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DIETER BESDO
TECHNICAL UNIVERSITY OF BRUNSWICK

EXAMPLES TO EXTREMUM AND VARIATIONAL PRINCIPLES IN MECHANICS

SEMINAR NOTES ÁCCOMPANING THE VOLUME
No. 54 BY H. LIPPMANN

COURSE HELD AT THE DEPARTMENT
OF GENERAL MECHANICS
OCTOBER 1970

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P R E F A C E

The following examples to extremum and variational principles in mechanics were delivered in a seminar which accompanied a lecture course of Professor Horst LIPPMANN, Brunswick. Therefore, the examples cannot stand for themselves, their main function was to illustrate the results of the lecture course and to demonstrate several interesting peculiarities of the single solution methods.

The problems are normally chosen to be quite simple so that numerical computations are not necessary. Nevertheless, sometimes, the calculations will only be mentioned and not worked out here.

The sections of the seminar-course are not identical with those of the lecture course. Especially, there are no examples to more or less theoretical sections of the lectures. Because of the close connection to the lectures, no separate list of references is given. Also the denotation is mostly the same as in the lecture-notes.

I say many thanks to Professor Horst LIPPMANN for his help during the preparation-time and to the International Centre for Mechanical Sciences for the invitation to deliver this seminar.

Brunswick, October 3I, I970

Dieter Besdo

1. EXTREMA AND STATIONARITIES OF FUNCTIONS

1.1. Simple problems (cf. sect. 1.2 of the lecture-notes)

In this sub-section, several simple problems have to demonstrate definite peculiarities which may occur if we want to calculate extrema of functions.

Problem 1.1.-I : Given a function f in an unlimited region :

$$f = 10x + 12x^2 + 12y^2 - 3x^3 - 9x^2y - 9xy^2 - 3y^3 .$$

Find out the extrema.

This problem has to illustrate the application of the necessary and the sufficient conditions for extrema of functions.

At first, we see that f is not bounded :

If $y = 0$ and x tends to infinity we see :

$$x \longrightarrow +\infty \quad : \qquad f \longrightarrow -\infty ,$$

$$x \longrightarrow -\infty \quad : \qquad f \longrightarrow +\infty .$$

Thus, there is no absolute extremum.

To find out relative extrema, we have to use the derivatives

$$f_{,x} \equiv \frac{\partial f}{\partial x} \; ; \quad f_{,y} \equiv \frac{\partial f}{\partial y} \; ; \quad f_{,xy} \equiv \frac{\partial^2 f}{\partial x \partial y} \; ;$$

$$f_{,xx} \equiv \frac{\partial^2 f}{\partial x^2} \; ; \quad f_{,yy} = \frac{\partial^2 f}{\partial y^2} \; :$$

$$f_{,x} = 10 + 24x - 9x^2 - 18xy - 9y^2 ,$$

$$f_{,y} = 24y - 9x^2 - 18xy - 9y^2 ,$$

$$f_{,xx} = 24 - 18(x+y) ,$$

$$f_{,xy} = -18(x+y) ,$$

$$f_{,yy} = 24 - 18(x+y) .$$

Necessary condition for an extremum of a continually differentiable function is stationarity :

$$f_{,x} = 0 \; , \quad f_{,y} = 0 .$$

This yields the two points :

$$\tilde{x}_1 = \frac{5}{8} \; ; \quad \tilde{y}_1 = 15/24 \; ; \quad \tilde{f}_1 = 725/72 ,$$

$$\tilde{x}_2 = -\frac{3}{8} \; ; \quad \tilde{y}_2 = 1/24 \; ; \quad \tilde{f}_2 = -139/72 .$$

But we do not know whether these points represent relative extrema. We examine the matrix

$$\frac{\partial^2 f}{\partial x^2} \equiv \begin{bmatrix} f_{,xx} & f_{,xy} \\ f_{,xy} & f_{,yy} \end{bmatrix}, \quad \text{taken at} \quad x = \tilde{x}, y = \tilde{y}, f = \tilde{f}.$$

If it is positive or negative definite we have a minimum or a maximum resp., if it is positive or negative semidefinite we possibly may have a minimum or maximum resp., but then we cannot be sure. If $\partial^2 f / \partial x^2$ is not semidefinite we have no extremum but a saddle-point. Applying this we see :

point 1 $$\left(\frac{\partial^2 f}{\partial x^2}\right)_1 = \begin{bmatrix} -6 & -30 \\ -30 & -6 \end{bmatrix} \equiv A_1.$$

We check the definiteness by a direct method. We introduce the vector $\eta = (\alpha \; \beta)$, then $g = \eta A < \eta >$ is examined :

$$g_1 = \eta A_1 < \eta > = -6(\alpha^2 + \beta^2) - 60\alpha\beta.$$

We see :

$$g_1 = -72 \quad \text{if} \quad \alpha = \beta = 1,$$

$$g_1 = +48 \quad \text{if} \quad \alpha = -\beta = 1.$$

A_1 is not definite or semidefinite: point 1 is a saddle-point.

point 2
$$\left(\frac{\partial^2 f}{\partial x^2}\right)_2 = \begin{bmatrix} 30 & 6 \\ 6 & 30 \end{bmatrix} \equiv A_2 \, .$$

This yields

$$g_2 = \eta \, A_2 < \eta > = 24\,(\alpha^2 + \beta^2) + 6\,(\alpha + \beta)^2 > 0 \quad \text{if } \alpha \neq 0 \text{ or } \beta \neq 0 \, .$$

Hence, A_2 is positive definite, point 2 represents a relative minimum.

The function f has only one minimum and no maximum. This is possible if it has the form which is sketched in Fig. 1.1.-1.

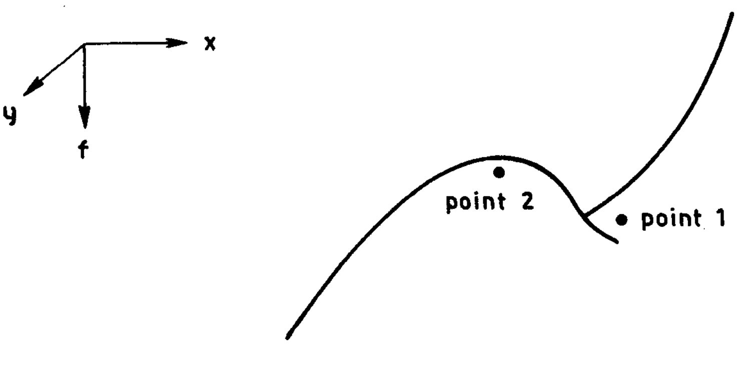


Fig. 1.1-1

Problem 1.1-2 : Given the function $f = 3x^2 + 4y^2 + z^2$ declaired in the region G where $g \equiv x^2 + y^2 + z^2 - 1 \leq 0$ (unit sphere), calculate the extrema, also boundary-extrema.

This problem has to show the curious effect that extrema can be lost if we are not careful enough when calculating extrema on boundaries.

First we try to find out extrema in the interior of the region G :

$$f_{,x} = 6x \; ; \quad f_{,y} = 8y \; ; \quad f_{,z} = 2z \; .$$

Hence, $f_{,x_i} = 0$ leads to $\tilde{x} = \tilde{y} = \tilde{z} = \tilde{f} = 0$.

Because of $f \geq 0$ this must be a minimum. There is no second extremum in the interior.

Inside of G, f is bounded. So there must be a maximum on the boundary.

The boundary is described by $g = 0$. Therefore, $z^2 = 1 - x^2 - y^2$ can be put into f instead of z^2. So we get the function f as a function $\overset{B}{f} = \overset{B}{f}(x,y)$ which is valid on the boundary :

$$\overset{B}{f} = 1 + 2x^2 + 3y^2$$

$\overset{B}{f}_{,x} = \overset{B}{f}_{,y} = 0$ yields $x = y = 0$, $f = \overset{B}{f} = 1$ which is a minimum of $\overset{B}{f}$ and, therefore, cannot be a maximum of f.

Two questions arise now :

1. Is $x = y = 1$, $f = 1$, $z = \pm 1$ a minimum of the function f in G?

The allowed region is given by $g \leq 0$. Thus, the gradient $\operatorname{grad} g$ represents a vector directed towards the outside of G. Then

$$(\operatorname{grad} f) \cdot (\operatorname{grad} g) \gtreqless 0$$

is a necessary condition for a minimum or a maximum on the boundary.

$(\operatorname{grad} f) \cdot (\operatorname{grad} g) \gtrless 0$ at a special point x_i is sufficient, if there is a minimum or maximum, resp., on the boundary.

$$(\operatorname{grad} g) \cdot (\operatorname{grad} f) = 12x^2 + 16y^2 + 4z^2 > 0$$

shows that we can find out only maxima on the boundary.

2. We have found out only two stationary points on the boundary. They did not represent maxima. But there must be a maximum. It seems to be lost. What is the reason?

We have lost the maximum because of the following mistake which we made : When establishing the function $\overset{B}{f}$, we did not notice that z^2 must be positive. This would have lead to the new boundary condition for $\overset{B}{f}$:

$$\overset{B}{g} \equiv x^2 + y^2 - 1 \leq 0 .$$

We see : the elimination of variables, especially of squared ones, may be dangerous.

The other way for the calculation of the stationarities of f on the boundary is the use of LAGRANGIAN multipliers. The problem : $f \Longrightarrow$ stationary under the side-condition $g = 0$, can be expressed as

$$h(x_i, \lambda) = 3x^2 + 4y^2 + z^2 + \\ + \lambda(x^2 + y^2 + z^2 - 1) \Longrightarrow \quad \text{stationary}$$

This method yields each stationary point on the boundary which, then, can be checked, whether it is an extremum .

$h \Longrightarrow$ stationary yields the conditions :

$$6\tilde{x} + 2\tilde{\lambda}\tilde{x} = 0 \quad \text{or} \quad (6 + 2\tilde{\lambda})\tilde{x} = 0 \tag{1}$$

$$8\tilde{y} + 2\tilde{\lambda}\tilde{y} = 0 \quad \text{or} \quad (8 + 2\tilde{\lambda})\tilde{y} = 0 \tag{2}$$

$$2\tilde{z} + 2\tilde{\lambda}\tilde{z} = 0 \quad \text{or} \quad (2 + 2\tilde{\lambda})\tilde{z} = 0 \tag{3}$$

$$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 1 . \tag{4}$$

Eq. (1) postulates $\tilde{x} = 0$ or $\tilde{\lambda} = -3$,
Eq. (2) yields $\tilde{y} = 0$ or $\tilde{\lambda} = -2$, and
Eq. (3) leads to $\tilde{z} = 0$ or $\tilde{\lambda} = -1$.

$\tilde{x} \neq 0$ is possible if $\tilde{\lambda} = -3$ but then we must have $\tilde{y} = \tilde{z} = 0$, $\tilde{x} = \pm 1$ (from eq. (4)). In the same way we get :
$\tilde{y} \neq 0$ leads to $\tilde{x} = \tilde{z} = 0$, $\tilde{y} = \pm 1$,
$\tilde{z} \neq 0$ brings out $\tilde{x} = \tilde{y} = 0$, $\tilde{z} = \pm 1$.

These are points of stationarity on the boundary.

The examination of their extremum properties shows that the points

$$\tilde{x} = \tilde{z} = 0 \;, \quad \tilde{y} = \pm 1 \;, \quad \tilde{f} = 4$$

represent the (absolute) maxima of f in G.

Problem 1.1-3 : We have a given plate of sheet metal and we want to produce with it a fixed number of tin-boxes. Calculate the optimum relation between the height h and the radius r of the tin boxes, if the volume of the boxes is to be maximized.

This simple problem was used as an additional problem to demonstrate the advantage of LAGRANGE-multipliers.

The volume

$$V = \pi r^2 h$$

(r = radius, h = height) has to be a maximum. On the other hand, the surface area of the plate per box

$$S = 2\pi r h + \gamma \pi r^2 \qquad (\gamma \geq 2)$$

is a given value. The quantity γ is introduced, because different cases will be examined : ideally no falling-off leads to $\gamma = 2$, falling-off as sketched in Fig. 1.1-2 belongs to $\gamma = 2.20$, realistic values of γ may be 2.30 to 2.60.

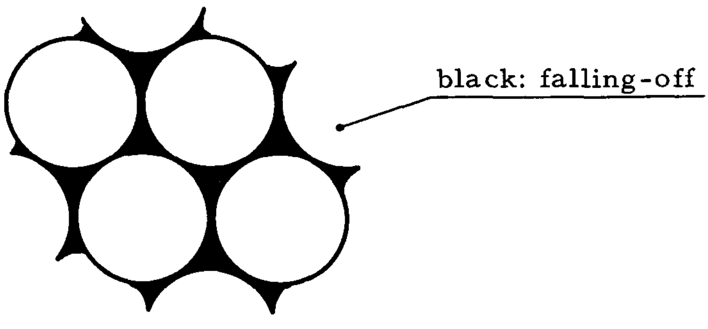


Fig. 1.1-2

The problem is fixed now :

$$V = \pi r^2 h \Longrightarrow \text{maximum}$$

under the side-condition

$$g = 2\pi r h + \gamma \pi r^2 - S = 0.$$

This leads to $H(r,h,\lambda)$ being

$$H(r,h,\lambda) = \pi r^2 h -$$

$$- \lambda(2\pi r h + \gamma \pi r^2 - S) \Longrightarrow \quad \text{stationary}$$

Hence, we get

(1) $$\tilde{r}\tilde{h} - \tilde{\lambda}\tilde{h} - \tilde{\lambda}\gamma\tilde{r} = 0\,,$$

(2) $$\tilde{r}^2 - \tilde{\lambda}2\tilde{r} = 0\,,$$

(3) $$2\pi\tilde{r}\tilde{h} + \gamma\pi\tilde{r}^2 - S = 0.$$

Eq. (2) yields : $\tilde{\lambda} = \frac{1}{2}\tilde{r}$.

Putting this $\tilde{\lambda}$ into eq. (1), we reach

$$\tilde{r}\tilde{h} - \frac{1}{2}\tilde{r}\tilde{h} - \frac{1}{2}\gamma\tilde{r}^2 = \frac{1}{2}(\tilde{r}\tilde{h} - \gamma\tilde{r}^2) = \frac{1}{2}\tilde{r}(\tilde{h} - \gamma\tilde{r}) = 0$$

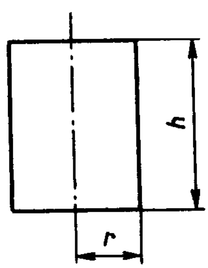
or : $\tilde{r} = 0$ or $\tilde{h} = \gamma\tilde{r}$.

$\tilde{r} = 0$ cannot be a maximum of V $(V = 0)$. Therefore, $\tilde{h} : \tilde{r} = \gamma$ must be the desired result. The ideal tin-box has the form sketched in Fig. 1.1-3 ($\gamma = 2\ 5$). Eq. (3) can be used for the determination of $\tilde{r}$, $\tilde{h}$ as functions of S :

$$2\gamma\pi\tilde{r}^2 + \gamma\pi\tilde{r}^2 = S\,,$$

$$\tilde{r} = \sqrt{\frac{S}{3\gamma\pi}}\;;\quad \tilde{h} = \sqrt{\frac{\gamma S}{3\pi}}$$

Fig. 1.1-3



1.2. Linear programming (no correspondence to the lectures)

In a lot of problems where a minimum or a maximum has to be calculated, the equations describing the boundaries and the functions which have to be optimized, are linear in their variables. Then, the method of "Linear Programming" can be applied. We will derive the theory by use of a simple example. Later on, this method will be applied for the calculation of the load-carrying capacity.

Problem 1.2-1 : A farmer has 100 ha (German unit of measurement, 1 ha = 10, 000 m^2) grounds on which he wants to cultivate four types of fruits (I to IV) in order to reach maximal profit. For this purpose, he has to use different means which are restricted : his capital and the working-time are not infinite. Further on, he has to use two machines A and B which he has to lend. This is possible for restricted times only.

We assume that costs and times depend linearly on the area which is cultivated with a special fruit. Also the profit (where all costs are subtracted already) is to be a linear function of the parts of the grounds cultivated with the different fruits. Then, the theory of linear programming can be applied.

Two problems will be handled :

a) Only two fruits (I and IV) are taken into account. Then, the two distributions of the profit α) and β) will be compared (cf. Table 1.2-1).

b) Four fruits are possibly cultivated. The profit-distribution α) is valid.

The values which are necessary for the calculation are printed in Table 1.2-1.

Table 1.2-1

means / fruits	money $\left[\frac{\text{lire}}{\text{ha}}\right]$	time $\left[\frac{\text{days}}{\text{ha}}\right]$ for work	machA	machB	profit $\left[\frac{\text{Lire}}{\text{ha}}\right]$ case α)	case β)
I	20,000	1	0	1/2	24,000	12,000
II	40,000	2	1	2	48,000	-
III	20,000	3	0	0	36,000	-
IV	30,000	4	1	0	54,000	60,000
restrictions for the sums:	[Lire] 2,750,000	[days] 230	50	75		

a) Two fruits (I and IV), profit given by α) and β)

The restrictions of the means lead to necessary conditions for every solution, given in Table 1.2-2 ($x = x_I$ = number of ha's cultivated by fruit I, $y = x_{IV}$).