Advanced Calculus for Applications

SECOND EDITION

Francis B. Hildebrand

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PRENTICE-HALL, INC.
Englewood Cliffs, New Jersey

Library of Congress Cataloging in Publication Data

HILDEBRAND, FRANCIS BEGNAUD.
Advanced calculus for applications.

Published in 1948 under title: Advanced calculus for engineers.
Bibliography: p.
Includes index.
1. Calculus. I. Title.
QA303.H55 1976 515 75-34473
ISBN 0-13-011189-9

10 9 8 7 6 5 4 3 2 1

Printed in the United States of America

PRENTICE-HALL INTERNATIONAL, INC., *London*
PRENTICE-HALL OF AUSTRALIA, PTY. LTD., *Sydney*
PRENTICE-HALL OF CANADA, LTD., *Toronto*
PRENTICE-HALL OF INDIA PRIVATE LIMITED, *New Delhi*
PRENTICE-HALL OF JAPAN, INC., *Tokyo*
PRENTICE-HALL OF SOUTHEAST ASIA (PTE.) LTD., *Singapore*

Preface

The purpose of this text is to present an integrated treatment of a number of those topics in mathematics which can be made to depend only upon a sound course in elementary calculus, and which are of common importance in many fields of application.

An attempt is made to deal with the various topics in such a way that a student who may not proceed into the more profound areas of mathematics still may obtain an intelligent working knowledge of a substantial number of useful mathematical methods, together with an appropriate awareness of the foundations, interrelations, and limitations of these methods. At the same time, it is hoped that a student who is to progress, say, into a rigorous course in mathematical analysis will be provided, in addition, with increased incentive and motivation. For both of these purposes, the phrase "It can be shown" is used occasionally, not only to exhibit a generalization of an established conclusion or a useful related fact, but also to introduce a needed basic result for which a rigorous demonstration would require what is believed to be an inappropriately excessive amount of detailed analysis or of prerequisite preparation.

This revision incorporates a large number of relatively minor changes for the purpose of increased clarity or precision or to supply a previously omitted proof, a substantial amount of added textual material (particularly in the later chapters), and about 250 additional problems.

The first four chapters are concerned chiefly with ordinary differential equations, including analytical, operational, and numerical methods of solution, and with special functions generated as solutions of such equations. In particular, the material of the first chapter can be considered as either a systematic review or an initial introduction to the elementary concepts and techniques, associated with linear equations and with special solvable types of nonlinear equations, which are needed in subsequent chapters. The fifth chapter deals with boundary-value problems governed by ordinary differential equations, with the associated characteristic functions, and with series and integral representations of arbitrary functions in terms of these functions.

Chapter 6 develops the useful ideas and tools of vector analysis; Chapter 7 provides brief introductions to some special topics in higher-dimensional calculus which are rather frequently needed in applications. The treatment here occasionally consists essentially of indicating the plausibility and practical significance of a result and stating conditions under which its validity is rigorously established in listed references.

In Chapter 8, certain basic concepts associated with the simpler types of partial differential equations are introduced, after which, in Chapter 9, full use is made of most of the tools developed in earlier chapters for the purpose of formulating and solving a variety of typical problems governed by the partial differential equations of mathematical physics. A new section deals with the application of the so-called method of variation of parameters to such problems.

Chapter 10 treats the basic topics in the theory of analytic functions of a complex variable, including contour integration and residue calculus. Although certain developments in preceding chapters could be made more elegant and more complete if they were made to depend upon this treatment, introduced at an earlier stage, it is felt that, in some cases, the knowledge based on a brief initial study of analytic functions may not be sufficiently firm to support significantly dependent treatments of the other topics, but that such knowledge then may better serve to clarify the other topics when subsequently provided. However, since most of the treatments of Chapter 10, as well as most of those of Chapters 6 and 7, are independent of the content of preceding chapters, material from these chapters can indeed be introduced at an earlier stage in a given course, at the discretion of the instructor. It has been considered reasonable to assume knowledge of certain elementary properties of complex numbers in the earlier chapters, even though the solution of the equation $x^4 + 1 = 0$ then may occasion a personal review on the part of the reader.

A new Chapter 11 considers some applications of analytic function theory to other fields, including the derivation of methods for the inversion of Laplace transforms (an expansion of material previously presented in annotated problems), an indication of the properties and uses of conformal mapping (formerly included in Chapter 10), and a new brief treatment of Green's functions as related to partial differential equations.

Extensive sets of problems are included at the end of each chapter, grouped in correspondence with the respective sections with which they are associated. In addition to more-or-less routine exercises, there are numerous annotated problems which are intended to guide the reader in developing results or techniques which extend or complement treatments in the text, or in dealing with a particularly challenging application. Such problems may serve as focal points for extended discussions or for the introduction of additional (or alternative) material into a chapter, permitting the text to serve somewhat more flexibly in courses of varied types. New problems of this sort now permit the consideration of topics such as one-dimensional Green's functions and applications of elliptic integrals, Fourier transforms, and associated Legendre functions. Answers to all problems are either incorporated into the statement of the problem or listed at the end of the book.

The author is particularly indebted to Professor E. Reissner for valuable collaboration in the preliminary stages of the preparation of the original edition and for many ideas which contributed to whatever useful novelty some of the treatments may possess, and to Professor G. B. Thomas for additional advice and help, as well as to a rather long list of other colleagues and students who have offered criticisms and suggestions leading to many of the modifications incorporated into this revision.

F. B. HILDEBRAND

Contents

5

Boundary-Value Problems and Characteristic-Function Representations 186

6

Vector Analysis 269

1

Ordinary Differential Equations

1.1. Introduction. A *differential equation* is an equation relating two or more variables in terms of derivatives or differentials. Thus the simplest differential equation is of the form

$$\frac{dy}{dx} = h(x), \tag{1}$$

where $h(x)$ is a given function of the independent variable x. The solution is obtained immediately by integration, in the form

$$y = \int h(x)\,dx + C, \tag{2}$$

where C is an arbitrary constant. Whether or not it happens that the integral can be expressed in terms of named or tabulated functions is incidental, in the sense that we accept as a *solution* of a differential equation any functional relation, *not involving derivatives or integrals of unknown functions*, the satisfaction of which implies the satisfaction of the differential equation. Similarly, in an equation of the form

$$F(x)G(y)\,dx + f(x)g(y)\,dy = 0, \tag{3}$$

we may *separate the variables* and obtain a solution by integration in the form

$$\int \frac{F(x)}{f(x)}\,dx + \int \frac{g(y)}{G(y)}\,dy = C, \tag{4}$$

if suitable account is taken of situations in which a divisor may vanish.

Usually we desire to obtain the *most general* solution of the differential equation; that is, we require *all* functional relations which imply the equation.

In the general case it may be difficult to determine when *all* such relations have indeed been obtained. Fortunately, however, this difficulty does not exist in the case of so-called *linear* differential equations, which are of most frequent occurrence in applications and which are to be of principal interest in what follows.

A differential equation of the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x) \qquad (5)$$

is said to be a *linear* differential equation of *order n*. The distinguishing characteristic of such an equation is the absence of products or nonlinear functions of the *dependent variable* (unknown function) y and its derivatives, the highest derivative present being of order n. The coefficients $a_0(x), \ldots, a_n(x)$ may be arbitrarily specified functions of the independent variable x.

For a linear equation of the *first* order,

$$a_0(x)\frac{dy}{dx} + a_1(x)y = f(x),$$

it is shown in Section 1.4 that if both sides of the equation are multiplied by a certain determinable function of x (an "integrating factor"), the equation always can be put in an equivalent form

$$\frac{d}{dx}[p(x)y] = F(x),$$

where $p(x)$ and $F(x)$ are simply expressible in terms of a_0, a_1, and f, and hence then can be solved directly by integration.

Although no such simple general method exists for solving linear equations of higher order, there are two types of such equations which are of particular importance in applications and which can be completely solved by direct methods. These two cases are considered in Sections 1.5 and 1.6. In addition, this chapter presents certain techniques that are available for treatment of more general linear equations.

Many of the basically useful properties of linear differential equations do not hold for *nonlinear* equations, such as

$$\frac{dy}{dx} = x + y^2, \qquad \frac{d^2y}{dx^2} + \sin y = 0, \qquad \frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 + y = e^x.$$

A few special types of solvable nonlinear equations are dealt with briefly in Section 1.12.

The equations to be considered in this chapter are known as *ordinary* differential equations, as distinguished from *partial* differential equations, which involve *partial derivatives* with respect to two or more independent variables. Equations of the latter type are treated in subsequent chapters.

Before proceeding to the study of linear ordinary differential equations, we next briefly introduce the notion of *linear dependence*, which is basic in this work.

1.2. Linear Dependence. By a *linear combination* of n functions $u_1(x)$, $u_2(x), \ldots, u_n(x)$ is meant an expression of the form

$$c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x) \equiv \sum_{k=1}^{n} c_ku_k(x), \tag{6}$$

where the c's are constants. When at least one c is not zero, the linear combination is termed *nontrivial.* The functions $u_1, u_2, \ldots, u_n$ are then said to be *linearly independent* over a given interval (say $a \leqq x \leqq b$) if over that interval no one of the functions can be expressed as a linear combination of the others, or, equivalently, if no *nontrivial* linear combination of the functions is identically zero over the interval considered. Otherwise, the functions are said to be *linearly dependent* over that interval.

As an example, the functions $\cos 2x$, $\cos^2 x$, and 1 are linearly dependent over *any* interval because of the identity

$$\cos 2x - 2\cos^2 x + 1 \equiv 0.$$

It follows from the definition that *two* functions are linearly dependent over an interval if and only if one function is a constant multiple of the other over that interval. The necessity of the specification of the interval in the general case is illustrated by a consideration of the two functions x and $|x|$. In the interval $x > 0$ there follows $x - |x| \equiv 0$, whereas in the interval $x < 0$ we have $x + |x| \equiv 0$. Thus the two functions are linearly dependent over any interval not including the point $x = 0$; but they are linearly independent over any interval including $x = 0$, since no single linear combination of the two functions is identically zero over such an interval.

Although in practice the linear dependence or independence of a set of functions generally can be established by inspection, the following result is of some importance in theoretical discussions. We assume that each of a set of n functions $u_1, u_2, \ldots, u_n$ possesses n finite derivatives at all points of an interval I. Then, if a set of constants exists such that

$$c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0$$

for all values of x in I, these same constants also satisfy the identities

$$\begin{aligned}
c_1\frac{du_1}{dx} + c_2\frac{du_2}{dx} + \cdots + c_n\frac{du_n}{dx} &= 0,\\
c_1\frac{d^2u_1}{dx^2} + c_2\frac{d^2u_2}{dx^2} + \cdots + c_n\frac{d^2u_n}{dx^2} &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
c_1\frac{d^{n-1}u_1}{dx^{n-1}} + c_2\frac{d^{n-1}u_2}{dx^{n-1}} + \cdots + c_n\frac{d^{n-1}u_n}{dx^{n-1}} &= 0.
\end{aligned}$$

Thus the n constants must satisfy n homogeneous linear equations. However, such a set of equations can possess *nontrivial* solutions only if its coefficient determinant vanishes. Thus it follows that *if the functions $u_1, u_2, \ldots, u_n$ are* *linearly dependent over an interval I, then the determinant*

$$W(u_1, u_2, \ldots, u_n) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ \frac{du_1}{dx} & \frac{du_2}{dx} & \cdots & \frac{du_n}{dx} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d^{n-1}u_1}{dx^{n-1}} & \frac{d^{n-1}u_2}{dx^{n-1}} & \cdots & \frac{d^{n-1}u_n}{dx^{n-1}} \end{vmatrix} \tag{7}$$

vanishes identically over I. This determinant appears frequently in theoretical work and is called the *Wronskian* (or Wronskian determinant) of the functions. Thus we see that *if the Wronskian of $u_1, u_2, \ldots, u_n$ is not identically zero over I, then the functions are linearly independent over I.*

To illustrate, since the value of the determinant

$$W(1, x, x^2, \ldots, x^n) = \begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^n \\ 0 & 1! & 2x & 3x^2 & \cdots & nx^{n-1} \\ 0 & 0 & 2! & 6x & \cdots & n(n-1)x^{n-2} \\ 0 & 0 & 0 & 3! & \cdots & n(n-1)(n-2)x^{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & n! \end{vmatrix}$$

is merely the product of the nonvanishing constants appearing in the principal diagonal and hence cannot vanish, it follows that the functions appearing in the first row are linearly independent (over *any* interval).

Unfortunately, the converse of the preceding theorem is *not* true since, in unusual cases, the Wronskian of a set of *linearly independent* functions also may vanish. That is, the vanishing of the Wronskian is *necessary* but not *sufficient* for linear dependence of a set of functions. (For an example establishing the insufficiency, see Problem 5.)

1.3. Complete Solutions of Linear Equations. The most general linear differential equation of the nth order can be written in the form

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = h(x). \tag{8}$$

Here it is assumed that both sides of the equation have been divided by the coefficient of the highest derivative. We will speak of this form as the *standard form* of the equation. This equation is frequently written in the abbreviated form

$$Ly = h(x), \tag{9}$$

where L here represents the *linear differential operator*

$$L = \frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{d}{dx} + a_n(x). \tag{10}$$

The problem of solving Equation (8) consists of determining the most general expression for y which, if substituted into the left-hand side of (8), or if operated on by (10), gives the prescribed right-hand side $h(x)$. When a relationship of the form $y = u(x)$ satisfies Equation (8), it is conventional to say that either the *relation* $y = u(x)$ or the *function* $u(x)$ is a *solution* of that equation.†

If all the coefficients $a_1(x), \ldots, a_n(x)$ were zero, the solution of Equation (8) would be accomplished directly by n successive integrations, each integration introducing an independent constant of integration. Thus it might be expected that the general solution of (8) also would contain n independent arbitrary constants. As a matter of fact, it is known that *in any interval I in which the coefficients are continuous, there exists a continuous solution to Equation* (8) *involving exactly n independent arbitrary constants; furthermore, there are no solutions of Equation* (8) *valid in I which cannot be obtained by specializing the constants in any such solution.*

It should be noticed that this is a property peculiar to *linear* differential equations. To illustrate, the nonlinear differential equation

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 4y = 4x - 1 \tag{11}$$

is of first order. A solution containing one arbitrary constant is of the form

$$y = x - (x - c)^2, \tag{12}$$

as can be verified by direct substitution. However, this is not the most general solution, since the function $y = x$ also satisfies the differential equation but cannot be obtained by specializing the arbitrary constant in the solution given. The additional solution $y = x$ is called a *singular solution*. Such solutions can occur only in the solution of nonlinear differential equations.

We consider first the result of replacing the function $h(x)$ by zero in Equation (8). The resulting differential equation, $Ly = 0$, is said to be *homogeneous*, since each term in the equation then involves the first power of y or of one of its derivatives. In this case, from the linearity of the equation, it is easily seen that any linear combination of individual solutions is also a solution. Thus, if n linearly independent solutions $u_1(x), u_2(x), \ldots, u_n(x)$ of the associated homogeneous equation

$$Ly_H = 0 \tag{13}$$

are known, the *general* solution of Equation (13) is of the form

$$y_H(x) = c_1u_1(x) + c_2u_2(x) + \cdots + c_nu_n(x) = \sum_{k=1}^{n} c_ku_k(x), \tag{14}$$

where the c's are the n required arbitrary constants. That is, all solutions of the homogeneous equation associated with (8) are obtained by suitably specializing the constants in Equation (14).

†Whereas a relationship of the implicit form $\phi(x, y) = 0$ also would be acceptable as a solution, there is no need for this generalization when the equation is *linear*.