INVERSE SEMIGROUPS

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I

PRELIMINARIES

1. A historical survey of the theory of inverse semigroups shows that there have been two foci of the origin and development of inverse semigroups: the Soviet and the Western schools, initiated by Wagner and Preston, respectively.

2. A list of needed concepts from the theory of partially ordered sets, and in particular lattices, is collected first and is used freely throughout the text. Simple properties of some of these concepts are also proved if they will be needed later.

3. Definitions related to semigroups in general such as identities and zeros, subsemigroups, idempotents, generation, orthogonal sum, and so on, are listed next. The semigroups of partial and full transformations are also introduced.

4. Homomorphisms and their close relatives congruences are introduced together with several ramifications. An explicit expression for the congruence generated by an arbitrary relation on a semigroup is derived.

5. Ideals and their variants are introduced, as well as the kernel of a semigroup. Certain simple properties of these concepts are also established including a characterization of 0-simple semigroups.

6. Green's relations are introduced and their most important properties are established. These include the structure of a $\mathcal{D}$-class and the behavior of idempotents in relation to $\mathcal{D}$-classes.

7. Regular elements and semigroups admit several characterizations; so do completely regular elements. Several auxiliary results are proved, including Lallement's lemma.

8. Concepts related to the translational hull are introduced, and a few of their properties are established.

9. A modest portion of the theory of ideal extensions is discussed, including general extensions, strict and pure extensions, and dense extensions. The relationship with the translational hull plays here an essential role.

10. Free semigroups and free semigroups with involution are defined and constructed, the latter by a construction which is part of the construction of a free group.

11. Identities and varieties are discussed in some detail. The expression for the join of two varieties, and for the variety generated by a semigroup, are explicitly found. The relationship of fully invariant congruences on a free semigroup with varieties is established.

12. The concepts related to amalgamation are introduced including the weak, special, and strong amalgamation properties. Two useful lemmas concerning these notions are proved.

13. The free product and the amalgamated free product of semigroups are discussed in some detail. Several universal properties of these concepts are established.

14. A short list of needed definitions from category theory is given.

I.1 INTRODUCTION

Inverse semigroups were introduced by Wagner in 1952 as regular semigroups with commuting idempotents. In 1953 Liber proved that Wagner's definition is equivalent to the requirement that every element has a unique inverse. Wagner called inverse semigroups "generalized groups" and he and some of his followers have used this term since that time. The term "inverse semi-groups" was introduced by Preston who independently discovered this class of semigroups in 1954.

From their inception to the present day, inverse semigroups have attracted a wide attention among workers in semigroups. Their popularity has several objective and subjective reasons.

In the first place, the closeness of inverse semigroups to groups made it possible to search for structure theorems vaguely modeled on those in group theory. Even though this approach had only a limited success, groups still play a decisive role in important structure theorems for various classes of inverse semigroups. Although the similarity of inverse semigroups and groups is not as substantial as it may appear on the first examination, there is an important analogy between them. Inverse semigroups represent an abstraction of the properties of sets of one-to-one partial transformations closed under composition and inversion just as groups play that type of role for permutation groups. This fact has actually been the leitmotiv and the focus of attention of the Saratov school of inverse semigroups (or should we say "generalized groups"?) headed and inspired by Wagner and by Schein.

In the second place, the simple and esthetically pleasing axioms for inverse semigroups have exercised a certain charm upon many researches in the field of semigroups. Many structure theorems and concepts for various classes of regular semigroups have much simpler formulations for the corresponding classes of inverse semigroups. Such inverse semigroups as the bicyclic semigroup (which has been rediscovered many times) and Brandt semigroups enjoy properties of great value in the study of other classes besides inverse semigroups. It is this abstract approach that was adopted by Preston and pursued by Munn, McAlister, Reilly, and others. The valuable contributions by Clifford antedate these authors and concern the structure of inverse semigroups belonging to certain special classes.

These are some of the principal movers of the theory and some of the objective reasons for the attention paid to inverse semigroups. The subjective reasons for the considerable development of the theory of inverse semigroups can be found in the magnetic personalities of the prime movers: Wagner and Schein created a following in the Soviet school of inverse semigroups, Preston and Munn in the school of the West.

Research activity in inverse semigroups has been both intensive and extensive. On the intensive side, deep structure theorems abound for special classes of inverse semigroups. Let us recall only a few jewels: Clifford's theorems for semilattices of groups and for Brandt semigroups, Reilly's theorem for bisimple ω-semigroups and McAlister's theorem for E-unitary inverse semigroups. In this special category belong the Wagner and the Munn representations. On the extensive side, the number of papers dealing entirely or partly with inverse semigroups is a large one. The bibliography at the end of this text represents an attempt to collect all the items dealing primarily with or bearing upon inverse semigroups.

The following is a concise discussion of the topics covered in various chapters.

I. An extensive collection of concepts, and some of their properties, concerning semigroups in general make up a chapter on preliminaries. In particular, the following topics are discussed: semigroups, congruences and homomorphisms, ideals, Green's relations, regularity, the translational hull, ideal extensions, free semigroups, varieties, amalgamation, and free products. The chapter starts with the needed definitions from the theory of partially ordered sets and ends with a list of needed concepts from category theory.

II. Some of the important, and widely researched, classes of inverse semigroups include Clifford semigroups, Brandt semigroups, strict inverse semigroups, Bruck semigroups over monoids, and Reilly semigroups. These classes harbor some of the most important constructions of the theory and provide suitable examples exhibiting various phenomena discussed in the succeeding chapters.

III. Congruences on inverse semigroups were first described by Preston; a different approach was later devised by Scheiblich. In the study of the congruence lattice, the initial steps of Reilly and Scheiblich play a significant role. Of all the classes of semigroups for which congruences have been investigated, the study of congruences on inverse semigroups has been most profitable. In fact, all the important structure theorems for inverse semigroups are based on various special congruences.

IV. Inverse semigroups admit an analogue of the Cayley theorem in group theory, namely the Wagner representation by one-to-one partial transformations. The Munn representation is a homomorphism of an inverse semigroup into the inverse semigroup of isomorphisms between principal ideals of its semilattice of idempotents. Congruence-free inverse semigroups admit several interesting characterizations. The general theory of representations of inverse semigroups by one-to-one partial transformations on a set, due to Schein, exhibits many features akin to those of group representations by permutations.

V. The translational hull of an inverse semigroup is again an inverse semigroup, a result first proved by Ponizovskiĭ. Related to the subject of the translational hull are the two hulls $C(S)$ and $\hat{S}$ designed by Schein and McAlister, respectively. The translational hull of a Clifford semigroup admits a suitable Clifford representation. For Brandt semigroups the translational hull is sufficiently transparent so that one may construct ideal extensions of these semigroups in great detail.

VI. Just as in the case of group extensions, it is natural to consider conjugate extensions of inverse semigroups. Treating these extensions, one is led to the conjugate hull of an inverse semigroup, analogous to the automorphism group of a group. Normal extensions of inverse semigroups represent a close analogue of the Schreier group extensions. They were initially studied by Petrich, but a general solution was furnished by Allouch. The theory of normal extensions runs somewhat parallel to the Schreier theory with the added complication of a partition of idempotents.

VII. The McAlister structure theorem for E-unitary inverse semigroups in terms of P-semigroups certainly dominates most treatments of general or special E-unitary inverse semigroups. An alternative construction for these semigroups was offered by Petrich, Reilly, and Žitomirskiĭ. The structure of F-inverse semigroups was described by McFadden and O'Carroll. This relatively new field is already rich in significant achievements.

VIII. Even though Wagner proved the existence of free inverse semigroups rather early, it was Scheiblich who provided for it a concrete construction. His work caused a burst of activity which produced improvements in his construction as well as other descriptions of free inverse semigroups. The underlying ideas of the McAlister P-theorem have much in common with Scheiblich's work. Jones established some remarkable properties of free inverse semigroups.

IX. Free monogenic inverse semigroups were first described by Gluskin. Much later, alternative descriptions followed, the simplest one being by Scheiblich as a subdirect product of two copies of the bicyclic semigroup. Congruences and various properties of these semigroups were investigated in some detail.

X. Bisimple inverse monoids exhibit many features reminiscent of groups. The first construction of these semigroups was offered by Clifford in an early paper. Since then they attracted the attention of many researches. Munn and McAlister provided alternative constructions, and Reilly modified Clifford's construction to describe bisimple inverse semigroups.

XI. Inverse semigroups whose idempotents form an ω-chain are said to be ω-regular. The main contributors to deciphering the structure of these semigroups were Reilly, Kočin, and Munn. Their structure is so well elucidated that one is able to answer many questions concerning these semigroups. These results stimulated much interest in inverse semigroups with some restrictions on idempotents.

XII. Varieties of inverse semigroups is a subject of recent origin but it already includes some deep results. The joins and meets of an inverse semigroup variety with the variety of groups provide the first insight into the structure of the lattice of inverse semigroup varieties. These results and the structure of the lowest three levels of the lattice are due to Kleiman. Djadčenko investigated the so-called small varieties and Reilly the completely semisimple varieties.

XIII. Almost all important results on amalgamation of inverse semigroups are due to Hall. He proved, in several different ways, that the class of inverse semigroups has the strong amalgamation property. In fact, he characterized precisely, up to group varieties, which inverse semigroup varieties have the (weak) strong amalgamation property. An alternative approach to the treatment of amalgamation of inverse semigroups was contributed by Howie.

XIV. One of the first attempts to "construct" all inverse semigroups is that of Schein by means of a Croisot groupoid, a partial order on it, and partial products. Meakin devised a similar approach based on a Croisot groupoid, a semilattice structure on its idempotents and "structure mappings" among some of the $\mathcal{R}$-classes of the groupoid. These constructions have theoretical, rather than practical, value showing to what extent some of the ingredients of an inverse semigroup determine the semigroup itself.

Various chapters may be grouped as follows. Chapter I consists of preliminaries. Chapter II provides basic special classes and constructions and concerns the structure of the semigroups in these classes. Chapters III and IV treat special aspects concerning all inverse semigroups; similarly Chapters V and VI concern several hulls and extensions of general inverse semigroups. Chapters VII to XI contain studies of the structures of inverse semigroups belonging to some special classes (hence are of the same general character as Chapter II). Chapters XII to XIV can be characterized as global analysis from three different points of view; they again concern all inverse semigroups.

Chapters are denoted by Roman numerals, sections by the chapter number and an Arabic numeral, and statements by yet another Arabic numeral. References within chapters indicate only the section and statement numbers, say 2.3; references to other chapters bear full information, say VII.2.3. The bibliography includes some papers dealing only marginally or not at all with inverse semigroups, but whose content may be of interest in our development. It excludes all announcements and conference reports, with a few exceptions when these items are of particular interest.

I.2 PARTIALLY ORDERED SETS

This is a brief compendium of concepts and simple properties related to partially ordered sets and more particularly lattices.

I.2.1 Definition

If X is any set, then any subset of the Cartesian product $X \times X$ is a *relation* on X. A *partially ordered set* $(X, \leq)$, to be simply denoted by X, is a pair where X is a nonempty set and $\leq$ is a reflexive, antisymmetric, and transitive relation on X.

Now let X be a partially ordered set. If the greatest lower bound (respectively least upper bound) of two elements α and β of X exists, we denote it by $\alpha \wedge \beta$ (respectively $\alpha \vee \beta$) and call this element the *meet* (respectively the *join*) of α and β. If any two elements of X have a lower bound in X, then X is *lower directed*. If any two elements of X have a meet, then X is a *lower semilattice*.

If Y is a nonempty subset of X which is a semilattice under the order induced on it by the order of X, then Y is a *subsemilattice* of X. If any two elements of X have a meet and a join, then X is a *lattice*. If Y is a subset of X which has a greatest lower bound, the latter is denoted by $\wedge Y$ or $\wedge_{\alpha \in Y} \alpha$ and is called the *meet* of Y; analogously for the *join* $\vee Y$ or $\vee_{\alpha \in Y} \alpha$.

Further, X is *linearly* (or *totally*) *ordered* if for any $x, y \in X$, either $x \leq y$ or $y \leq x$; in such a case X is a *chain*.

I.2.2 Lemma

Let X be a partially ordered set. If for some $\alpha, \beta, \gamma \in X$, $(\alpha \wedge \beta) \wedge \gamma$ and $\alpha \wedge (\beta \wedge \gamma)$ exist, then they are equal.

Proof. The proof of this lemma is left as an exercise.

For functions on partially ordered sets, we have the following concepts.

I.2.3 Definition

Let X and X' be partially ordered sets. A function $\varphi : X \to X'$ *preserves order* (or is *order preserving*) if for any $\alpha, \beta \in X$, $\alpha \leq \beta$ implies $\alpha\varphi \leq \beta\varphi$; φ *inverts order* (or is *order inverting*) if for any $\alpha, \beta \in X$, $\alpha \leq \beta$ implies $\beta\varphi \leq \alpha\varphi$. A bijection φ of X onto X' is an *order isomorphism* if both φ and φ^{-1} preserve order; in the case that $X = X'$, φ is an *order automorphism* of X. A bijection φ of X onto X' is an *order antiisomorphism* if both φ and φ^{-1} invert the order.

The following subsets of a partially ordered set are of particular interest.

I.2.4 Definition

Let X be a partially ordered set. A nonempty subset Y of X is an (*order*) *ideal* of X if for any $\alpha \in Y$, $\beta \in X$, $\beta \leq \alpha$ implies $\beta \in Y$. For any $\alpha \in X$, the set

$$[\alpha] = \{\beta \in X | \beta \leq \alpha\}$$

is the *principal* (*order*) *ideal of X generated by α*. If Y is an ideal of X such that for every $\alpha \in X$, $Y \cap [\alpha]$ is a principal ideal, then Y is a *p-ideal*. An ideal Y of X is *essential* if for any $\alpha \in X$, there exists $\beta \in Y$ such that $\beta \leq \alpha$. For any $\alpha, \beta \in X$, $\alpha \leq \beta$, the set

$$[\alpha, \beta] = \{\gamma \in X | \alpha \leq \gamma \leq \beta\}$$

is an *interval* of X. If $\alpha, \beta \in X$ are such that $\alpha < \beta$ and $\alpha < \gamma < \beta$ for no $\gamma \in Y$, then *β covers α*, or *α is covered by β*, which we denote by $\alpha \prec \beta$. We denote by X^1 the partially ordered set obtained from X by adjoining to it an element which plays the role of the greatest element of X^1.

We now turn to lattices.

I.2.5 Definition

Let L be a lattice. Then L is *distributive* if

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \qquad (\alpha, \beta, \gamma \in L);$$

we may equivalently interchange the signs $\wedge$ and $\vee$. A weaker condition is: L is *modular* if

$$\alpha \leq \gamma \Rightarrow \alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge \gamma \qquad (\alpha, \beta, \gamma \in L).$$

The lattice L is *complete* if every nonempty subset of L has a meet and a join. A sublattice V of L which is complete under the order induced on it by the order of X and whose meets and joins coincide with those of X is a *complete sublattice* of L. A subset V of L is a *complete* $\wedge$*-sublattice* of L if V is a complete lattice whose meets coincide with those in L. A *complete* $\vee$*-sublattice* has an analogous meaning.

There is a simple criterion for completeness which is often useful.

I.2.6 Lemma

Let X be a partially ordered set. If X has a greatest element and each nonempty subset of X has a meet, then X is a complete lattice.

Proof. Let Y be a nonempty subset of X. The set Z of all upper bounds of Y is nonempty since the greatest element of X is an element of Z. By hypothesis, $\alpha = \wedge Z$ exists, and thus $\alpha = \vee Y$, that is, Y has a join.

For mappings on a lattice, we have the following concepts.

I.2.7 Definition

Let L and L' be lattices. A mapping $\varphi : L \to L'$ is a *homomorphism* if

$$(\alpha \wedge \beta)\varphi = \alpha\varphi \wedge \beta\varphi, \qquad (\alpha \vee \beta)\varphi = \alpha\varphi \vee \beta\varphi \qquad (\alpha, \beta \in L);$$

in such a case, we say that φ *preserves meets and joins*, and this definition can be extended to arbitrary meets and joins in an obvious way. If φ is also a bijection of L onto L', it is an *isomorphism* of L onto L'. If L and L' are complete lattices and $\varphi : L \to L'$ preserves arbitrary meets (respectively joins), then φ is a *complete* $\wedge$*-homomorphism* (respectively *complete* $\vee$*-homomorphism*); the conjunction of the two conditions makes a *complete homomorphism*.

An equivalence ρ on L is a *congruence* on L if

$$\alpha\rho\beta \Rightarrow (\alpha \wedge \gamma)\rho(\beta \wedge \gamma), \quad (\alpha \vee \gamma)\rho(\beta \vee \gamma) \qquad (\alpha, \beta, \gamma \in L).$$

Note that the correspondence of congruences and homomorphisms in lattices is the same as in any universal algebra. The following lemma will be useful.

I.2.8 Lemma

If φ is an order isomorphism of a lattice L onto a lattice L', then φ is a (lattice) isomorphism.

Proof. Let φ be as in the statement of the lemma, and let $a, b \in L$. Then $a \wedge b \leq a$ implies $(a \wedge b)\varphi \leq a\varphi$ and analogously $(a \wedge b)\varphi \leq b\varphi$ so that $(a \wedge b)\varphi \leq a\varphi \wedge b\varphi$. Further, $a\varphi \wedge b\varphi \leq a\varphi$ which yields $(a\varphi \wedge b\varphi)\varphi^{-1} \leq a$ and symmetrically $(a\varphi \wedge b\varphi)\varphi^{-1} \leq b$. Hence $(a\varphi \wedge b\varphi)\varphi^{-1} \leq a \wedge b$ and thus $a\varphi \wedge b\varphi \leq (a \wedge b)\varphi$. Consequently, $(a \wedge b)\varphi = a\varphi \wedge b\varphi$. A dual argument shows that $(a \vee b)\varphi = a\varphi \vee b\varphi$.

We now discuss binary relations.

I.2.9 Definition

Let X be any set. We denote by $\mathfrak{B}(X)$ the *set of all relations on* X. Then $\mathfrak{B}(X)$ is a complete lattice under inclusion with least element the *empty relation* $\varnothing$ and greatest element the *universal relation* ω on X. The *equality* (or *identical*) *relation* ε is the least reflexive relation on X. Further, $\mathfrak{B}(X)$ is provided with a multiplication defined by

$$x\alpha\beta y \Leftrightarrow x\alpha z,\ z\beta y \qquad \text{for some } z \in X.$$

The notation ε and ω will be used consistently; only in the case of possible confusion, we will write ε_X and ω_X instead of ε and ω, respectively. Simple verification shows that the multiplication of binary relations is associative. We thus may write products without parentheses and define α^n as the nth iterate $\alpha\alpha \cdots \alpha$.

1.2.10 Definition

Let $\rho \in \mathfrak{B}(X)$. The relation ρ^{-1} defined by

$$x\rho^{-1}y \Leftrightarrow y\rho x \qquad (x, y \in X)$$

is the *inverse relation* of ρ. The relation $\rho^t = \cup_{n=1}^{\infty}\rho^n$ is the *transitive closure* of ρ; explicitly

$$x\rho^t y \Leftrightarrow \text{there exist } z_1, z_2, \ldots, z_n \in X \text{ such that}$$

$$x = z_1, \qquad z_i\rho z_{i+1}, \qquad i = 1, 2, \ldots, n-1, \qquad z_n = y.$$

We then have the following simple result.

1.2.11 Lemma

For any $\rho \in \mathfrak{B}(X)$, the following statements hold.

(i) ρ^t is the least transitive relation on X containing ρ.
(ii) $(\rho \cup \rho^{-1} \cup \varepsilon)^t$ is the least equivalence relation on X containing ρ.

Proof. The proof of this lemma is left as an exercise.

The intersection of equivalence relations on X is evidently an equivalence relation. Since also ω is an equivalence relation, 2.6 implies that the partially ordered set of all equivalence relations on X is a complete $\cap$-sublattice of $\mathfrak{B}(X)$. The join of a nonempty family $\mathfrak{F}$ of equivalence relations on X is of the form $(\cup\mathfrak{F})^t$ according to 2.11. The following special case is of particular interest.

1.2.12 Lemma

Let α and β be equivalence relations on a set X. Then $\alpha\beta$ is an equivalence relation if and only if $\alpha\beta = \beta\alpha$, in which case $\alpha\beta = \alpha \vee \beta$ in the lattice of equivalence relations on X.

Proof. Since $\alpha \cup \beta \subseteq \alpha\beta \subseteq \alpha \vee \beta$, if $\alpha\beta$ is an equivalence relation, then $\alpha\beta = \alpha \vee \beta$ and

$$\alpha\beta = (\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1} = \beta\alpha.$$

Conversely, if $\alpha\beta = \beta\alpha$, then

$$(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1} = \beta\alpha = \alpha\beta,$$

$$\alpha\beta\alpha\beta = \alpha\alpha\beta\beta = \alpha\beta,$$

so $\alpha\beta$ is symmetric and transitive, and it is obviously reflexive, so it is an equivalence relation.

We will need the following simple result.

I.2.13 Lemma

If L is a lattice of commuting equivalence relations on a set X, then L is modular.

Proof. In view of 2.12, for any $\alpha, \beta \in L$, we have $\alpha \vee \beta = \alpha\beta$. Now let $\alpha, \beta, \gamma \in L$ be such that $\alpha \subseteq \gamma$. Let $a[(\beta\alpha) \cap \gamma]b$. Then $a\beta\alpha b$ and $a\gamma b$ and thus $a\beta c$, $c\alpha b$ for some $c \in S$. Since $\alpha \subseteq \gamma$, we get $c\gamma b$. But then $a\gamma b$ and $b\gamma c$ which implies $a\gamma c$. Now $a\beta c$ and $a\gamma c$ yield $a(\beta \cap \gamma)c$, which together with $c\alpha b$ gives $a(\beta \cap \gamma)\alpha b$. Consequently, $(\alpha\beta) \cap \gamma \subseteq \alpha(\beta \cap \gamma)$. In the lattice notation this reads $(\alpha \vee \beta) \wedge \gamma \leq \alpha \vee (\beta \wedge \gamma)$; since $\alpha \leq \gamma$, the opposite inclusion is true in any lattice. Therefore L is modular.

The following construction will be needed.

I.2.14 Definition

Let P and Q be partially ordered sets. On $P \times Q$ introduce a relation $\leq$ by

$$(p, q) \leq (p', q') \text{ if } p = p', q \leq q' \quad \text{or} \quad p < p'.$$

One verifies easily that $\leq$ is a partial order in $P \times Q$; $\leq$ is the *lexicographic order* on $P \times Q$. The partially ordered set $(P \times Q, \leq)$ is the *ordinal product* of P and Q, to be denoted by $P \circ Q$.

I.2.15 Exercises

(i) Let Y be a lower semilattice and I be an order p-ideal of Y. For every $\alpha \in Y$, define $\bar{\alpha}$ by the condition $[\alpha] \cap I = [\bar{\alpha}]$. What can be said about the mapping $\alpha \to \bar{\alpha}$ $(\alpha \in Y)$?

(ii) Show that the lattice of all normal subgroups of any group is modular. Give an example of a group in which the lattice of all subgroups is not modular.