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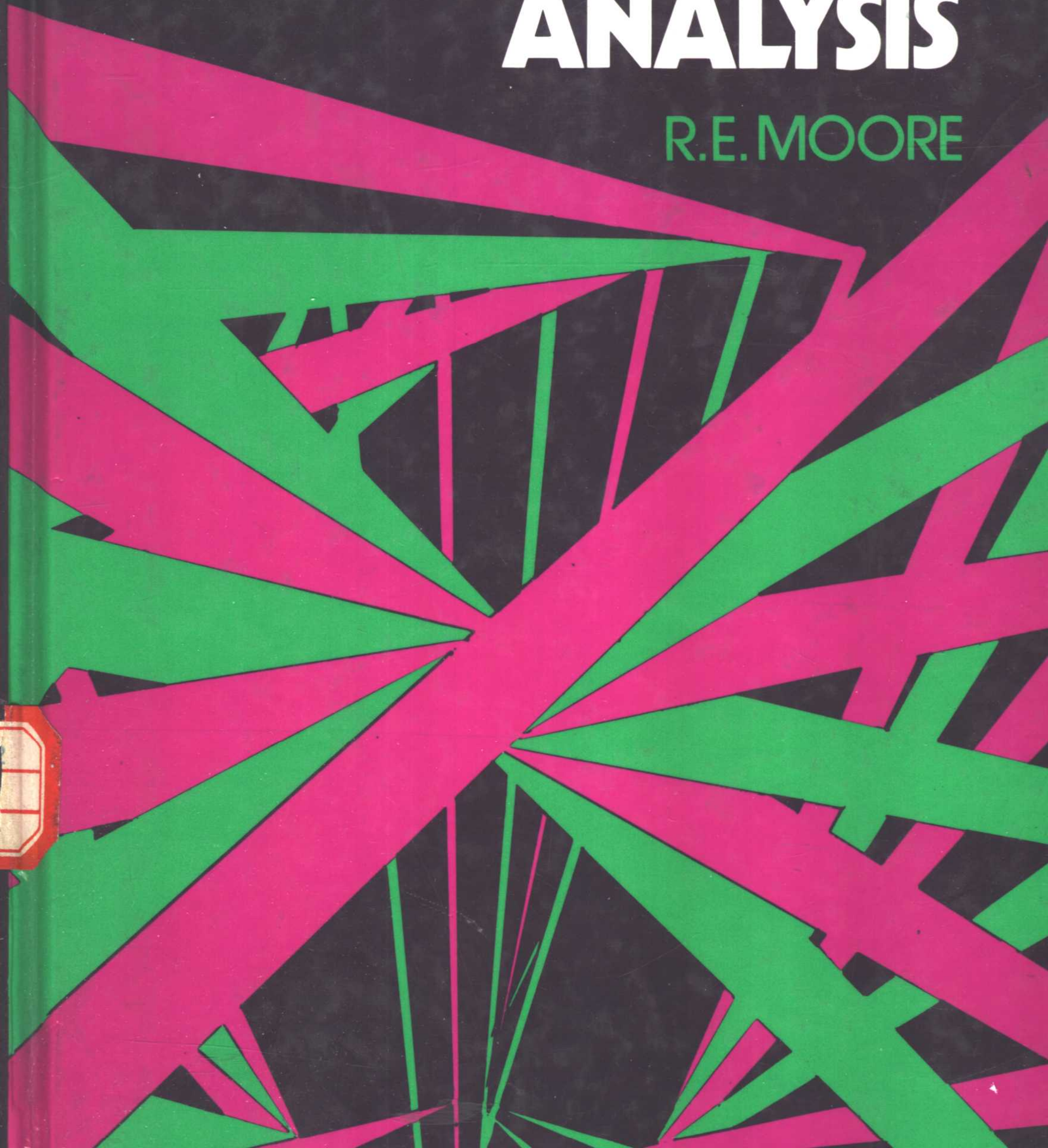
COMPUTATIONAL FUNCTIONAL ANALYSIS

R.E. MOORE



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for Devin and Claire

Preface

The term 'functional analysis' now refers to a fruitful and diversified branch of mathematics which includes the study of set-theoretic, topological, algebraic, geometric, order, and analytic properties of mappings in finite and infinite dimensional spaces. It is characterized by a generality and elegance which is lacking in classical analysis. Computational mathematics and numerical analysis now rely heavily on results from this theory.

In these lecture notes, the main emphasis is on numerical methods for operator equations – in particular, on the analysis of approximation error in various methods for obtaining approximate solutions to equations and systems of equations. These might be algebraic, linear, non-linear, differential, integral, or other types of equations.

An important part of functional analysis is the extension of techniques for dealing with finite dimensional problems to the infinite dimensional case. This allows us to obtain results which apply at the same time to finite systems of algebraic equations or equally to differential and integral equations.

In mathematics, there is often a trade-off between generality and precision. As a result, in any specific application of functional analysis to a particular numerical problem, there is always the possibility of sharpening results by making use of special properties of the particular problem. In spite of this, the methods of functional analysis are, at the very least, an excellent starting point for any practical problem.

This text is designed for a one-semester introduction at the first year graduate level; however, the material can easily be expanded to fill a two-semester course. It has been taught both ways by the author at the University of Wisconsin-Madison and as a one-semester course at the University of Texas at Arlington. By adding a little additional detail and proceeding at a leisurely pace, Chapters 1–9 and 11–13 can serve as the first semester's material concentrating on *linear* operator equations. The remaining material, concentrating on *nonlinear* operator equations, can serve as the second semester's material, again with a little additional detail and proceeding at a comfortable pace. The material as written can be covered in one semester as a concentrated introduction for students who are willing to work hard to acquire, in a short period, the rudiments of a powerful discipline.

An easy way to expand the material to fit a two-semester course is for the instructor to discuss in detail every one of the more than 100 exercises in the text *after* the students have had a try at them.

It is no more possible to acquire mathematical strength and skills by simply sitting in a lecture room and listening to someone talk about mathematics than it is to acquire physical strength and skills by sitting in a living room and watching football on television. Therefore, it is essential for the education of the students that they try all the exercises, which are designed to help them learn how to discover mathematics for themselves.

The usual practise of numbering equations along with frequent cross-references to equations on distant pages has been dropped in this text as an unnecessary encumbrance.

I am grateful for the helpful suggestions of an anonymous referee, who carefully read the first draft.

CHAPTER 1

Introduction

The outcome of any numerical computation will be a finite set of numbers. The numbers themselves will be finite decimal (or binary) expansions of rational numbers. Nevertheless, such a set of numbers can represent a *function* in many ways: as coefficients of a polynomial; as coefficients of a piecewise polynomial function (for example a spline function); as Fourier coefficients; as left and right hand endpoints of interval coefficients of an interval valued function; as coefficients of each of the components of a vector valued function; as values of a function at a finite set of argument points; etc.

The concepts and techniques of functional analysis we will study will enable us to design and apply methods for the approximate solution of operator equations (differential equations, integral equations, and others). We will be able to compute numerical representations of approximate solutions and numerical estimates of error. Armed with convergence theorems, we will know that, by doing enough computing, we will be able to obtain approximate solutions of any desired accuracy, and know when we have done so.

Since no previous knowledge of functional analysis is assumed here, a number of introductory topics will be discussed at the beginning in order to prepare for discussion of the computational methods.

The literature in functional analysis is now quite extensive, and only a small part of it is presented here – that which seems most immediately relevant to computational problems. This is an introductory study. It is hoped that the reader will be brought along far enough to be able to begin reading the more advanced literature and to apply the techniques to practical problems.

Some knowledge of linear algebra and differential equations will be assumed. Previous study of numerical methods and some experience in computing will help in understanding the applications to be discussed. No background in measure theory is assumed; in fact, we will make scant use of those concepts.

In the first part of the study, we will introduce a number of kinds of topological spaces suitable for investigations of computational methods for solving linear operator equations. These will include Hilbert spaces, Banach spaces, and metric spaces. Linear functionals will play an important role, especially in

Hilbert spaces. In fact, these mappings are the source of the name 'functional analysis'. We will see that the Riesz representation theorem plays an important role in computing when we operate in reproducing kernel Hilbert spaces.

The study of order relations in function spaces leads to important computing methods based on interval valued mappings. We will see how interval analysis fits into the general framework of functional analysis.

In the second part of the study, we will turn our attention to methods for the approximate solution of linear operator equations.

In the third part of the study, we will investigate methods for the approximate solution of nonlinear operator equations.

CHAPTER 2

Linear spaces

We begin with an introduction to some basic concepts and definitions in linear algebra. These are of fundamental importance for linear problems in functional analysis, and are also of importance for many of the methods for nonlinear problems, since these often involve solving a sequence of linear problems related to the nonlinear problem.

The main ideas are these: We can regard real valued functions, defined on a continuum of arguments, as points (or vectors) in the same way as we regard n-tuples of real numbers as points; that is, we can define addition and scalar multiplication. We can take linear combinations. We can form larger or smaller linear spaces containing or contained in them; and we can identify equivalent linear spaces, differing essentially only in notation.

Many numerical methods involve finding approximate solutions to operator equations (for example differential equations or integral equations) in the form of polynomial approximations (or other types of approximations) which can be computed in reasonably simple ways. Often the exact solution cannot be computed at all in finite real time, but can only be approximated as the limit of an infinite sequence of computations.

Thus, for numerical approximation of solutions as well as for theoretical analysis of properties of solutions, linear spaces are indispensable.

The basic properties of relations are introduced in this chapter, since they will be met in many different contexts throughout the subsequent chapters.

An understanding of the material in the exercises will be assumed as the text proceeds.

Definition

A *linear space*, or *vector space*, over the field **R** of real numbers is a set X, of elements called points, or vectors, endowed with the operations of addition and scalar multiplication having the following properties:

(1) $\forall x, y \in X$ and $\forall a, b \in \mathbf{R}$: [$\forall$ = for all; $\in$ = in the set]

$$x + y \in X,$$

$$
\begin{aligned}
a\,x &\in X, \\
1\,x &= x, \\
a(b\,x) &= (a\,b)\,x, \\
(a+b)\,x &= a\,x + b\,x, \\
a\,(x+y) &= a\,x + a\,y;
\end{aligned}
$$

(2) $(X, +)$ is a commutative group; that is, $\forall\, x, y, z \in X$:

$\exists\; 0 \in X$ such that $0 + x = x$,

$\exists\; (-x) \in X$ such that $x + (-x) = 0$, [$\exists$ = there exists]

$x + y = y + x$,

$x + (y + z) = (x + y) + z$.

Examples

(1) $X = \mathbf{R}$ with addition and scalar multiplication defined in the usual way for real numbers;

(2) $X = E^n$, n-dimensional Euclidean vector space, with componentwise addition and scalar multiplication;

(3) X = polynomials, with real coefficients, of degree not exceeding n, with addition defined by adding coefficients of monomials of the same degree and scalar multiplication defined by multiplication of each coefficient;

(4) all polynomials, with real coefficients, with addition and scalar multiplication as in (3);

(5) continuous real valued functions on $\mathbf{R}$ with pointwise addition and scalar multiplication: $(x + y)\,(t) = x(t) + y(t)$ and $(a\,x)\,(t) = a\,x(t)$;

(6) all real valued functions on $\mathbf{R}$ with addition and scalar multiplication as in (5).

Exercise 1 Can we define addition and scalar multiplication for n-by-n matrices with real coefficients so that they form a linear space? Check all the required properties.

Definition

A *linear manifold*, or *subspace*,† of a linear space X, is a subset Y of X which is algebraically closed‡ under the operations of addition and scalar multiplication for elements of X. Thus, Y is itself a linear space.

Exercise 2 Show that the zero element of a linear space X is also an element of every subspace of X.

---

† $x + y$ and $a\,x$ are in Y for all x and y in Y and all real a.

‡ In a *topological* linear space, a subspace is defined as *closed* linear manifold; see Chapter 5.

Exercise 3 Show that examples (3), (4), and (5) are, respectively, subspaces of examples (4), (5), and (6). Can you find any other subspaces of example (6)?

Definition
Two linear spaces X and Y are *isomorphic* if there is a one-one, *linear* mapping of X onto Y: $m(x+y) = m(x) + m(y), m(a\,x) = a\,m(x)$.

Exercise 4 Show that such a mapping has an inverse which is also linear.

Exercise 5 Let T be an arbitrary set with n distinct elements. Show that the linear space of real valued functions on T with pointwise addition and scalar multiplication is isomorphic to E^n.

NOTE: Unless otherwise stated, all linear spaces considered in this text will be over the real scalar field.

Definition
The *Cartesian product* (or *direct sum*) of two linear spaces X and Y, denoted by $X \times Y$ (or $X \oplus Y$), is the set of ordered pairs (x, y) with $x \in X$ and $y \in Y$, endowed with componentwise addition and scalar multiplication:

$$(x, y) + (u, v) = (x + u, y + v)$$
$$a(x, y) = (a\,x, a\,y) \ .$$

Exercise 6 Show that E^n is isomorphic to $E^{n-1} \times \mathbf{R}$.

Definitions
A *relation*, r, in a set X, is a subset of $X \times X$. If (x, y) belongs to the relation r, we write $x\,r\,y$.

A relation is called *transitive* if

$$\forall x, y, z : x\,r\,y \text{ and } y\,r\,z \text{ implies } x\,r\,z \ .$$

A relation is called *reflexive* if

$$\forall x : x\,r\,x \ .$$

A relation is called *symmetric* if

$$\forall x, y : x\,r\,y \text{ implies } y\,r\,x \ .$$

An *equivalence* relation is a relation that is transitive, reflexive, and symmetric. An equivalence relation in a set X factors X into *equivalence classes*. Denote by C_x the equivalence class to which x belongs. Thus $y \in C_x$ means that $y\,r\,x$.

Exercise 7 Show that two equivalence classes in a set X are either disjoint or they coincide.

Exercise 8 Suppose r is an equivalence relation in a linear space X. Suppose further that $x' \in C_x$ and $y' \in C_y$ imply that $x' + y' \in C_{x+y}$ and $a\,x' \in C_{ax}$ for all real a. Show that the set of equivalence classes is again a linear space with

$$C_x + C_y = C_{x+y}$$

and

$$a\,C_x = C_{ax} \; .$$

Examples

(1) Suppose that Y is a subspace of a linear space X. We can define an equivalence relation in X by

$$\forall x, y \in X \;, \; x\,r\,y \text{ if and only if } x - y \text{ is in } Y.$$

The linear space of equivalence classes defined in this way is called the *factor space, X modulo Y*, written X/Y. The elements of X/Y can be regarded as parallel translations of the subspace Y, since each element of X/Y except for the equivalence class of 0 is disjoint from Y (does not intersect Y). Each element of X/Y is a set in X of the form $x + Y$, that is, the set of all elements of X which are the sum of x and an element of Y.

(2) Let X be the set of all real valued continuous functions on an interval $[a, b]$ in the real line. Let Y be the subspace of functions which vanish at the endpoints a and b. Then X/Y consists of the sets of functions which have given values at a and b.

Exercise 9 Let $X = E^2$ and let Y be a one-dimensional subspace of X. Sketch the elements of X/Y.