_2$ on a binary Picasso image[1] by means of (1.1). The problem of shape restoration consists of designing a set operator Ψ such that

$$\hat{\mathbf{X}} = \Psi(\mathbf{Y}) = \Psi((\mathbf{X} \setminus \mathbf{N}_1) \cup \mathbf{N}_2) \,, \tag{1.2}$$

is "optimally" close to $\mathbf{X}$, in some sense. Refer to [3], [4] for more information on this subject and for specific "optimal" techniques for shape restoration by means of random set modeling.

Since shapes are combined by means of unions and intersections, it is natural to consider *morphological image operators* Ψ in (1.2) [5]. This leads to a popular technique for shape processing and analysis, known as *mathematical morphology*, that is briefly described in Section 2. Our main purpose here is to provide an introduction to the problem of processing random shapes by means of mathematical morphology. Compatibility issues with mathematical morphology suggest that shapes should be modeled as

[1] Pablo Picasso, *Pass with the Cape*, 1960.

random closed sets [6]. However, this approach is limited by theoretical and practical difficulties, as explained in Section 3. In Section 4, *morphological sampling* is employed so as to transform a random closed set into a much simpler *discrete random set*. It is argued that, in many applications, morphological sampling of a random closed set is a sensible thing to do. Discrete random sets are introduced in Section 5. Three useful random set models are then presented in Section 6, and concluding remarks are finally presented in Section 7.

2. Mathematical morphology. A popular technique for shape processing and analysis is mathematical morphology. This technique was originally introduced by Matheron [6] and Serra [7] as a tool for investigating geometric structure in binary images. Although extensions of mathematical morphology to grayscale and other images (e.g., multispectral images) exist (e.g., see the book by Heijmans [5]), we limit our exposition here to the binary case. In the following, a binary image X will be first considered to be a subset of the two-dimensional Euclidean space $\mathbb{R}^2$.

Morphological shape operators are defined by means of a *structuring element* $A \subset \mathbb{R}^2$ (shape mask) which interacts with a binary image X so as to enhance or extract useful information. The type of interaction is determined by testing whether the translated structuring element $A_v = \{a+v \mid a \in A\}$ *hits* or *misses* X; i.e., testing whether $X \cap A_v \neq \emptyset$ (A_v hits X) or $X \cap A_v = \emptyset$ (A_v misses X). This is the main idea behind the most fundamental morphological operator, known as the *hit-or-miss transform*, given by

$$X \otimes (A, C) \; = \; \{v \in \mathbb{R}^2 \mid A_v \subseteq X, X \cap C_v = \emptyset\}\,, \tag{2.1}$$

where A, C are two structuring elements such that $A \cap C = \emptyset$. Although the hit-or-miss transform satisfies a number of useful properties, perhaps the most striking one is the fact that any *translation invariant* shape operator Ψ (i.e., an operator for which $\Psi(X_v) = [\Psi(X)]_v$, for every $v \in \mathbb{R}^2$) can be written as a union of hit-or-miss transforms (e.g., see [5]).

When $C = \emptyset$ in (2.1), the hit-or-miss transform reduces to a morphological operator known as *erosion*. The erosion of a binary image X by a structuring element A is given by

$$X \ominus A \; = \; \{v \in \mathbb{R}^2 \mid A_v \subseteq X\}.$$

It is clear that erosion comprises of all points v of $\mathbb{R}^2$ for which the structuring element A_v, located at v, fits inside X. The dual of erosion, with respect to set complement, is known as *dilation*. The dilation of a binary image X by a structuring element A is given by

$$X \oplus A \; = \; (X^c \ominus \check{A})^c \; = \; \{v \in \mathbb{R}^2 \mid X \cap \check{A}_v \neq \emptyset\}\,, \tag{2.2}$$

where $\check{A} = \{-v \mid v \in A\}$ is the *reflection* of A about the origin. Therefore, dilation comprises of all points v of $\mathbb{R}^2$ for which the translated structuring

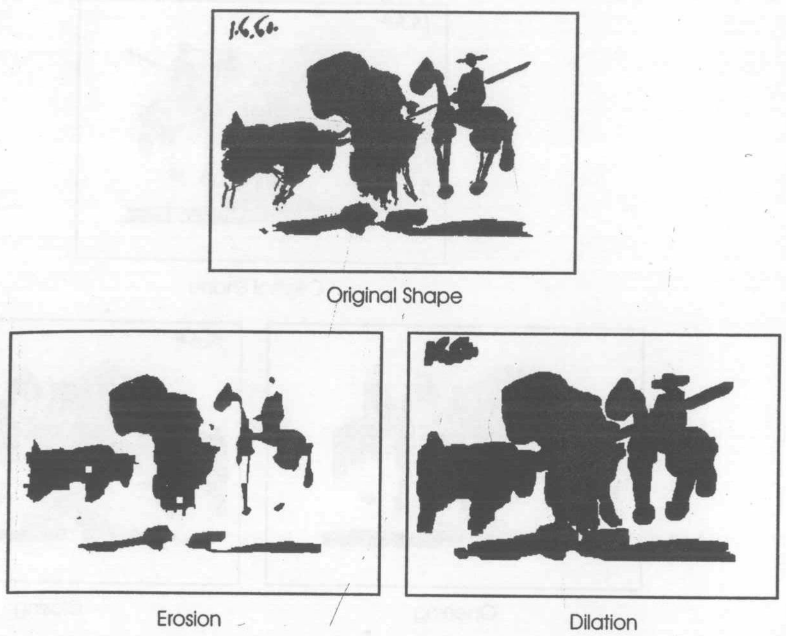

FIG. 3. *Erosion and dilation of the Picasso image X depicted in Fig. 2 by means of a 5×5 SQUARE structuring element A. Notice that erosion comprises of all pixels v of X for which the translated structuring element A_v fits inside X, whereas dilation comprises of all pixels v in $\mathbb{R}^2$ for which the translated structuring element A_v hits X.*

element $\check{A}_v$ hits X. From (2.1) and (2.2) it is clear that $X \otimes (\emptyset, \check{A}) = (X \oplus A)^c$, and dilation is therefore the set complement of the hit-or-miss transform of X by $(\emptyset, \check{A})$. It can be shown that erosion is *increasing* (i.e., $X \subseteq Y \Rightarrow X \ominus A \subseteq Y \ominus A$) and *distributes* over intersection (i.e., $(\cap_{i \in I} X_i) \ominus A = \cap_{i \in I}(X_i \ominus A)$), whereas, dilation is increasing and distributes over union (i.e., $(\cup_{i \in I} X_i) \oplus A = \cup_{i \in I}(X_i \oplus A)$). Furthermore, if A contains the origin, then erosion is *anti-extensive* (i.e., $X \ominus A \subseteq X$) whereas dilation is *extensive* (i.e., $X \subseteq X \oplus A$). The effects of erosion and dilation on the binary Picasso image are illustrated in Fig. 3.

Suitable composition of erosions and dilations generates more complicated morphological operators. One such composition produces two useful morphological operators known as *opening* and *closing*. The opening of a binary image X by a structuring element A is given by

$$X \circ A \; = \; (X \ominus A) \oplus A\,,$$

whereas the closing is given by

$$X \bullet A \; = \; (X \oplus A) \ominus A\,.$$

It is not difficult to show that $X \circ A = \bigcup_{v:\, A_v \subseteq X} A_v$ and, therefore, the

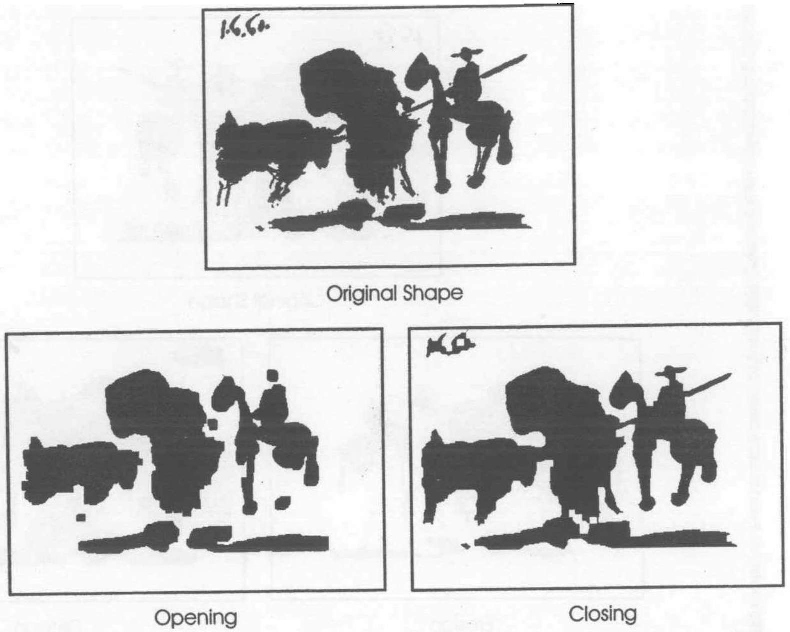

FIG. 4. *Opening and closing of the Picasso image X depicted in Fig. 2, by means of a 5×5 SQUARE structuring element A. Notice that opening comprises of the union of all translated structuring elements A_v that fit inside X, whereas closing is the set complement of the union of all translated structuring elements A_v that fit inside X^c.*

opening $X \circ A$ is the union of all translated structuring elements A_v that fit inside shape X. On the other hand, the closing is the dual of opening in the sense that $X \bullet A = (X^c \circ \check{A})^c$. It can be shown that opening is increasing, anti-extensive, and *idempotent* (i.e., $(X \circ A) \circ A = X \circ A$), whereas closing is increasing, extensive, and idempotent. Figure 4 depicts the effects of opening and closing on the binary Picasso image. Notice that the opening $X \circ A$ behaves as a *shape filter*, in the sense that it eliminates all components of X that cannot contain a translated copy of A. In fact, opening and closing are special cases of *morphological filters* [8]. By definition, a morphological filter is any image operator that is increasing and idempotent. Clearly, opening and closing are morphological filters whereas erosion and dilation are not, since they are not idempotent.

3. Random sets. A *random set* (RS) on $\mathbb{R}^2$ is a random element that takes values in a collection $\mathcal{S}$ of subsets of $\mathbb{R}^2$. If $(\Omega, \Sigma(\Omega), \mu)$ is a probability space [9], then a RS $\mathbf{X}$ is a measurable mapping from Ω into $\mathcal{S}$, that is

$$\{\omega \in \Omega \mid \mathbf{X}(\omega) \in \mathcal{A}\} \in \Sigma(\Omega), \quad \forall \mathcal{A} \in \Sigma(\mathcal{S}),$$

where $\Sigma(\mathcal{S})$ is an appropriate σ-field in $\mathcal{S}$. The RS $\mathbf{X}$ defines a *probability distribution* P_X on $\Sigma(\mathcal{S})$ by

$$P_X[\mathcal{A}] = \mu[\{\omega \in \Omega \mid \mathbf{X}(\omega) \in \mathcal{A}\}], \quad \forall \mathcal{A} \in \Sigma(\mathcal{S}).$$

A common choice for $\mathcal{S}$ is the *power set* $\mathcal{P} = \mathcal{P}(\mathbb{R}^2)$ of $\mathbb{R}^2$ (i.e., the collection of all subsets of $\mathbb{R}^2$) with $\Sigma(\mathcal{S}) = \Sigma(\mathcal{P})$ being the σ-field in $\mathcal{P}$ generated by sets of the form $\{X \in \mathcal{P} \mid v_i \notin X, i = 1,2,...,m;\ w_j \in X, j = 1,2,...,n\}$, where $v_i, w_j \in \mathbb{R}^2$, and $m, n \geq 0$ are integers. It is worthwhile noticing here that $\Sigma(\mathcal{P})$ is also generated by the simple family $\{\{X \in \mathcal{P} \mid X \cap \{v\} = \emptyset\}, v \in \mathbb{R}^2\}$. Consider the *finite-dimensional distribution functions* of RS $\mathbf{X}$, given by

$$p_{v_1,v_2,...,v_n}(x_1, x_2, ..., x_n) = P_X[\, \mathrm{I}_{\mathbf{X}}(v_i) = x_i,\ i = 1,2,...,n\,],$$

where

$$\mathrm{I}_{\mathbf{X}}(v) = \begin{cases} 1, & \text{if } v \in \mathbf{X} \\ 0, & \text{otherwise} \end{cases},$$

is the *indicator function* of $\mathbf{X}$ and $v_i \in \mathbb{R}^2$, $x_i \in \{0,1\}$. As a direct consequence of *Kolmogorov's theorem* [9], the probability distribution of a RS $\mathbf{X}$: $\Omega \to \mathcal{P}$ is uniquely determined from a collection of finite-dimensional distribution functions $\{p_{v_1,v_2,...,v_n}(x_1, x_2, ..., x_n);\ v_i \in \mathbb{R}^2, x_i \in \{0,1\}, n \geq 1\}$ that satisfy Kolmogorov's conditions of *symmetry* and *consistency* [9] (see also [10]). Therefore, a random set $\mathbf{X}$: $\Omega \to \mathcal{P}$ is uniquely specified by means of its finite-dimensional distribution functions.

A question that immediately arises here is whether the previous choices for $\mathcal{S}$ and $\Sigma(\mathcal{S})$ lead to a definition for a RS that is compatible with mathematical morphology. To be more precise, let us concentrate on the problem of transforming a RS by means of a morphological operator Ψ (that, in this paper, is limited to an erosion, dilation, opening, or closing). If $\mathbf{X}$ is a random set, it is expected that $\Psi(\mathbf{X})$ will also be a random set. This translates to the requirement that morphological operators need to be measurable with respect to $\Sigma(\mathcal{S})$. For example, if $\mathbf{X}$ is a RS, we expect that the dilation $\mathbf{X} \oplus K$ of $\mathbf{X}$ by a *compact* (i.e., topologically *closed* and *bounded*) structuring element K is a RS as well. However, it is not difficult to verify that $\{X \in \mathcal{P} \mid v \notin X \oplus K\} = \{X \in \mathcal{P} \mid X \cap (\check{K} \oplus \{v\}) = \emptyset\}$, which is clearly not an element of $\Sigma(\mathcal{P})$, since K is not necessarily finite. Hence, it is not in general possible to determine the probability $P_{X \oplus K}[\mathrm{I}_{X \oplus K}(v) = 0]$ that the dilated RS $\mathbf{X} \oplus K$ does not contain point v from the probability distribution of RS $\mathbf{X}$. In other words, the previous probabilistic description of RS $\mathbf{X}$ is not sufficiently rich to determine the probability distribution of a morphologically transformed RS $\mathbf{X} \oplus K$. Therefore, the previously discussed choices for $\mathcal{S}$ and $\Sigma(\mathcal{S})$ are not compatible with mathematical morphology. If we assume that shapes include their boundary (which is the most common case in practice), then we can set $\mathcal{S} = \mathcal{F}$, where $\mathcal{F}$ is the collection of all closed subsets of $\mathbb{R}^2$, and consider a σ-field $\Sigma(\mathcal{F})$ containing sets of the form $\{X \in \mathcal{F} \mid X \cap K = \emptyset\}$, for $K \in \mathcal{K}$, where $\mathcal{K}$ is the collection of all compact subsets of $\mathbb{R}^2$. It can be shown that the smallest such σ-field is the one generated by the family $\{\{X \in \mathcal{F} \mid X \cap K = \emptyset\}, K \in \mathcal{K}\}$ as well as by the family $\{\{X \in \mathcal{F} \mid X \cap G \neq \emptyset\}, G \in \mathcal{G}\}$, where $\mathcal{G}$ denotes the collection of all (topologically) *open* subsets of $\mathbb{R}^2$. This leads to modeling random shapes by means of *random closed sets* (RACS). A RACS $\mathbf{X}$ is a measurable mapping from Ω into $\mathcal{F}$, that is [6], [11]

$$\{\omega \in \Omega \mid \mathbf{X}(\omega) \in \mathcal{A}\} \in \Sigma(\Omega), \quad \forall \mathcal{A} \in \Sigma(\mathcal{F}) .$$

The RACS $\mathbf{X}$ defines a probability distribution P_X on $\Sigma(\mathcal{F})$ by

$$P_X[\mathcal{A}] = \mu[\{\omega \in \Omega \mid \mathbf{X}(\omega) \in \mathcal{A}\}], \quad \forall \mathcal{A} \in \Sigma(\mathcal{F}) .$$

An alternative to specifying a RACS by means of a probability distribution, that is defined over classes of sets in $\Sigma(\mathcal{F})$, is to specify the RACS by means of its *capacity functional*, defined over compact subsets of $\mathbb{R}^2$. The capacity functional T_X of a RACS $\mathbf{X}$ is defined by

$$T_X(K) = P_X[\mathbf{X} \cap K \neq \emptyset], \quad \forall K \in \mathcal{K} .$$

This functional satisfies the following five properties:

PROPERTY 3.1. Since no closed set hits the empty set, $T_X(\emptyset) = 0$.

PROPERTY 3.2. Being a probability, T_X satisfies $0 \leq T_X(K) \leq 1$, for every $K \in \mathcal{K}$.

PROPERTY 3.3. The capacity functional is increasing on $\mathcal{K}$; i.e.,

$$K_1, K_2 \in \mathcal{K} \text{ and } K_1 \subseteq K_2 \;\Rightarrow\; T_X(K_1) \leq T_X(K_2) .$$

PROPERTY 3.4. The capacity functional is *upper semi-continuous* (u.s.c.) on $\mathcal{K}$, which is equivalent to

$$K_n \downarrow K \text{ in } \mathcal{K} \;\Rightarrow\; T_X(K_n) \downarrow T_X(K) ,$$

where $A_n \downarrow A$ means that $\{A_n\}$ is a decreasing sequence such that $\inf A_n = A$.

PROPERTY 3.5. If, for $K, K_1, K_2, \ldots \in \mathcal{K}$,

$$Q_X^{(0)}(K) = Q_X(K) = P_X[\mathbf{X} \cap K = \emptyset] = 1 - T_X(K) , \tag{3.1}$$

and

$$Q_X^{(n)}(K; K_1, K_2, \ldots, K_n) = Q_X^{(n-1)}(K; K_1, K_2, \ldots, K_{n-1}) - Q_X^{(n-1)}(K \cup K_n; K_1, K_2, \ldots, K_{n-1}) ,$$

for $n = 1, 2, \ldots$, then

$$0 \leq Q_X^{(n)}(K; K_1, K_2, \ldots, K_n) = P_X[\mathbf{X} \cap K = \emptyset;\ \mathbf{X} \cap K_i \neq \emptyset,\ i = 1, 2, \ldots, n] \leq 1 , \tag{3.2}$$

for every $n \geq 1$.

A functional T_X that satisfies properties 3.3–3.5 above is known as an *alternating capacity of infinite order* or a *Choquet capacity* [12]. Therefore, the capacity functional of a RACS is a Choquet capacity that in addition satisfies properties 3.1 and 3.2. As a direct consequence of the *Choquet-Kendall-Matheron theorem* [6], [12], [13], the probability distribution of a RACS $\mathbf{X}$ is uniquely determined from a Choquet capacity $T_X(K), K \in \mathcal{K}$, that satisfies properties 3.1 and 3.2. It can be shown that

$$T_X(K_1 \cup K_2) \leq T_X(K_1) + T_X(K_2) , \quad \forall K_1, K_2 : K_1 \cap K_2 = \emptyset .$$

The capacity functional is therefore only subadditive and hence not a measure. However, knowledge of $T_X(K)$, for every $K \in \mathcal{K}$, allows us determine the probability distribution of $\mathbf{X}$. Functional $Q_X(K)$ in (3.1) is known as the *generating functional* of RACS $\mathbf{X}$, whereas, functional $Q_X^{(n)}(K; K_1, K_2, \ldots, K_n)$ is the probability that the RACS $\mathbf{X}$ misses K and hits K_i, $i = 1, 2, \ldots, n$ (see (3.2)).

Let us now consider the problem of morphologically transforming a RACS. As we mentioned before, if $\Psi\colon \mathcal{F} \to \mathcal{F}$ is a measurable operator with respect to $\Sigma(\mathcal{F})$, then $\Psi(\mathbf{X})$ will be a RACS, provided that $\mathbf{X}$ is a RACS. In simple words, the probability distribution of $\Psi(\mathbf{X})$ can be in principle determined from the probability distribution of $\mathbf{X}$ and knowledge of operator Ψ. It can be shown that erosion, dilation, opening, and closing of a closed set, by means of a compact structuring element, are all measurable with respect to $\Sigma(\mathcal{F})$. Therefore, erosion, dilation, opening, and closing of a RACS, by means of a compact structuring element, is also a RACS. Understanding the effects that morphological transformations have on random sets requires statistical analysis. We would therefore need to relate statistics of $\Psi(\mathbf{X})$ with statistics of $\mathbf{X}$. This can be done by relating the capacity functional $T_{\Psi(X)}$ of $\Psi(\mathbf{X})$ with the capacity functional T_X of $\mathbf{X}$. In general, a simple closed-form relationship is feasible only in the case of dilation, in which case [6], [14]

$$T_{X \oplus A}(K) = T_X(K \oplus \check{A}) , \quad \forall A, K \in \mathcal{K} . \tag{3.3}$$

However, it can be shown that [15]

$$T_{X \ominus A}(K) = 1 - \sum_{K' \subseteq K} (-1)^{|K'|} R_X(K' \oplus A) , \; \forall A, K \in \mathcal{K}_o , \tag{3.4}$$

with

$$R_X(K) = P_X[\mathbf{X} \supseteq K] = \sum_{K' \subseteq K} (-1)^{|K'|} [\, 1 - T_X(K') \,] , \quad \forall K \in \mathcal{K}_o, \tag{3.5}$$

where $\mathcal{K}_o \subset \mathcal{K}$ is the collection of all *finite* subsets of $\mathbb{R}^2$. Therefore, a closed-form relationship between $T_{X\ominus A}(K)$ and $T_X(K)$ can be obtained, by means of (3.4), (3.5), when both A and K are finite. Furthermore, and as a direct consequence of (3.3)–(3.5), we have that [15]

$$(3.6) \qquad T_{X\bigcirc A}(K) = 1 - \sum_{K' \subseteq K \oplus \check{A}} (-1)^{|K'|} R_X(K' \oplus A) \, , \; \forall A, K \in \mathcal{K}_o \, ,$$

whereas

$$(3.7) \qquad T_{X\bullet A}(K) = 1 - \sum_{K' \subseteq K} (-1)^{|K'|} R_{X\oplus A}(K' \oplus A) \, , \; \forall A, K \in \mathcal{K}_o \, ,$$

with

$$(3.8) \qquad R_{X\oplus A}(K) = \sum_{K' \subseteq K} (-1)^{|K'|} [\, 1 - T_X(K' \oplus \check{A}) \,] \, , \; \forall K \in \mathcal{K}_o \, .$$

It is worthwhile noticing that (3.4)–(3.8) are related to the well known *Möbius transform* of combinatorics (e.g., see [16]). If W is a finite set, $\mathcal{P}(W)$ its power set, and $U(K)$ is a real-valued functional on $\mathcal{P}(W)$, then the Möbius transform of U is a functional $V(K)$ on $\mathcal{P}(W)$, given by

$$(3.9) \qquad V(K) = \sum_{K' \subseteq K} U(K') \, , \; \forall K \in \mathcal{P}(W) \, .$$

Referring to (3.4), (3.5), it is clear that $1 - T_{X\ominus A}(K)$ is the Möbius transform of functional $(-1)^{|K|} R_X(K \oplus A)$, whereas $R_X(K)$ is the Möbius transform of functional $(-1)^{|K|}[1 - T_X(K)]$. Similar remarks hold for (3.6)–(3.8). Notice that $U(K)$ can be recovered from $V(K)$ by means of the *inverse Möbius transform*, given by

$$(3.10) \qquad U(K) = \sum_{K' \subseteq K} (-1)^{|K \setminus K'|} V(K') \, , \; \forall K \in \mathcal{P}(W) \, .$$

Direct implementation of (3.4)–(3.8) is hampered by substantial storage and computational requirements. However, the storage scheme and the *fast Möbius transform* introduced in [17] can be effectively employed here so as to ease such requirements. We should also point-out here that the capacity functional of a RACS is the same as the *plausibility functional* used in the theory of evidence in expert systems [18], and that $R_X(K)$ in (3.5) is known as the *commonality functional* [17]. Finally, there is a close relationship between random set theory and expert systems, as is nicely explained by Nguyen and Wang in [19] and Nguyen and Nguyen in [20].

4. Discretization of RACSs. From our previous discussion, it is clear that the capacity functionals $T_{X\ominus A}(K)$, $T_{X\bigcirc A}(K)$, and $T_{X\bullet A}(K)$ can be evaluated from the capacity functional $T_X(K)$ only when A and K are finite. It is therefore desirable to: (a) consider finite structuring elements A, and (b) make sure that RACSs $\mathbf{X} \ominus A$, $\mathbf{X} \bigcirc A$, and $\mathbf{X} \bullet A$ are uniquely specified by means of their capacity functionals only over $\mathcal{K}_o$. Requirement (b) is not true in general, even if A is finite. However, we may consider discretizing $\mathbf{X}$, by sampling it over a sampling grid S, in order to obtain a *discrete random set* (DRS) $\mathbf{X}_d = \sigma(\mathbf{X})$, where σ is a sampling operator. It will soon become apparent that a DRS is uniquely specified by means of its capacity functional only over finite subsets of $\mathbb{R}^2$. It is therefore required that erosion, dilation, opening, or closing of a RACS $\mathbf{X}$ by a compact structuring element A be discretized. Moreover, it is desirable that the resulting discretization produces an erosion, dilation, opening, or closing of a DRS $\mathbf{X}_d = \sigma(\mathbf{X})$ by a finite structuring element $A_d = \sigma(A)$. In this case, the discretized morphological transformations $\mathbf{X}_d \ominus A_d$, $\mathbf{X}_d \oplus A_d$, $\mathbf{X}_d \bigcirc A_d$, and $\mathbf{X}_d \bullet A_d$ will be DRSs, whose capacity functional can be evaluated from the capacity functional of $\mathbf{X}_d$, by means of (3.3)–(3.8). Notice however that this procedure should be done in such a way that the resulting discretization is a good approximation (in some sense) of the original continuous problem. We study these issues next.

Let S be a *sampling grid* in $\mathbb{R}^2$, such that

$$S = \{k_1 e_1 + k_2 e_2 \mid k_1, k_2 \in \mathbb{Z}\} \, ,$$

where $e_1 = (1, 0)$, $e_2 = (0, 1)$ are the two linearly independent unit vectors in $\mathbb{R}^2$ along the two coordinate directions and $\mathbb{Z}$ is the set of all integers. Consider a bounded open set C, given by

$$C = \{x_1 e_1 + x_2 e_2 \mid -1 < x_1, x_2 < 1\} \, ,$$

known as the *sampling element*. Let $\mathcal{P}(S)$ be the power set of S. Then, an operator σ: $\mathcal{F} \to \mathcal{P}(S)$, known as the *sampling operator*, is defined by

$$(4.1) \qquad \sigma(X) = \{s \in S \mid C_s \cap X \neq \emptyset\} = (X \oplus C) \cap S \, , \; X \in \mathcal{F} \, ,$$

whereas, an operator ρ: $\mathcal{P}(S) \to \mathcal{F}$, known as the *reconstruction operator*, is defined by

$$(4.2) \qquad \rho(V) = \{v \in \mathbb{R}^2 \mid C_v \cap S \subseteq V\} \, , \; V \in \mathcal{P}(S) \, .$$

See [5], [21], [22] for more details. The combined operator $\pi = \rho\sigma$ is known as the *approximation operator*. When operator σ is applied on a closed set $X \in \mathcal{F}$ it produces a discrete set $\sigma(X)$ on S. On the other hand, application of operator ρ on a discrete set $\sigma(X)$ produces a closed set $\pi(X) = \rho\sigma(X)$ that approximates X. The effects that operators σ, ρ, and π have on a closed set X are illustrated in Fig. 5.

Whether or not a closed set X is well approximated by $\pi(X)$ depends on how fine X is sampled by the sampling operator σ. To mathematically

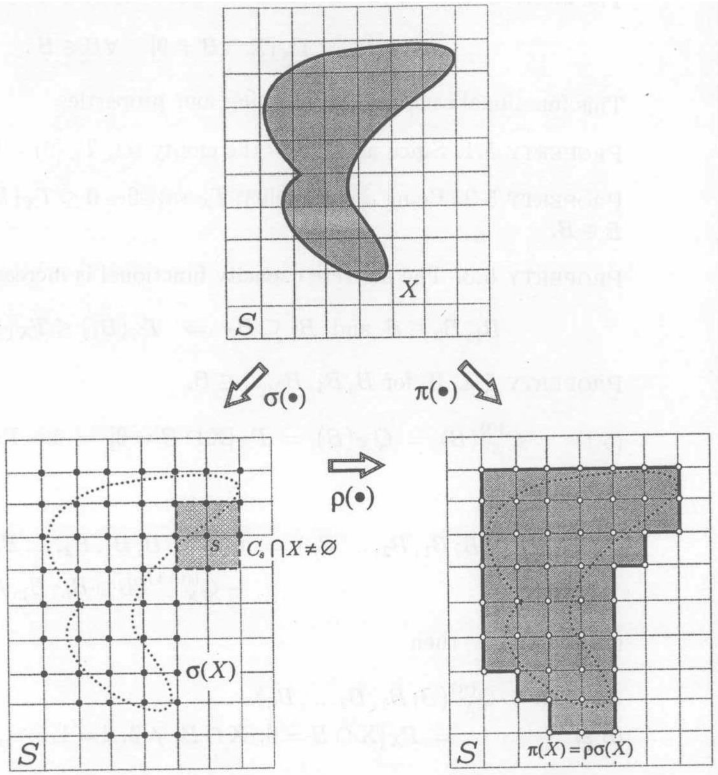

FIG. 5. *The effects of morphological sampling σ, reconstruction ρ, and approximation π on a closed subset X of $\mathbb{R}^2$.*

quantify this, consider sequences $\{S_n\}_{n\geq 1}$ and $\{C_n\}_{n\geq 1}$ of sampling grids and sampling elements, such that

$$S_{n+1} = \frac{1}{2} S_n,\ n \geq 1,\ S_1 = S \quad \text{and} \quad C_{n+1} = \frac{1}{2} C_n,\ n \geq 1,\ C_1 = C\,,$$

where $cX = \{cx \mid x \in X\}$. We then define sampling and reconstruction operators σ_n and ρ_n, by replacing S and C in (4.1), (4.2), by S_n and C_n. This determines a sequence of increasingly fine discretizations of a closed set X, denoted by $\mathcal{D} = \{S_n, \sigma_n, \rho_n\}_{n\geq 1}$, that is known as the *covering discretization* [5], [22]. It can be shown that, for $X \in \mathcal{F}$,

$$X \subseteq \cdots \subseteq \pi_{n+1}(X) \subseteq \pi_n(X) \subseteq \pi_{n-1}(X) \subseteq \cdots \subseteq \pi_1(X)\,,$$

and

$$\bigcap_{n\geq 1} \pi_n(X) = X\,,$$

which means that the approximation $\pi_n(X)$ of X *monotonically* converges to X from above (this is also denoted by $\pi_n(X) \downarrow X$), which implies that $\pi_n(X) \xrightarrow{\mathcal{F}} X$, where $\xrightarrow{\mathcal{F}}$ denotes convergence in the *hit-or-miss topology* (see [5], [6] for more information about the hit-or-miss topology).

For every $n = 1, 2, \ldots$, define a sequence $\mathbf{X}_{d,n}$ by (recall (4.1))

$$\mathbf{X}_{d,n} \;=\; \sigma_n(\mathbf{X}) \;=\; (\mathbf{X} \oplus C_n) \cap S_n\,,$$

where $\mathbf{X}$ is a RACS, and a sequence $\mathbf{X}_n$ by (recall (4.2))

$$\mathbf{X}_n = \rho_n(\mathbf{X}_{d,n}) = \rho_n\sigma_n(\mathbf{X}) = \pi_n(\mathbf{X}) = \{v \in \mathbb{R}^2 \mid (C_n)_v \cap S_n \subseteq \mathbf{X}_{d,n}\}\,.$$

$\mathbf{X}_{d,n}$ *almost surely* (a.s.) contains a countable number of points and is therefore a DRS. On the other hand, it is known that $\mathbf{X}_n$ is an a.s. closed set, whereas it has been shown in [10] that π_n is a measurable mapping; therefore, $\mathbf{X}_n$ is a RACS. In fact, it is not difficult to show that $\mathbf{X}_n \downarrow \mathbf{X}$, a.s., which implies that $\mathbf{X}_n \xrightarrow{\mathcal{F}} \mathbf{X}$, a.s., as well. Furthermore, if A is a *$\mathcal{D}$–regular compact structuring element*, for which

(4.3) $$A \;=\; \pi_N(A)\,,\ \text{ for some } 1 \leq N < \infty\,,$$

then it can be shown that [10], [23]

$$\rho_n(\sigma_n(\mathbf{X}) \ominus \sigma_n(A)) \xrightarrow{\mathcal{F}} \mathbf{X} \ominus A\,,\ \ a.s.\,,$$

$$\rho_n(\sigma_n(\mathbf{X}) \oplus \sigma_n(A)) \xrightarrow{\mathcal{F}} \mathbf{X} \oplus A\,,\ \ a.s.\,,$$

$$\rho_n(\sigma_n(\mathbf{X}) \circ \sigma_n(A)) \xrightarrow{\mathcal{F}} \mathbf{X} \circ A\,,\ \ a.s.\,,$$

$$\rho_n(\sigma_n(\mathbf{X}) \bullet \sigma_n(A)) \xrightarrow{\mathcal{F}} \mathbf{X} \bullet A\,,\ \ a.s.\,.$$

This means that the covering discretization guarantees that erosion, dilation, opening, or closing of a RACS $\mathbf{X}$ by a $\mathcal{D}$–regular compact structuring element A (i.e., a structuring element that satisfies (4.3)) can be well approximated by an erosion, dilation, opening, or closing, respectively, of a DRS $\sigma_n(\mathbf{X})$ by a finite structuring element $\sigma_n(B)$, for some large n, as is desirable. The requirement that A is a $\mathcal{D}$-regular compact structuring element is not a serious limitation since a wide collection of structuring elements may satisfy this property [23].

The previous results focus on the a.s. convergence of discrete morphological operators to their continuous counterparts. However, results concerning convergence of the associated capacity functionals also exist. It has been shown in [10] that the capacity functional of the approximating RACS $\pi_n(\mathbf{X})$ monotonically converges from above to the capacity functional of RACS $\mathbf{X}$; i.e.,

$$T_{\pi_n(\mathbf{X})}(K) \downarrow T_{\mathbf{X}}(K)\,,\ \ \forall K \in \mathcal{K}\,.$$

Furthermore, it has been shown that the capacity functional of RACS $\rho_n(\sigma_n(\mathbf{X}) \ominus \sigma_n(A))$ converges to the capacity functional of RACS $\mathbf{X} \ominus A$, in the limit, as $n \to \infty$, provided that A is a $\mathcal{D}$–regular compact structuring element, with a similar convergence result being true for the case of dilation, opening, and closing. Finally, it has been shown that

$$\lim_{n\to\infty} T_{\mathbf{X}_{d,n}}(\sigma_n(K)) = T_{\mathbf{X}}(K), \ \forall K \in \mathcal{K},$$

and

$$\lim_{n\to\infty} T_{\mathbf{X}_{d,n}\ominus\sigma_n(A)}(\sigma_n(K)) = T_{\mathbf{X}\ominus A}(K), \ \forall K \in \mathcal{K},$$

provided that A is a $\mathcal{D}$–regular compact structuring element, with a similar convergence result being true for dilation, opening, and closing. Therefore, and for sufficiently large n, the continuous morphological transformations $\mathbf{X} \ominus A$, $\mathbf{X} \oplus A$, $\mathbf{X} \circ A$, and $\mathbf{X} \bullet A$, can be well approximated by the discrete morphological transformations $\mathbf{X}_{d,n} \ominus \sigma_n(A)$, $\mathbf{X}_{d,n} \oplus \sigma_n(A)$, $\mathbf{X}_{d,n} \circ \sigma_n(A)$, and $\mathbf{X}_{d,n} \bullet \sigma_n(A)$, respectively, provided that A is a $\mathcal{D}$–regular compact structuring element. This shows that, in most practical situations, it will be sufficient enough to limit our interest to a DRS $\mathbf{X}_d = \sigma(\mathbf{X}) = (\mathbf{X} \oplus C) \cap S$ instead of RACS $\mathbf{X}$, for a sufficiently fine sampling grid S, with the benefit (among some other benefits) of relating the capacity functional of a morphologically transformed DRS $\Psi(\mathbf{X}_d)$ to the capacity functional of $\mathbf{X}_d$, by means of (3.3)–(3.8). It can be shown that

$$T_{\mathbf{X}_d}(K) = P_{\mathbf{X}}[\mathbf{X} \cap ((K \cap S) \oplus C) \neq \emptyset], \ \forall K \in \mathcal{K},$$

which shows that the capacity functional of the DRS $\mathbf{X}_d$ need to be known only over finite subsets of S. Finally, it has been shown in [10] that

$$T_{\mathbf{X}_d}(B) = \sup\{T_{\mathbf{X}}(K);\ K \in \mathcal{K},\ K \subset B \oplus C\}, \ \forall B \in \mathcal{I},$$

where $\mathcal{I}$ is the collection of all *bounded* subsets of S, which relates the capacity functional of the DRS $\mathbf{X}_d = \sigma(\mathbf{X})$ with the capacity functional of RACS $\mathbf{X}$.

5. Discrete random sets. Following our previous discussion, given a probability space $(\Omega, \Sigma(\Omega), \mu)$, a DRS $\mathbf{X}$ on $\mathbb{Z}^2$ is a measurable mapping from Ω into $\mathcal{Z}$, the power set of $\mathbb{Z}^2$, that is

$$\{\omega \in \Omega \mid \mathbf{X}(\omega) \in \mathcal{A}\} \in \Sigma(\Omega), \ \forall \mathcal{A} \in \Sigma(\mathcal{Z}),$$

where $\Sigma(\mathcal{Z})$ is the σ-field in $\mathcal{Z}$ generated by the simple family $\{\{X \in \mathcal{Z} \mid X \cap B = \emptyset\}, B \in \mathcal{B}\}$, where $\mathcal{B}$ is the collection of all *bounded* subsets of $\mathbb{Z}^2$. A DRS $\mathbf{X}$ defines a probability distribution $P_{\mathbf{X}}$ on $\Sigma(\mathcal{Z})$ by

$$P_{\mathbf{X}}[\mathcal{A}] = \mu[\{\omega \in \Omega \mid \mathbf{X}(\omega) \in \mathcal{A}\}], \ \forall \mathcal{A} \in \Sigma(\mathcal{Z}).$$

The *discrete capacity functional* of a DRS $\mathbf{X}$ is defined by

$$T_{\mathbf{X}}(B) = P_{\mathbf{X}}[\mathbf{X} \cap B \neq \emptyset], \ \forall B \in \mathcal{B}.$$

This functional satisfies the following four properties:

PROPERTY 5.1. Since no set hits the empty set, $T_{\mathbf{X}}(\emptyset) = 0$.

PROPERTY 5.2. Being a probability, $T_{\mathbf{X}}$ satisfies $0 \leq T_{\mathbf{X}}(B) \leq 1$, for every $B \in \mathcal{B}$.

PROPERTY 5.3. The discrete capacity functional is increasing on $\mathcal{B}$; i.e.,

$$B_1, B_2 \in \mathcal{B} \text{ and } B_1 \subseteq B_2 \ \Rightarrow\ T_{\mathbf{X}}(B_1) \leq T_{\mathbf{X}}(B_2).$$

PROPERTY 5.4. If, for $B, B_1, B_2, \ldots \in \mathcal{B}$,

$$Q_{\mathbf{X}}^{(0)}(B) = Q_{\mathbf{X}}(B) = P_{\mathbf{X}}[\mathbf{X} \cap B = \emptyset] = 1 - T_{\mathbf{X}}(B), \tag{5.1}$$

and

$$\begin{aligned} Q_{\mathbf{X}}^{(n)}(B; B_1, B_2, \ldots, B_n) = {} & Q_{\mathbf{X}}^{(n-1)}(B; B_1, B_2, \ldots, B_{n-1}) \\ & - Q_{\mathbf{X}}^{(n-1)}(B \cup B_n; B_1, B_2, \ldots, B_{n-1}), \end{aligned}$$

for $n = 1, 2, \ldots$, then

$$\begin{aligned} 0 \leq {} & Q_{\mathbf{X}}^{(n)}(B; B_1, B_2, \ldots, B_n) \\ & = P_{\mathbf{X}}[\mathbf{X} \cap B = \emptyset;\ \mathbf{X} \cap B_i \neq \emptyset,\ i = 1, 2, \ldots, n] \leq 1, \end{aligned} \tag{5.2}$$

for every $n \geq 1$.

As a special case of the Choquet-Kendall-Matheron theorem, the probability distribution of a DRS is uniquely determined by a discrete capacity functional $T_{\mathbf{X}}(B)$, $B \in \mathcal{B}$, that satisfies properties 5.1–5.4 above. Functional $Q_{\mathbf{X}}(B)$, $B \in \mathcal{B}$, in (5.1) is known as the *discrete generating functional* of $\mathbf{X}$, whereas, functional $Q_{\mathbf{X}}^{(n)}(B; B_1, B_2, \ldots, B_n)$, B, B_1, B_2, ..., B_n $\in \mathcal{B}$, is the probability that the DRS $\mathbf{X}$ misses B and hits B_i, $i = 1, 2, \ldots, n$ (see (5.2)).

In practice, images are observed through a finite-size window W, $|W| < \infty$, where $|A|$ denotes the *cardinality* (or *area*) of set A. Therefore, it seems reasonable to consider DRSs whose realizations are limited within W. Let $\mathcal{B}_W$ be the collection of all (bounded) subsets of $\mathbb{Z}^2$ that are included in W. A DRS $\mathbf{X}$ is called an *a.s. W–bounded DRS* if $P_{\mathbf{X}}[\mathbf{X} \in \mathcal{B}_W] = 1$. It is not difficult to see that an a.s. W–bounded DRS is uniquely specified by means of a discrete capacity functional $T_{\mathbf{X}}(B)$, $B \in \mathcal{B}_W$. Furthermore, if $M_{\mathbf{X}}(X)$, $X \in \mathcal{B}_W$, is the *probability mass function* of $\mathbf{X}$, i.e., if

$$M_{\mathbf{X}}(X) = P_{\mathbf{X}}[\mathbf{X} = X], \ X \in \mathcal{B}_W,$$