Questions on Principles of Physics

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Second edition

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St. Martin's School, Northwood

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Preface

Aims.

Any school physics book published nowadays needs to justify itself on at least two counts. First, it must use SI units exclusively, and second, it must offer something different from the great number of books already available. The majority of problems to be found at the ends of chapters in textbooks, and indeed in many books devoted exclusively to questions, are collected from GCE examination papers. Such questions are written specifically to test examination candidates' understanding at the end of their two year course, and therefore are not necessarily suitable as vehicles for introducing a new topic to the inexperienced student.

The questions in this book have been written with the express aim of developing confidence, understanding and interest. We have tried to make many of them simple to answer by structuring them in such a way that the student is led easily from one step to the next. Where a more difficult idea is introduced we have sometimes given a hint for the solution. It must be emphasized that this book is *not* linked in any way to examination *technique*, but we hope that its effect will be to improve examination performance by developing a real *understanding* of the ideas of physics. We hope that this understanding will give the student the necessary ability to tackle a university course with confidence.

While writing the questions we have put emphasis on the desirability of the student acquiring (*a*) a feeling for *orders of magnitude*, (b) a realization of the importance of *energy* as a linking concept, and (*c*) a familiarity with microscopic ideas and the behaviour of *electrons, atoms* and *molecules*.

Content

For convenience we have planned the organization of this book to match that of our A/S level textbook *Essential Principles of Physics*, but we have been careful to ensure that it can be used independently. We have included few questions of the type set in examinations because these are becoming available in ever-increasing quantity from the various examining boards in book form. Similarly there are few full scholarship problems. We have tried to make a systematic collection in which no topic of importance has been omitted.

We have separated the questions in nearly every chapter into two groups, *Qualitative* and *Quantitative*. The former are intended for discussion in class: seldom do they ask a question which can be answered by appealing directly to the concise text that we visualize many students will have. They do not require them to reproduce the sort of description that they will find in a detailed text, but more often they ask the question '*Why?*' Not all of them have clear-cut answers. Some are intended to develop the basic bookwork beyond that which students need *learn*, and are presented so that they involve useful application of those ideas with which they must become familiar. Many will require them to make use of more detailed monographs in the library. Others are written expressly to develop creative thinking, especially with regard to orders of magnitude.

It is regrettable that much of a science student's work at school is analytical rather than creative. For this reason we have included an extensive chapter on **Essays**. While some topics are historical and biographical, most of them have been chosen to promote reading and thought in a subject area of interest both to the student and to the future development of physics.

Details

(*a*) ***Classification of questions.*** The subject matter has been broken down into detailed chapters, and further subdivided within a chapter where it was thought to be helpful. Where a question aims to develop a particular point this has been indicated, and an index has been included to assist the location of such questions. Questions marked † are particularly easy, whereas those marked * either deal with a simple topic in an advanced way, or treat what is usually regarded as a scholarship topic. Questions marked **O-M** (standing for *order of magnitude*) are discussed on page 19.

(*b*) Material in *italic* is of two kinds: (*i*) *hints* for solving more sophisticated problems, and (*ii*) *teaching information* to point out the relevance of a problem, its connection with other topics, or perhaps the significance of the order of magnitude of the answer.

(*c*) In many minor ways we have tried to keep the working of the problems simple. For the most part information is presented to two significant figures, and where possible in round numbers. We have quoted volumes and areas rather than radii, weights rather than the ubiquitous (mass $\times$ $\boldsymbol{g}_0$) etc., since many students lose sight of the physical principles when they are faced by a lengthy numerical computation.

(*d*) ***Answers.*** Where a question has been broken down into several steps, we have sometimes given selected answers only. If the method of solution is so simple that it is indicated by the answers, then that answer is omitted.

Acknowledgements

(*a*) All the questions were written expressly for this book, but naturally few of them are original in concept. We have referred widely to current textbooks and examination papers, and we have collected from them the ideas on which these questions have been based. We are not aware that any question here duplicates that in any other publication, but freely acknowledge our indebtedness to the books listed in the Bibliography of *Essential Principles of Physics*. In particular we would like to pay tribute to *Physics*, by Halliday and Resnick, from which we derived the idea of the short but penetrating qualitative questions.

(*b*) Mrs E. G. Hodgson again earns our sincere thanks for taking on the demanding job of preparing the typescript, a task which she performed with notable accuracy.

(*c*) We would like to thank Kenneth Pinnock and Howard Jay of John Murray who advised us at all stages of the production of both this book and its companion, and who were most cooperative in following our wishes on details of layout and illustration.

(*d*) Dr J. W. Warren of Brunel University kindly undertook the uninviting task of scrutinizing the questions for errors. This attention to detail has been very helpful in reducing minor mistakes and obscurities, and we are most grateful.

(*e*) Once again we thank Tim Robinson for the first class job that he has made of the diagrams.

(*f*) One of us (PMW) would like to thank the Warden and Fellows of Merton College, Oxford, for their generous hospitality during his term as a schoolmaster student at the College; the Headmaster of Sherborne School for allowing him leave of absence; and his wife Sue for enduring for some years now what W. M. Gibson has delightfully called a 'severe case of authorship'.

A book of this nature is more than usually prone to minor mistakes, both numerical and otherwise. We encourage readers, students particularly, to write and tell us about those that they discover.

August, 1972

P.M.W.

Sherborne, Dorset

M.J.H.

Canterbury, Kent

Publisher's note:

After a long illness, courageously borne, Paddy Whelan died in May 1980 at the age of 42.

Preface to this Edition

Since the first edition a number of questions have been rewritten in the interests of clarity, and any that have not proved helpful have been removed. For this edition there is a 12% increase in the number of questions, particularly in the chapters on *Newton's Laws*, momentum, equilibrium, elasticity, electric current, electronics and radioactivity. There are two main areas of expansion:

(*a*) a significant increase in elementary (easy) questions in many chapters to help bridge the gap between the level required for GCSE Physics and the level of knowledge and understanding expected of students embarking on an A level course,

(*b*) additional questions demanded by changes of the A level syllabuses since the last edition.

The examination syllabuses remain different in many detailed aspects, hence some reluctance to remove questions from this book. There are more questions here than any one person is likely to use but, hopefully, there will be sufficient choice for each individual to meet his or her requirements by suitable selection.

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Introduction

About this Book

Before reading this section you are advised to refer to the *Preface*, in which we discuss the purpose of the book, and the methods that we adopt to achieve it. In particular you should refer to the ways in which we have classified the different types of questions.

Many of the questions require numerical values for the fundamental constants, or for particular physical properties, and in these problems we have indicated at the end which are to be used. For convenience the values of these constants are printed on the front and back end-papers. No *other* information should be assumed in answering these problems. For example the quantities R, N_A, k and the ideal-gas molar volume at s.t.p. are all closely related, and to assume constants other than those indicated could destroy the point of the question.

Some of the questions are marked **O-M**, which is an abbreviation for *order of magnitude*. For these questions you will have to make your own assumptions. More details about them, and a worked example, are given on p. 19.

If you make reference to older literature you will find information expressed in non-SI units. Conversion factors will be found in *Tables of Physical and Chemical Constants*, by Kaye and Laby (*Longman*).

The Meanings of Words in Questions

Although this book is not concerned with examination questions, you will note the recurrent use of words habitually used in such questions, and you are advised to clarify in your own mind exactly what they mean. Here are some examples.

(*a*) **Define** means 'Give a short but exact formal statement'. Thus the definition of a physical quantity is best given by an algebraic equation. The symbols used in the equation must first be defined. Definitions are discussed further on p. 3.

(*b*) **Explain** (or '*What is meant by . . . ?*') requires much more detail. There would be a considerable difference between a formal statement *defining* pressure, and a microscopic *explanation* as to the nature and causes of the pressure at a point in a fluid. (Nevertheless it is often helpful if this explanation is developed from the definition.)

(*c*) The word **physical** in a question usually implies that the answer must include an explanation of the *mechanism* by which some process occurs. Very often this will include a description involving the imagined behaviour (according to some particular model) of submicroscopic particles such as electrons or molecules.

Suppose, for example, that we are asked to give a physical explanation for the reason why a metal generally conducts more heat under specified conditions than a non-metal. It would be no answer at all to quote the defining equation for thermal conductivity λ, and then to point out that the value of λ for a metal is generally larger than that for a non-metal. We would need to go into the microscopic description of the physical processes by which the two types of substances transport energy through themselves.

(*d*) Other words which occur frequently in questions are *Describe*, *Discuss*, *Compare*, *Contrast*, etc., and you must always check, before answering such questions, that you are doing just what is asked. For the latter two it is not sufficient to draw up a pair of corresponding lists which leave the reader to make the necessary comparisons.

Prefixes for SI Units

Certain prefixes have been agreed for use with the basic and derived SI units. All those listed below are of the form 10^{3n}, where n is a whole positive or negative number. Not all these prefixes have been used widely in the past, but it seems probable that they will be in the future. For that reason we have used them extensively in the questions of this book. It is essential that you become familiar with them as soon as possible, and to help you a number of concrete examples are given in the table below. The values of the prefixes will also be found on the front and back end-papers.

Prefix	*Symbol*	*Meaning*	*Common example*	
atto	a	10^{-18}	the electronic charge	$e = -0.16$ aC
femto	f	10^{-15}	energy of a typical X-ray photon	$W = 1$ fJ
pico	p	10^{-12}	capacitance of a mica tuning capacitor	$C = 6.0$ pF
nano	n	10^{-9}	the wavelength of sodium light	$\lambda = 589$ nm
micro	μ	10^{-6}	the leakage current in a junction diode	$I = 3.5$ μA
milli	m	10^{-3}	inductance of an air-cored solenoid	$L = 5.7$ mH
kilo	k	10^{3}	power of an electric heater	$P = 2.5$ kW
mega	M	10^{6}	standard pressure	$p_0 = 0.10$ MPa
giga	G	10^{9}	the half-life of radium decay	$T_{\frac{1}{2}} = 51$ Gs
tera	T	10^{12}	typical frequency of middle i.r. waves	$f = 10$ THz
peta	P	10^{15}	threshold frequency for caesium Gkγ	$\nu_0 = 0.45$ PHz
exa	E	10^{18}	pull of Earth on Moon	$F = 100$ EN

Some Useful Relationships

Each section in this book starts with a page of useful relationships; these are generally of the first three categories of the previous paragraph. Defining equations mention explicitly the quantity being defined and its symbol, but to save space other symbols are not defined. Should ambiguity arise the reader is referred to the authors' *Essential Principles of Physics* for clarification. We emphasize how important it is to note the conditions under which a particular equation applies in the form quoted.

The Nature and Status of Physical Equations

The solution of physics problems is too often regarded as a question of selecting an appropriate formula into which numbers can be substituted. While this may *sometimes* be a quick method of arriving at an answer, it is no way to learn the concepts of physics. Nevertheless physics is primarily an exact (quantitative) discipline, and this means that much of our work is concerned with equations.

It is the aim of this paragraph to indicate the relative importance of equations, so that you can decide for yourself what has to be *learned*. Not every equation can be fitted into a neat classification, but we can distinguish the following.

(*a*) ***Defining equations.*** These equations summarize the series of operations by which we relate a new physical quantity to other pre-existing quantities, and thus by implication to the seven fundamental quantities. For example we define electric potential V by relating it to the electric p.e. W of a test charge Q_0 through the defining equation $V = W/Q_0$.

In one sense a defining equation is a convention, and, as such, cannot be wrong: nevertheless private conventions are less useful than universally observed conventions, and much confusion is avoided if we all use the same convention. Therefore *defining equations must be learned.*

(*b*) ***Laws.*** A law is a statement that summarizes, with great precision and simplicity, ideas of fundamental importance. It is usually expressed in the form of an equation relating symbols that represent experimental observations, or, more simply, physical quantities. Thus the equation $F = Gm_1m_2/r^2$, once we have defined explicitly all the symbols involved, represents *Newton's Law* of universal gravitation. *Equations representing laws must* *be learned*, or at least a physicist must be able to write down the equation that symbolizes a verbal statement of a law.

(*c*) ***Principles and useful results.*** There are many equations in physics that are the consequence of applying laws and defining equations to situations which are frequently encountered. These results are then used so often that much time is saved if they are committed to memory. The following well-known equations are a few examples selected at random:

$$Fs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2$$

$$p + \tfrac{1}{2}\rho v^2 + h\rho g = \text{constant}$$

$$p = \tfrac{1}{3}\rho\overline{c^2}$$

$$n = \frac{\sin[(A + D_{\text{min}})/2]}{\sin A/2}$$

$$V = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q}{r}$$

$$\boldsymbol{E} = \sigma/\varepsilon_0$$

$$\mathscr{E} = I(R + r)$$

It should be stressed that it is not strictly *necessary* to learn any of these equations, but the derivation of (e.g.) $p = \frac{1}{3}\rho\overline{c^2}$ is an exercise which many beginning students find difficult and time consuming. Thus *it is useful to learn these equations*, provided that one remembers that the ideas behind the derivation of the equations are what really matter.

If you are uncertain whether you have remembered a derived equation correctly, you can always do a quick mental check that your version is at least dimensionally consistent.

Example: suppose you thought that the energy stored by an inductance L was of the form $\frac{1}{2}LI$. This has the unit $(\text{V s A}^{-1}) \times (\text{A}) = \text{V s} = \text{J s C}^{-1}$. To obtain the unit J we must multiply by C s^{-1}, i.e. A. We deduce that the equation is more likely to be of the form

$$W = \tfrac{1}{2}LI^2$$

(We should immediately recognize that stored energy can be represented by an expression in which a variable is *squared*.)

(*d*) ***Special results.*** There are occasions when you may read in a textbook an analysis whose end-product is an equation applicable to one special situation only. For example it can be shown that the loss of electrical p.e. ΔW when two capacitors are joined is given by the equation

$$\Delta W = \frac{\frac{1}{2}(C_2Q_1 - C_1Q_2)^2}{C_1C_2(C_1 + C_2)}$$

The effort spent in learning such an equation is better used in mastering the principles used to derive it. It is a common fallacy to imagine that the learning of such equations is equivalent to an understanding of the ideas behind them, and very often they are misquoted, and used even in circumstances to which they do not apply. *These equations should not be learned.*

Of course there are some results which a specialist will use a large number of times. For example he may want to use the equation

$$\lambda = \frac{h}{\sqrt{2m_e eV}} = \frac{1.23\ \text{nm}}{\sqrt{V/\text{volts}}}$$

He would not *learn* this equation (although no doubt he would become familiar with it by repetitive use), but rather he would write it down and refer to it when necessary.

Summary

When you first meet a particular equation, ask yourself whether it is

(*a*) a definition,
(*b*) a law
(*c*) an important *idea* which you will want to use time and time again, or
(*d*) applicable to one situation only.

Do not learn equations of type (*d*).

Mathematics

Significant Figures

No physical measurement is exact, and if a reported figure is to make an appropriate impression, then its uncertainty must also be given. Thus a mass might be quoted as $m = (1.25 \pm 0.02)$ kg. This implies that the best estimate of the third significant figure is 5, but that it is also recognized that during the measurement of this mass there was an experimental uncertainty of 0.02 kg, and that the true value of m could be as high as 1.27 kg or as low as 1.23 kg. On this basis, if we used the quoted value of m for calculation purposes, we would not be confident that the result was accurate to better than about 1 part in 60. When the results of experimental work are to be published, it is essential that the extent of the experimental uncertainty should be quoted explicitly if the results are to be of value to others.

A way of *implying* the precision and accuracy of a measurement is simply to quote a sensible number of significant figures. If we write $m = 1.25$ kg, then there are two implications. One is that 5 is our *best estimate* of the third significant figure. The other is that we have not ruled out the possibility that the third figure might be (say) 1. We are not, by writing three significant figures, claiming an uncertainty of ± 0.01 kg. When we use this value of m for calculation purposes, it is pointless to express further results to more than three significant figures (except in special circumstances). There exist extensive formal rules for analysing probable errors, and the technique of their application is an important weapon of the physicist's armoury. Nevertheless there is no point in subjecting every calculation in a book of this type to such an analysis.

Accordingly in this book we adopt the following procedure. With few exceptions the numerical *information* in the questions is given to *two* significant figures (to save time being spent on laborious calculation). You are recommended to work each stage of your calculation to about *three* significant figures, and then to present the final conclusion rounded back to *two* significant figures. It is recognized that on occasion this procedure is not fully justified, but it is felt that there is little to be gained from extreme rigour, and much time will be saved.

You should be on the look-out for problems in which the nature of the calculation reduces considerably the number of figures which should be quoted in the answer. Consider as an extreme example the calculation of the difference of two close quantities, as in

$$\Delta\lambda = \lambda_2 - \lambda_1 = 589.59 \text{ nm} - 589.00 \text{ nm}$$
$$\underline{\Delta l = 0.59 \text{ nm}}$$

Five-figure information has resulted in a quantity whose uncertainty is at least 1 part in 60.

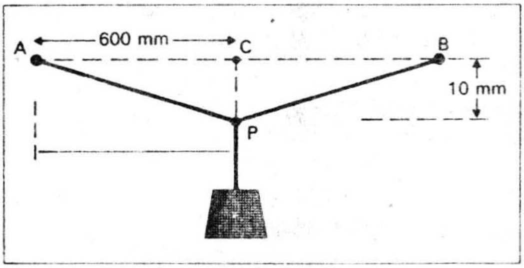
Approximations

Very often a result can be obtained far more easily, or with greater accuracy, by making simplifying approximations in the course of a calculation. No hard and fast rules can be given because every situation must be treated on its own merits, and experience enables one to judge what is justifiable in any particular problem.

The examples which follow are intended to guide you in your thinking. They show that the obvious way of dealing with a problem, involving perhaps a fair amount of four-figure arithmetic, sometimes gives less insight into the physical processes involved.

Example 1-1   A small length change

Refer to the diagram. A load has been added at P, *the mid-point of a wire* APB, *and has depressed it by* 10 mm. *What is the extension of half the wire?*



(*a*) Suppose the new length AP = l. Then by *Pythagoras*

$$l^2 = (600\text{ mm})^2 + (10\text{ mm})^2$$
$$= 36.01 \times 10^4\text{ mm}^2$$

In four-figure tables we find l = 600.1 mm.
The extension of half the wire is 0.1 mm.

(*b*) Let Δx be the extension of half the wire, so that AP = (600 mm + Δx). Then

$$(600\text{ mm} + \Delta x)^2 = (600\text{ mm})^2 + (10\text{ mm})^2$$
$$1200\,\Delta x\text{ mm} + \Delta x^2 = 100\text{ mm}^2$$

At once the 10 mm shows itself as the important quantity which determines the size of Δx. Since $\Delta x \sim 0.1$ mm, it follows that $\Delta x^2 \sim 0.01\text{ mm}^2$, and we may neglect 10^{-2} in comparison with 10^2.

$$\therefore \quad 1200\,\Delta x \approx 100\text{ mm},$$

giving $\quad \Delta x \approx 0.083\text{ mm}$

Notes

(*i*) Unintelligent use of the four-figure tables gave us an answer whose percentage error was ~20%.

(*ii*) The idea used in (*b*) is similar to that which enables us to write (for small α)

$$\sqrt{1 + 2\alpha} \approx 1 + \alpha$$

Example 1-2 The difference between two nearly equal quantities

The quantity $(1/R_1 + 1/R_2)$ *for a thin lens is* $1.00 \times 10^{-2}\text{ mm}^{-1}$. *Calculate the difference between its focal lengths for light of the* C (*red*) *and* F (*blue*) *Fraunhöfer lines, for which* $n_C = 1.514$ *and* $n_F = 1.524$.

(*a*) We can calculate the focal lengths as follows:

$$1/f_C = (n_C - 1)(1/R_1 + 1/R_2)$$
$$= (0.514) \times (10^{-2}\text{ mm}^{-1})$$
$$f_C = 19(5)\text{ mm}$$

Similarly $\quad f_F = 19(1)\text{ mm}$

Hence the difference between the focal lengths

$$\Delta f = f_C - f_F = 4\text{ mm}$$

Note that there is an uncertainty in Δf of about 2 mm, which is a percentage error of about 50%. Careless work could easily have led to answers of 2 mm or 6 mm.

(*b*) A better procedure is to calculate the *difference* between the focal lengths *directly*. Since

$1/f = k(n - 1)$, where $k = 1.00 \times 10^{-2}\text{ mm}^{-1}$

differentiating, we have

$$-\delta f/f^2 = k\delta n$$

so $\quad \Delta f \approx -kf^2\,\Delta n$

The minus sign tells us that f increases as n decreases (as is seen from the reciprocal). Using the *mean* value for f of 193 mm (above)

$$\Delta f \approx -(1.00 \times 10^{-2}\text{ mm}^{-1}) \times (3.72 \times 10^4\text{ mm}^2)(1.514 - 1.524)$$

from which $\quad \Delta f = 3.7\text{ mm}$

Careless work in method (*b*) would have led to an error less than the uncertainty implied by the fact that Δn is known to only one part in 10.

Some Useful Mathematics

A number of results are collected here for reference.

Algebra

Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}x^3 + \cdots$$

If $x \ll 1$ $\quad (1 + x)^n \approx 1 + nx$

$$(1 + x)^{-n} \approx 1 - nx$$

These useful approximations are valid when x^2 is negligible.

Quadratic Equations

The equation $\quad ax^2 + bx + c = 0$

has the solutions $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Logarithms

Common logarithms are those to base 10.

Thus if $\quad x = 10^y$

$$y = \lg x$$

where $\lg x$ implies $\log_{10} x$.

Natural logarithms are those to base e.

Thus if $\quad x = e^y$

then $\quad y = \ln x$

where $\ln x$ implies $\log_e x$. $\quad$ e is defined by

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.718\ldots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Natural and common logarithms are related by

$$\ln x = (2.303)\lg x, \text{ and}$$
$$\lg x = (0.434)\ln x$$

Trigonometry

The angle θ subtended by a length s of arc of a circle of radius r is given by

$$\theta[\text{rad}] = s/r$$

from which it follows that

$$1 \text{ rad} = 180°/\pi \approx 57.3°$$

If $x = \sin\theta$, then we say 'θ is the angle whose sine is x', and we write

$$\theta = \arcsin x$$

Useful Relationships

Signs: particular trig. functions are positive as follows:

first quadrant, all: second, sine: third, tangent: fourth, cosine.

$$\tan\theta = \sin\theta/\cos\theta$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$$

Sine law: (for any triangle)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine law: (for any triangle)

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Since
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots,$$
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$
and
$$\tan\theta = \theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

it follows that when $\theta \ll 1$ (say ~ 0.1 rad)

$$\sin\theta \approx \tan\theta \approx \theta, \text{ but } \cos\theta \approx 1$$

Calculus

Derivatives and Integrals

$y = f(x)$	$\frac{dy}{dx} = f'(x)$	$\int y\,dx$
$y = x^n$	nx^{n-1}	$\frac{x^{n+1}}{n+1} + C(n \neq -1)$
$y = x^{-1}$	$-x^{-2}$	$\ln x + C$
$y = \ln x$	x^{-1}	$x\ln x - x + C$
$y = \sin x$	$\cos x$	$-\cos x + C$
$y = \cos x$	$-\sin x$	$\sin x + C$

C is the constant of integration, whose value is determined by the limits of integration.

Average Value of a Function

The **mean** or **average value** $\langle y\rangle$ of $y = f(x)$ over the interval $x = a$ to $x = b$ is given by

$$\langle y\rangle = \frac{1}{b-a}\int_a^b y\,dx$$

Similarly
$$\langle y^2\rangle = \frac{1}{b-a}\int_a^b y^2\,dx$$

The **root mean square** (r.m.s.) value of y is given by

$$y_{\text{r.m.s.}} = \sqrt{\langle y^2\rangle}$$

For a periodic function the interval $(b - a)$ is understood to be taken over an integral number of periods or half periods.

*Differential Equations in Physics

Their Origin

Differential equations arise in many branches of physics, as the following examples show.

(*a*) ***Mechanics.*** The solution of many problems in dynamics depends upon the equation

$$\boldsymbol{F} = m\boldsymbol{a} = m(d^2\boldsymbol{x}/dt^2)$$

$\boldsymbol{F}$ may be a function of t, $\boldsymbol{x}$ and/or $\dot{x}$, and so we might have to integrate to find $\boldsymbol{x}$ as a function of t.

(*b*) ***Oscillations and waves.*** When $\boldsymbol{F}$ can be written in the form $-k\boldsymbol{x}$, the equation becomes

$$m(d^2\boldsymbol{x}/dt^2) + k\boldsymbol{x} = 0,$$

which is referred to as the **differential equation of s.h.m.** Its solution is discussed below. Wave motions have a similar important differential equation: in one dimension

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

(*c*) ***Heat flow.*** In one dimension the rate of heat flow through a material is described by the differential equation

$$dQ/dt = -\lambda A(d\theta/dx)$$

If, in a given situation, dQ/dt is constant, and both λ and A depend upon x, then we would have to integrate to find the variation of θ with x.

(*d*) ***D.C. circuits.*** The growth of current in a d.c. LR circuit is described by the equation

$$\mathscr{E} = L(dI/dt) + IR$$

The decay of charge on the plates of a capacitor in a d.c. CR circuit is described by the equation

$$IR + Q/C = 0$$
$$R(\mathrm{d}Q/\mathrm{d}t) + Q/C = 0$$

(*e*) ***A.C. circuits.*** The application of the law of energy conservation in an a.c. circuit leads to the differential equation

$$\mathscr{E}_0 \cos \omega t = L(\mathrm{d}I/\mathrm{d}t) + Q/C + IR$$

Since $I = \mathrm{d}Q/\mathrm{d}t$, we may write

$$\mathscr{E}_0 \cos \omega t = L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q,$$

a differential equation which can be solved to find Q (and I) in terms of t.

(*f*) ***Radioactive decay.*** The disintegration of a radioactive nucleus is an entirely *random* process. This means that $-\mathrm{d}N/\mathrm{d}t$, the rate of decay of a particular sample, is proportional to the (large) number N of active nuclei that it contains. We write

$$\mathrm{d}N/\mathrm{d}t = -\lambda N$$

If a radioisotope is being created in a nuclear reactor at the constant rate C, but at the same time the new nuclide disintegrates at the rate $-\lambda N$, then we can calculate the net rate of increase of nuclei from

$$\mathrm{d}N/\mathrm{d}t = C - \lambda N$$

The number of nuclei present after a given time interval must be found by solving this differential equation. (Note the formal similarity between

$$\mathrm{d}N/\mathrm{d}t + \lambda N - C = 0, \text{ and}$$
$$\mathrm{d}I/\mathrm{d}t + (R/L)I - (\mathscr{E}/L) = 0 \text{ (above)}$$

Similarities of this kind enable us to solve many differential equations by comparing them with others whose solutions are already known.)

The Solution of Differential Equations

There exists a considerable scheme for classifying differential equations as to their order and degree, and also a formal set of rules for their solution. Most of the differential equations which we meet can be solved by one of two methods.

(*a*) ***By separation of variables.*** If the variables can be separated onto the two sides of the equation, then the solution will be obtained by direct integration. Several examples are given below.

(*b*) ***By inspection.*** Because the equations that we deal with tend to treat a small number of distinct physical situations, previous experience and physical intuition will often enable us to *guess the form of the solution*. We can then test our proposed solution by substituting it into the original differential equation.

Example: the LC circuit. Experience teaches us that an equation of the form

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = -\left(\frac{1}{LC}\right)Q$$

describes an oscillatory situation, since it closely resembles

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\left(\frac{k}{m}\right)x$$

Suppose we write $(1/LC) = \omega^2$, and then suggest

$$Q = A \sin \omega t + B \cos \omega t$$

as our solution of

$$\mathrm{d}^2Q/\mathrm{d}t^2 = -\omega^2 Q$$

A and B are arbitrary constants. Differentiating twice, we have

(1) $\mathrm{d}Q/\mathrm{d}t = A\omega \cos \omega t - B\omega \sin \omega t$

(2) $\mathrm{d}^2Q/\mathrm{d}t^2 = -A\omega^2 \sin \omega t - B\omega^2 \cos \omega t$
$= -\omega^2 Q$

Since this is the original equation, we have verified that

$$Q = A \sin \omega t + B \cos \omega t$$

is a solution.

Constants

The above solution can also be expressed in the form

$$Q = C \sin(\omega t + \delta)$$

where C and δ are two further arbitrary constants related to A and B. To eliminate *two* such constants, and hence arrive back at the original equation, it was necessary to differentiate *twice*. This illustrates a general rule:

> *The **general solution** of an n th order equation (involving $\mathrm{d}^n y/\mathrm{d}x^n$) will have n constants.*

A **particular solution** is one in which the arbitrary constants have been given specified values. Thus to obtain the *particular* solution of a differential equation describing s.h.m. we need to have *two* separate items of information, or **boundary conditions**. For example we might be told that $\dot{x} = 0$ and $x = +a$ when $t = 0$, and this would enable us to write a particular solution involving no arbitrary constants.

Worked Examples

Example 1–3 Vertical motion under gravity

A body is projected vertically upwards from the Earth's surface at a speed u. What will be its speed v at a distance r