Variational and Quasivariational Inequalities

Applications to Free Boundary Problems

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Preface

In the last fifteen years, variational inequalities have gained importance in analysis, both from the theoretical and the practical points of view.

This book, some chapters of which made up the courses of functional analysis taught by C. Baiocchi at the University of Pavia in the academic years 1974–75 and 1975–76, is devoted to variational and quasivariational inequalities of elliptic type, and to the applications of this theory to the study of free-boundary problems. The book is far from being a complete treatment: subjects such as inequalities of evolution, numerical treatment of inequalities, and other types of applications of inequalities, are either completely missing or have been only barely mentioned. We have, however, tried, in the limited field which we have considered, to present an organic and self-sufficient treatment with a considerable bibliography.

The book is divided into three parts. The first two parts deal respectively with problems of variational and of quasivariational type; an understanding of these two parts requires a knowledge of calculus, the basic elements of measure theory and Lebesgue integration, and the elementary properties of Hilbert and Banach spaces. The third part is devoted to some collateral subjects which are outside the scope of the topics mentioned above but which are nevertheless necessary for the development of the theory (e.g. Green's formulae, seminorms, the maximum principle, . . .); sometimes, in parts I and II, we refer to this part III for specific results and notations.

For some of the topics, for example Sobolev spaces, we have not even tried to be self-sufficient, but in these cases we have provided the reader with precise references to enable him to enlarge and complete his knowledge of the subject.

We are grateful to the Istituto di Analisi Numerica del Consiglio Nazionale delle Ricerche (Pavia), and the Unione Matematica Italiana who made the publication of the Italian version of this book possible, and to our colleagues of the Istituto di Analisi Numerica del Consiglio Nazionale delle Ricerche and of the Istituto di Matematica dell'Università di Pavia for constructive criticism and correction of printing errors in the Italian version.

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January 1983

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PART I

Variational Problems

1

Introduction

1.1 EXAMPLE OF A VARIATIONAL PROBLEM. THE NOTION OF WELL-POSED PROBLEM

The obstacle problem

The following example, based on a problem of physics, allows us to prese a first problem of variational kind—the reason for this particular name will be seen later on. For simplicity we present a one-dimensional example, which will therefore lead to ordinary inequalities: substitution of the elastic string by an elastic membrane and the two-dimensional obstacle by a three-dimensional obstacle will give us an example in which partial derivatives are necessary.

Let us consider a body $A \subset \mathbb{R}^2$, which we shall call the *obstacle*, and two points P_1 and P_2 not belonging to A (see fig. 1.1); let us connect P_1 to P_2 by a weightless *elastic string* whose points cannot penetrate A: we are interested in studying the shape assumed by the string. With this aim we introduce a system of Cartesian axes $0xy$ with respect to which P_1 and P_2, respectively, have coordinates $(0, 0)$ and $(l, 0)$. Let us suppose that, with respect to this system of axes, the 'lower part' of the boundary of obstacle A (in the zone in which we are interested, i.e. in $[0, l]$) is a Cartesian curve of equation $y=\psi(x)$. Experience tells us that if $y=u(x)$ is the shape assumed by the string then

$$u(0)=u(l)=0 \tag{1.1}$$

since the string connects P_1 and P_2,

$$u(x) \leqslant \psi(x) \tag{1.2}$$

because the string does not penetrate the obstacle,

$$u''(x) \geqslant 0 \tag{1.3}$$

because the string being elastic and weightless must assume a convex shape, and

$$u(x) < \psi(x) \Rightarrow u''(x)=0 \tag{1.4}$$

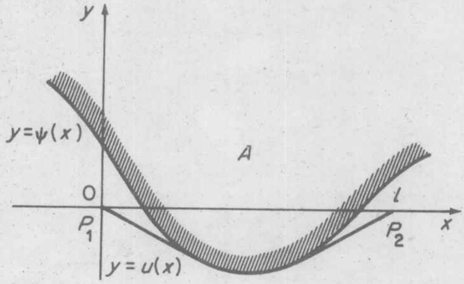

Figure 1.1

(i.e. where it does not touch the obstacle the string takes a linear shape) since the string tends to assume the shape with the minimum possible length (in particular were there no obstacle this would be l); (1.1), (1.2), (1.3), and (1.4) are equivalent to (1.1), (1.2), (1.3), and (1.5), with

$$[u(x)-\psi(x)]u''(x)=0, \tag{1.5}$$

because if (1.4) is true then either $u(x)-\psi(x)=0$ or $u''(x)=0$, and hence (1.5) is true, and if this is true then when $u(x)\neq\psi(x)$ we must have $u''(x)=0$, and (1.2) implies that $u(x)\neq\psi(x)$ only if $u(x)<\psi(x)$.

Expressions (1.1), (1.2), (1.3), and (1.5) constitute a mathematical formulation of the physical problem that we are dealing with, and the search for the function u that satisfies them constitutes a *variational problem*, with which, as we will see later, a *variational inequality* is associated. In the future we will refer to this problem as **problem 1.0**.

Spaces of data and spaces of unknowns

Problem 1.0, however, is not *complete* since we have not indicated the regularity which the unknown u must satisfy (i.e., we have not defined the function space to which u must belong) nor the regularity which we attribute to the data function ψ (i.e., the function space from which ψ is taken). It is clear that given ψ we can determine its regularity by simple inspection, but if we want to solve once and for all the 'problem of the string and the obstacle' we cannot fix ψ but only indicate some of its properties—i.e. indicate the space to which it belongs.

The matter of completeness of a problem—a concept not to be confused with that of well-posedness which we will speak of later—is anything but academic, as can be seen from the following considerations. Physics tells us that u must be continuous (otherwise there would be a break in the string) but not, for example, that it must be differentiable: we will have a non-differentiable shape of the string in the case of fig. 1.2, and if in this case we look for u in $C^0([0, l])$, as would be natural, (1.3) for example makes no sense—the shape of the string continues to be convex but the mathematical

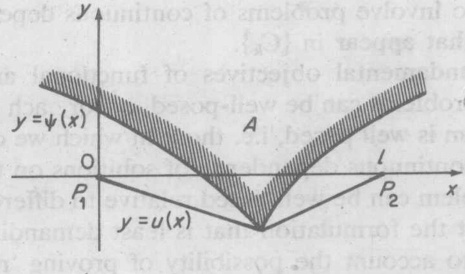

Figure 1.2

statement of this fact has to be changed. It is then most important to indicate the spaces in which the problem is posed, so that one knows how the written expressions are to be interpreted. Thus, for example, if ψ is given in $C^2([0, l])$ and we look for u in $C^2([0, l])$ then the expressions (1.1), (1.2), (1.3), and (1.5) can be interpreted in the usual sense; if, on the other hand, ψ is given in $C^1([0, l])$ and we look for u among the functions of $C^1([0, l])$ which have absolutely continuous derivative, then (1.1) and (1.2) can be still interpreted in the usual sense but (1.3) and (1.5) must be taken to be valid almost everywhere (a.e.) (we notice that if $u \in C^1([0, l])$ and $u' \in AC([0, l])$ then in fact u'' exists a.e.). The problems '*given* $\psi \in C^2([0, l])$ (or: $\psi \in C^1([0, l])$) *find* $u \in C^2([0, l])$ (or: $u \in \{u \in C^1([0, l]) : u' \in AC([0, l])\}$) *which satisfies* (1.1), (1.2), (1.3), *and* (1.5) *in the usual sense* (resp.: (1.1) *and* (1.2) *in the usual sense and* (1.3) *and* (1.5) *almost everywhere*)' are complete problems, even if in both cases the spaces chosen are not the most suitable ones for the treatment of the problem (which so formulated may not have a solution—on the other hand, the assumptions on ψ are too restrictive).

Well-posed problems

We observe, however, that in order to have a well formulated problem it is not enough to define the spaces where the expressions make sense. Thus we introduce the:

Notion of well-posed problem. The problem 'given the set $\{d_n \in \mathcal{D}_n\}_{n=1,\dots,N}$ of data d_n in topological spaces $\mathcal{D}_n$, find a set $\{u_m \in \mathcal{U}_m\}_{m=1,\dots,M}$ of unknowns u_m in topological spaces $\mathcal{U}_m$, which satisfy a set $\{C_k\}_{k=1,\dots,K}$ of conditions which connect, in the sense of the spaces $\mathcal{D}_n$ and $\mathcal{U}_m$, the unknowns to the data' is *well-posed* if for every set of data $\{d_n\}$ just one set $\{u_m\}$ of unknowns exists (which will be called solutions) that satisfy the conditions $\{C_k\}$, and further $\{u_m\}$ varies continuously with $\{d_n\}$ relative to the topologies of the spaces $\prod_{n=1}^{N} \mathcal{D}_n$ and $\prod_{m=1}^{M} \mathcal{U}_m$.

Note that in many problems (generally in nonlinear problems) the data $\{d_n\}$ can be 'hidden' in the conditions $\{C_k\}$; on the other hand, linear problems can also involve problems of continuous dependence of solutions on 'coefficients' that appear in $\{C_k\}$.

One of the fundamental objectives of functional analysis is to define spaces in which problems can be well-posed, or for each problem find spaces where the problem is well posed, i.e. those in which we can prove existence, uniqueness and continuous dependence of solutions on the data. It must be noted that a problem can be well posed relative to different spaces: we must in principle select the formulation that is least demanding on the data, but we must take into account the possibility of proving 'regularity results' (a concept which will be explained later).

1.2 SOME EXISTENCE RESULTS

We now consider some examples of problems for which we can ensure the existence of solutions, and at the same time mention some fundamental results which will be useful later; for the results which are given without proof see any textbook of functional analysis (e.g. Yosida, 1971 or Kolmogorov–Fomin, 1957).

Hahn–Banach theorem

The first problem that we will consider is the following:

PROBLEM 1.1. Let B be a Banach space and let $y \in B\backslash\{0\}$; find $L \in B'$ such that $L(y) = 1$ (as usual we denote by B' the dual of B).

That this problem has at least one solution is an elementary corollary of

THEOREM 1.1 (Hahn–Banach theorem). *If B is a normed space and V one of its linear varieties then each continuous linear functional on V can be extended as a continuous linear functional on B with the same norm.*

The Hahn–Banach theorem also solves (as is mentioned in its enunciation) the problem of the extension of continuous linear functionals: we will exploit it frequently.

Riesz theorem

Let us proceed to the problem of the representation of continuous linear functionals.

PROBLEM 1.2. Let H be a Hilbert space and $L \in H'$; find $u_L \in H$ such that $\forall v \in H\ L(v) = (u_L, v)_H$.

The question of the existence of a solution of this problem is solved by

THEOREM 1.2 (Riesz theorem). If H is a Hilbert space, for each $L \in H'$ there exists one and only one $u_L \in H$ such that $\forall v \in H\ L(v) = (u_L, v)_H$; further u_L satisfies $\|u_L\|_H = \|L\|_{H'}$.

It is interesting to note that this theorem not only gives us the existence and the uniqueness of the solution of problem 1.2 but also the continuous dependence on the data (since the application that to every L associates u_L is an isometry): the problem of the representation of continuous linear functionals is well-posed in each Hilbert space with the topology generated by the norm.

Banach's fixed-point theorem

The very important theorems of Hahn–Banach and of Riesz are results of 'vectorial-topological type': let us now consider a result which does not depend on a vectorial structure.

Let (S, d) be a complete non-empty metric space and $T: S \to S$ a *contraction operator*, i.e. an operator such that

$$\exists k < 1 \quad \forall x, y \in S \qquad d(T(x), T(y)) \leq kd(x, y). \tag{1.6}$$

With these data let us consider the

PROBLEM 1.3. Find $\bar{x} \in S$ such that $\bar{x} = T(\bar{x})$ (i.e. determine a fixed point for T).

This problem has a solution, as can be seen from

THEOREM 1.3 (Banach theorem for contractions). If (S, d) is a complete non-empty metric space and $T: S \to S$ is a contraction, there is one and only one fixed point for T.

Proof. Let us begin by showing the uniqueness, which we do by *reductio ad absurdum*. Let us suppose that T has two distinct fixed points $\bar{x}$ and $\bar{\bar{x}}$: we can write $\bar{x} = T(\bar{x})$ and $\bar{\bar{x}} = T(\bar{\bar{x}})$ and, since T is a contraction, $d(\bar{x}, \bar{\bar{x}}) = d(T(\bar{x}), T(\bar{\bar{x}})) \leq kd(\bar{x}, \bar{\bar{x}}) < d(\bar{x}, \bar{\bar{x}})$, which is absurd. To show existence let us consider the sequence defined by the recurrence relation

$$x_n \triangleq T(x_{n-1})(n = 1, 2, \ldots), \tag{1.7}$$

and show that no matter how we fix $x^* = x_0$ this sequence converges to a fixed point for T. Now, noting that

$$x_n = T(x_{n-1}) = T^2(x_{n-2}) = \ldots = T^n(x_0) \tag{1.8}$$

and that from (1.6), for each n,

$$\forall x, y \in S \qquad d(T^n(x), T^n(y)) \leq k^n d(x, y), \tag{1.9}$$

we can write successively, with $m \geq n$,

$$d(x_n, x_m) = d(T^n(x_0), T^m(x_0)) \leq k^n d(x_0, x_{m-n})$$
$$\leq k^n[d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_{m-n-1}, x_{m-n})]$$
$$\leq k^n d(x_0, x_1)[1 + k + \ldots + k^{m-n-1}] = d(x_0, x_1)\frac{k^n - k^m}{1-k}, \tag{1.10}$$

which shows that $\{x_n\}$ is a Cauchy sequence. Since (S, d) is complete, x_n converges: let $\bar{x}$ be its limit. To see that $\bar{x}$ is a fixed point for T is is enough to note that this operator is continuous (from (1.6)) and therefore $T(\bar{x}) = T(\lim_n x_n) = \lim_n T(x_n) = \lim_n x_{n+1} = \bar{x}$. ■

Note that the Banach theorem tells us that problem 1.3 is well-posed in non-empty complete metric spaces, because it demonstrates the existence and the uniqueness of the fixed point and, since there are no data, the problem of continuous dependence of the solution on the data does not arise (see, however, what we say immediately after the notion of well-posed problem).

It is interesting to note that the proof of the existence of a fixed point has a very important characteristic: it is of constructive type. In fact, we have shown the existence of the fixed point indicating a process by which it can be found. It is true that the solution, $\bar{x}$, is obtained through countable many operations, but we can approximate $\bar{x}$ as closely as we want with a finite number of operations and we can estimate the error which we make: so, if we take as an approximate solution $\bar{\bar{x}} = x_n$, the error $d(\bar{x}, \bar{\bar{x}})$ is not greater than $d(x_1, x_0)k^n/1-k$.

As a first application of the Banach theorem we note that from it follows the proof of the following result (Picard's theorem): '*given* Ω, *open set in* $\mathbb{R}^2$, $f: \Omega \to \mathbb{R}$ *continuous in the pair of variables and Lipschitz continuous with respect to y, and given* $(x_0, y_0) \in \Omega$, *the Cauchy problem* "$y' = f(x, y)$, $y(x_0) = y_0$" *is well posed locally*' (i.e., there exists $\varepsilon > 0$ such that in $]x_0 - \varepsilon, x_0 + \varepsilon[$ there exists a unique solution of the problem and this solution depends continuously on the data). In Chapter 3 we will see other applications of the Banach theorem.

In Chapter 9 we will present other fixed point theorems.

2

Minimization of Convex Functionals

2.1 FUNDAMENTAL THEOREM

The general problem of minimization

In this section we will look for the sufficient conditions such that the following problem of minimization has one (or one and only one) solution.

PROBLEM 2.1. Given a real vector space E, a function $f: E \to \mathbb{R}$ and a set $X \subset E$, find the minimum of f in X, i.e. find $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$.

The conditions we are looking for will naturally involve hypotheses on E, f, and X.

The search of 'good' hypothesis

In the particular case when $E = \mathbb{R}^n$, if f is continuous and X is compact, the theorem of Weierstrass ensures the existence of a minimum of f in X. In fact this theorem also ensures the existence of a maximum, which leads us to suspect that its assumptions are too strong; let us analyse them in order to obtain some useful information for our problem.

The function f is assumed to be continuous, and hence $\mathbb{R}^n$ must be taken to be a topological space. This presents the first problem: 'independently of whether the assumption *f is continuous* is necessary or not, will it be strictly necessary to exploit the topological structure of $\mathbb{R}^n$?'. The function $f_1: \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = |x|$ for $x \neq 0$ and $f(0) = 1$ (see fig. 2.1) enables us to give an answer in the affirmative: f_1 does not have a minimum in $\mathbb{R}$ (the greatest lower bound of $f_1(\mathbb{R})$ is 0, which is not achieved) and the characterization of functions such as f_1 requires a topological structure. We conclude, therefore, that we must provide E with a topology; it will be useful to have a structure of a vector space on E too, so that we require:

(H1) E is a topological vector space