The Finite Element Method in Partial Differential Equations

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Preface

There is no longer any need to sell the finite element method as a technique for solving partial differential equations. This is particularly so in the case of elliptic equations where at the moment it has taken over from the finite difference method. It is a good example of a topic which transcends many boundaries and its development has only been made possible by cooperation between engineers, mathematicians and numerical analysts. Because of the breadth of interests of its devotees it is easy to convince oneself that there is not a suitable text on the finite element method, a point of view which has led to a rapidly growing literature on the subject. The material in the present book is intended to bridge the gap between the well known works of Zienkiewicz (1971) and Strang and Fix (1973), which represent the finite element interests of engineers and mathematicians respectively. At no time do the present authors take sides in the long-standing controversy regarding the relative merits of finite difference and finite element methods. It is sufficiently gratifying to know that two such powerful techniques exist for the numerical solution of partial differential equations.

Most of the book is aimed at final-year undergraduate and first-year postgraduate students in mathematics and engineering. No specialized mathematical knowledge is required for understanding the material presented beyond what is normally taught in undergraduate courses on vector spaces and advanced calculus. An exception to this is Chapter 5, which can be omitted on a first reading of the book. Hilbert space and functional analytic concepts are introduced throughout the book mainly from the point of view of unifying material. Only a working knowledge of partial differential equations is assumed since anything beyond this would seriously limit the usefulness of the book. Since a variational principle rather than a partial differential equation is often our starting point, a chapter on variational principles is included with suitable references to more advanced works on the subject.

We hope that practical users of the finite element method will also find the book useful. For their benefit we have covered as many variants of the finite element method as possible, viz. Ritz, Galerkin, least squares and collocation, and in Chapter 4 we have given a large selection of possible basis functions to be used with any of the above methods. To balance this overcoverage of material in particular areas we have omitted eigenvalue problems. Our reason (or excuse) is that these are more than adequately covered in Chapter 6 of Strang and Fix (1973).

The list of references is restricted to those texts actually referred to in the book. For a more complete list of references see *A Bibliography for Finite Elements* by Whiteman (1975). Some recent texts and conference proceedings devoted mainly to finite element methods are listed in the references for the convenience of interested readers. These are Zienkiewicz (1971), Aziz (1972), Oden (1972), Strang and Fix (1973), Gram (1973), Lancaster (1973), Miller (1973, 1975), Whiteman (1973, 1976), Watson (1974, 1976), De Boor (1974), Oden, Zienkiewicz, Gallagher and Taylor (1974), and Prenter (1975).

Much of the material in this book has been presented in the form of lectures to Honours and M.Sc. mathematics students in the Universities of Dundee and Liverpool. Also at the invitation of the Institutt for Atomenergi the former author lectured on the material of Chapters 2 and 4 at the Nato Advanced Study Institute held in Kjeller, Norway in 1973, and the latter author lectured on the material of Chapters 3, 5 and 6 at the Technical University of Denmark during an invited stay there in 1973.

In the preparation of this book, the authors have benefited greatly from discussions with colleagues and former students. Special thanks are due to Bob Barnhill, Lothar Collatz, David Griffiths, Dirk Laurie, Jack Lambert, Peter Lancaster, Robin McLeod, Gil Strang, Gene Wachspress, Jim Watt and Olek Zienkiewicz. Final thanks are due to Ros Dudgeon and Doreen Manley for their expert typing of the manuscript.

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Chapter 1

Introduction

1.1 APPROXIMATION BY PIECEWISE POLYNOMIALS

Consider initially the problem of approximating a real-valued function $f(x)$ over a finite interval of the x-axis. A simple approach is to break up the interval into a number of non-overlapping subintervals and to interpolate linearly between the values of $f(x)$ at the end points of each subinterval (see Figure 1(a)). If there are n subintervals denoted by $[x_i, x_{i+1}]$ $(i = 0,1,2, \ldots, n-1)$, then the piecewise linear approximating function depends only on the function values f_i $(= f(x_i))$ at the nodal points x_i $(i = 0,1,2, \ldots, n)$. In a problem where $f(x)$ is given implicitly by an equation (differential, integral, functional, etc.), the values f_i are the unknown parameters of the problem. In the problem of interpolation, the values f_i are known in advance.

In the subinterval $[x_i, x_{i+1}]$, the appropriate part of the linear

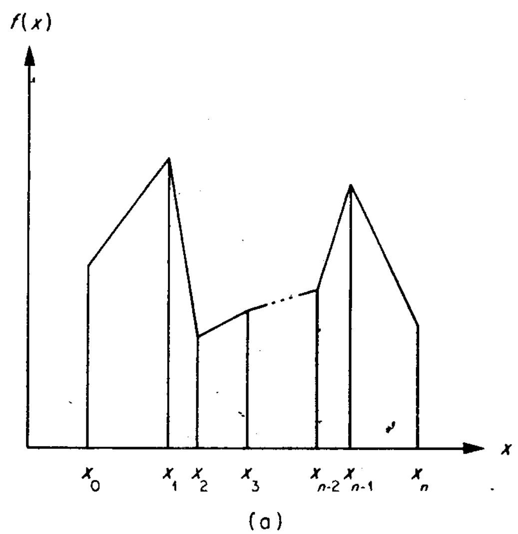


Figure 1a

approximating function is given by

$$p_1^{(i)}(x) = \alpha_i(x)f_i + \beta_{i+1}(x)f_{i+1} \qquad (x_i \leqslant x \leqslant x_{i+1}), \tag{1.1}$$

where

$$\alpha_i(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} \quad \text{and} \quad \beta_{i+1}(x) = \frac{x - x_i}{x_{i+1} - x_i} \; (i = 0,1,2, \ldots, n-1).$$

Hence the piecewise approximating function over the interval $x_0 \leqslant x \leqslant x_n$ is given by

$$p_1(x) = \sum_{i=0}^{n} \varphi_i(x) f_i, \tag{1.2}$$

where

$$\varphi_0(x) = \begin{cases} \dfrac{x_i - x}{x_i - x_0} & (x_0 \leqslant x \leqslant x_1) \\ 0 & (x_1 \leqslant x \leqslant x_n), \end{cases}$$

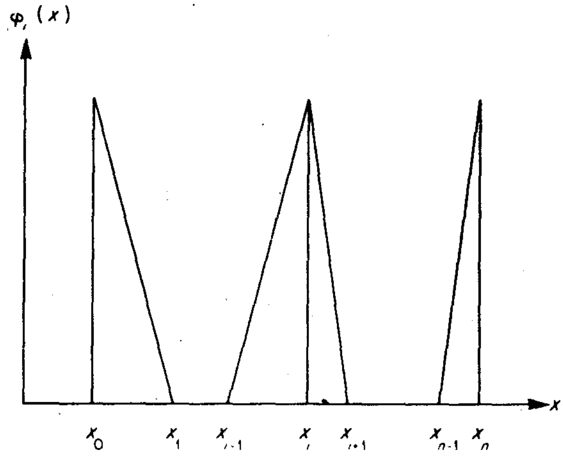
$$\varphi_i(x) = \begin{cases} 0 & (x_0 \leqslant x \leqslant x_{i-1}) \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & (x_{i-1} \leqslant x \leqslant x_i) \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & (x_i \leqslant x \leqslant x_{i+1}) \\ 0 & (x_{i+1} \leqslant x \leqslant x_n), \end{cases} \tag{1.3}$$

and

$$\varphi_n(x) = \begin{cases} 0 & (x_0 \leqslant x \leqslant x_{n-1}) \\ \dfrac{x - x_{n-1}}{x_n - x_{n-1}} & (x_{n-1} \leqslant x \leqslant x_n) \end{cases}$$

are pyramid functions illustrated in Figure 1(b). The pyramid functions given by (1.3) represent an elementary type of basis function. In particular the basis functions $\varphi_i(x)$ $(i = 1,2, \ldots, n-1)$ are identically zero except for the range $x_{i-1} \leqslant x \leqslant x_{i+1}$, and are said to have local support. Throughout this text, basis functions will be constructed of varying degrees of complexity but always with local support. A fundamental property of most basis functions is that they take the value unity at a particular nodal point and are zero at most of the other nodal points.

In general, the first derivatives of the piecewise approximating polynomial $p_1(x)$ given by (1.1) are not the same as $f(x)$ even at the nodes. Consequently we now look at the possibility of constructing a piecewise



(b)

Figure 1b

approximating function which has the same values of function and first derivative as $f(x)$ at the nodal points x_i $(i = 0,1,2, \ldots, n)$. In mathematical terms, we have to construct a piecewise cubic polynomial $p_3(x)$ such that

$$D^k f(x_i) = D^k p_3(x_i) \qquad (k = 0,1;\ i = 0,1,2, \ldots, n),$$

where $D = d/dx$. In the subinterval $[x_i, x_{i+1}]$, the appropriate part of the approximating cubic polynomials is given by

$$p_3^{(i)}(x) = \alpha_i(x) f_i + \beta_{i+1}(x) f_{i+1} + \gamma_i(x) f_i' + \delta_{i+1}(x) f_{i+1}', \tag{1.4}$$

where

$$\alpha_i(x) = \frac{(x_{i+1} - x)^2 [(x_{i+1} - x_i) + 2(x - x_i)]}{(x_{i+1} - x_i)^3},$$

$$\beta_{i+1}(x) = \frac{(x - x_i)^2 [(x_{i+1} - x_i) + 2(x_{i+1} - x)]}{(x_{i+1} - x_i)^3},$$

$$\gamma_i(x) = \frac{(x - x_i)(x_{i+1} - x)^2}{(x_{i+1} - x_i)^3} \tag{1.5}$$

and

$$\delta_{i+1}(x) = \frac{(x - x_i)^2 (x - x_{i+1})}{(x_{i+1} - x_i)^3}$$

$(i = 0,1,2, \ldots, n-1)$ and where $'$ denotes differentiation with respect to x. The piecewise approximating function over the interval

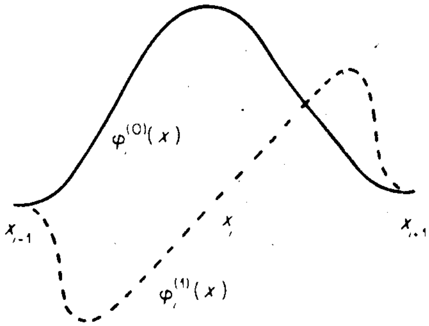


Figure 2

$x_0 \leqslant x \leqslant x_n$ is given by

$$p_3(x) = \sum_{i=0}^{n} [\varphi_i^{(0)}(x)f_i + \varphi_i^{(1)}(x)f_i'], \tag{1.6}$$

where the cubic polynomials $\varphi_i^{(0)}(x)$, $\varphi_i^{(1)}(x)$ ($i = 0,1,2, \ldots, n$) are easily obtained from (1.5). The basis functions $\varphi_i^{(0)}(x)$ and $\varphi_i^{(1)}(x)$ ($i = 1,2, \ldots, n-1$) are illustrated in Figure 2.

The basis functions in (1.2) and (1.6) arise from particular cases of *piecewise Hermite interpolation* (or approximation) for a partitioned interval. In more general terms, let $\Pi: a = x_0 < x_1 < \ldots < x_n = b$ denote any partition of the interval $R = [a, b]$ on the x-axis. For a positive integer m, and a partition Π of the interval, let $H = H^{(m)}(\Pi, R)$ be the set of all real-valued piecewise polynomial functions $w(x)$ defined on R such that $w(x) \in C^{m-1}(R)$ and $w(x)$ is a polynomial of degree $2m-1$ on each subinterval $[x_i, x_{i+1}]$ of R. Given any real-valued function $f(x) \in C^{m-1}(R)$, then its unique piecewise Hermite interpolate is the element $p_{2m-1}(x) \in H$ such that

$$D^k f(x_i) = D^k p_{2m-1}(x_i) \quad \begin{cases} (0 \leqslant k \leqslant m-1) \\ (0 \leqslant i \leqslant n). \end{cases} \tag{1.7}$$

The particular cases $m = 1,2$ have already been dealt with and produce the basis functions given by (1.2) and (1.6) respectively. Error estimates for piecewise Hermite interpolates are given by Birkhoff, Schultz and Varga (1968).

In problems where only $f(x)$ has to be determined, it is often undesirable to introduce derivatives of $f(x)$ as additional parameters and so cause a considerable increase in the order of the system of equations to be solved. Consequently a very desirable property in piecewise functions might be continuity of derivatives at the points at which pieces of the polynomials meet without introducing the values of the derivatives as additional unknown parameters. The simplest example of this approach is the fitting in each subinterval $[x_i, x_{i+1}]$ $(i = 0,1,2, \ldots, n-1)$ of a quadratic such that the first derivatives are continuous at each internal nodal point x_i $(i = 1,2, \ldots, n-1)$. A convenient form for this piecewise approximate, known as the quadratic *spline* is

$$S_2^{(i)}(x) = f_i + \frac{f_{i+1} - f_i}{x_{i+1} - x_i}(x - x_i) + c_i(x - x_i)(x - x_{i+1})$$

$$(i = 0,1,2, \ldots, n-1) \tag{1.8}$$

and the continuity of the first derivatives leads to

$$c_i + c_{i-1} = \frac{1}{h^2}(f_{i+1} - 2f_i + f_{i-1}) \quad (i = 1,2, \ldots, n-1), \tag{1.9}$$

where the nodal points in the interval have been taken equally spaced, distance h apart. Equation (1.9) gives $(n-1)$ linear relations between the n unknown coefficients c_i $(i = 0,1,2, \ldots, n-1)$, and so in the case of the quadratic spline there is one free coefficient. Since $S_2^{(i)''}(x) = 2c_i$ $(i = 0,1,2, \ldots, n-1)$, a knowledge of the second derivative at any point completely solves the problem.

The most popular form of the spline is the cubic spline. Here, given the values of f_i $(i = 0,1,2, \ldots, n)$, we fit cubic polynomials between successive pairs of nodal points and require continuity of both first and second derivatives at all internal nodal points. In this case, if $S_3^{(i)}(x)$ $(i = 0,1,2, \ldots, n-1)$ is the required cubic spline, then $S_6^{(i)''}(x)$ must be linear in $[x_i, x_{i+1}]$, and so

$$S_3^{(i)''}(x) = c_i \frac{x_{i+1} - x}{x_{i+1} - x_i} + c_{i+1} \frac{x - x_i}{x_{i+1} - x_i} \quad (i = 0,1,2, \ldots, n-1)$$

where c_i, c_{i+1} are the values of the second derivatives at x_i, x_{i+1} respectively. This form ensures continuity of the second derivative at the internal nodal points. After applying the further conditions

$$\left.\begin{aligned} S_3^{(i)}(x_i) &= f_i \\ S_3^{(i)}(x_{i+1}) &= f_{i+1} \end{aligned}\right\} \quad (i = 0,1,2, \ldots, n-1)$$

and

$$S_3^{(i-1)'}(x_i) = S_3^{(i)'}(x_i) \quad (i = 1,2, \ldots, n-1),$$

the cubic spline is obtained in the form

$$S_3^{(i)}(x) = \frac{c_i}{6h}(x_{i+1} - x)^3 + \frac{c_{i+1}}{6h}(x - x_i)^3 + \left(\frac{f_i}{h} - \frac{hc_i}{6}\right)(x_{i+1} - x)$$

$$+ \left(\frac{f_{i+1}}{h} - \frac{hc_{i+1}}{6}\right)(x - x_i) \quad (i = 0,1,2, \ldots, n-1), \tag{1.10}$$

where the nodal points are equally spaced, and the $(n + 1)$ coefficients c_i $(i = 0,1,2, \ldots, n)$ are given by the $(n - 1)$ linear relations

$$c_{i+1} + 4c_i + c_{i-1} = \frac{6}{h^2}(f_{i+1} - 2f_i + f_{i-1}) \quad (i = 1,2, \ldots, n-1). \tag{1.11}$$

The two free parameters in the case of the cubic spline are often removed by taking $c_0 = c_n = 0$, and hence the other parameters are uniquely defined by (1.11).

A more natural form of the cubic spline for equally spaced nodal points in the interval $I = [0, b]$ is

$$S_I\left(\frac{x}{h}\right) = \alpha_0 + \alpha_1\left(\frac{x}{h}\right) + \alpha_2\left(\frac{x}{h}\right)^2 + \alpha_3\left(\frac{x}{h}\right)^3 + \sum_{s=1}^{n-1}\beta_s\left(\frac{x}{h} - s\right)_+^3, \tag{1.12}$$

where

$$\left(\frac{x}{h} - s\right)_+ = \begin{cases} 0 & \left(\frac{x}{h} \leqslant s\right) \\ \frac{x}{h} - s & \left(\frac{x}{h} > s\right). \end{cases}$$

It can easily be verified that $S_I(x/h)$ and all its derivatives except the third are continuous at the $(n - 1)$ internal nodal points x_i $(i = 1,2, \ldots, n-1)$ for all values of the $(n + 3)$ coefficients $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_s$ $(s = 1,2, \ldots, n-1)$. Applying the condition

$$S_I\left(\frac{x_i}{h}\right) = f_i \quad (i = 0,1,2, \ldots, n), \tag{1.13}$$

there are $(n + 1)$ linear relations for the $(n + 3)$ coefficients, and so there are two free parameters. The system of linear equations reduces to the form given in Exercise 4. If the cubic spline (1.12) involving two arbitrary parameters is now expressed in the form

$$S_I\left(\frac{x}{h}\right) = \sum_{i=0}^{n} f_i C_i\left(\frac{x}{h}\right), \tag{1.14}$$

where $C_i(x_i/h) = 1$, $C_i(x_j/h) = 0$ $(j \neq 1)$, $(i,j = 0,1,2, \ldots, n)$, the cardinal splines $C_i(x/h)$ obtained do not have local support and are not practical basis functions.

Cubic spline functions with local support of $4h$ were introduced as suitable basis functions by Schoenberg (1969). At nodal points $x = ih$ $(i = 2,3, \ldots, n-2)$, away from the ends of the interval, these take the form

$$B_i\left(\frac{x}{h}\right) = \frac{1}{4}\left[\left\{\frac{x}{h} - (i-2)\right\}_+^3 - 4\left\{\frac{x}{h} - (i-1)\right\}_+^3 + 6\left\{\frac{x}{h} - i\right\}_+^3 - 4\left\{\frac{x}{h} - (i+1)\right\}_+^3 + \left\{\frac{x}{h} - (i+2)\right\}_+^3\right]. \tag{1.15}$$

These functions and their first two derivatives are zero for $-\infty < x/h \leqslant i-2$ and $i+2 \leqslant x/h < +\infty$. Also

$$B_i(i-1) = B_i(i+1) = \frac{1}{4},\ B_i(i) = 1 \qquad (i = 2,3, \ldots, n-2).$$

The remaining functions $B_0(x/h)$, $B_1(x/h)$, $B_{n-1}(x/h)$, $B_n(x/h)$ require special consideration. By setting

$$S_I\left(\frac{x}{h}\right) = \sum_{i=0}^{n} \gamma_i B_i\left(\frac{x}{h}\right), \tag{1.16}$$

and matching the right-hand sides of (1.14) and (1.16), a tridiagonal system of linear equations is obtained which enables the coefficients γ_i $(i = 0,1,2, \ldots, n)$ in (1.16) to be obtained. The majority of the equations in this system are given by

$$\gamma_i + \frac{1}{4}(\gamma_{i+1} + \gamma_{i-1}) = f_i \quad (i = 2,3, \ldots, n-2).$$

Bivariate approximation

We now consider the problem of approximating a real-valued function of two variables by piecewise continuous functions over a bounded region R with boundary ∂R. The region is divided up into a number of elements and the particular shapes of region considered at this stage are (1) rectangular and (2) polygonal.

(1) *Rectangular region*. The sides of this region are parallel to the x- and y-axes, and the region is subdivided into similar rectangular elements by drawing lines parallel to the axes. Let the rectangular region be $[x_0, x_m] \times [y_0, y_n]$ and a typical element be $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, where $x_{i+1} - x_i = h_1$ and $y_{j+1} - y_j = h_2$ $(0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant n-1)$ (see Figure 3). The bilinear form which interpolates $f(x,y)$ over the rectangular element is

$$p_1^{(i,j)}(x,y) = \alpha_{i,j}(x,y)f_{i,j} + \beta_{i+1,j}(x,y)f_{i+1,j} + \gamma_{i,j+1}(x,y)f_{i,j+1} + \delta_{i+1,j+1}(x,y)f_{i+1,j+1}, \tag{1.17}$$

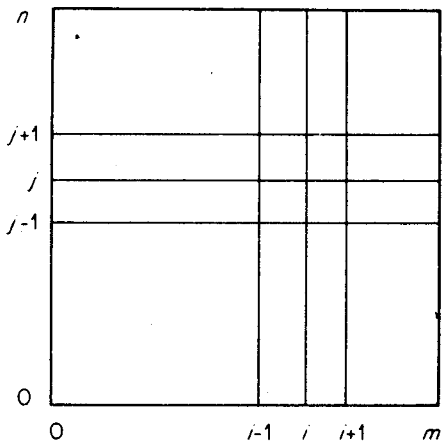


Figure 3

where

$$\alpha_{i,j}(x,y) = \frac{1}{h_1 h_2}(x_{i+1} - x)(y_{j+1} - y),$$

$$\beta_{i+1,j}(x,y) = \frac{1}{h_1 h_2}(x - x_i)(y_{j+1} - y),$$

$$\gamma_{i,j+1}(x,y) = \frac{1}{h_1 h_2}(x_{i+1} - x)(y - y_j)$$

and

$$\delta_{i+1,j+1}(x,y) = \frac{1}{h_1 h_2}(x - x_i)(y - y_j)$$

$(0 \leqslant i \leqslant m-1; 0 \leqslant j \leqslant n-1)$. The piecewise approximating function over the region $[x_0, x_m] \times [y_0, y_n]$ is given by

$$p_1(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \varphi_{i,j}(x,y) f_{i,j}. \tag{1.18}$$

The basis functions $\varphi_{i,j}(x,y)$ $(1 \leqslant i \leqslant m-1; 1 \leqslant j \leqslant n-1)$ are identically zero except for the rectangular region $[x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$, and so have local support (see Exercise 6 and Figure 3).

The case just considered is the simplest example of piecewise *bivariate* Hermite interpolation (or approximation) over a rectangular region subdivided into rectangular elements. In more general terms, for any positive integer l, and any subdivision of the rectangle R into rectangular elements, let $H = H^{(l)}(R)$ be the collection of all real-valued

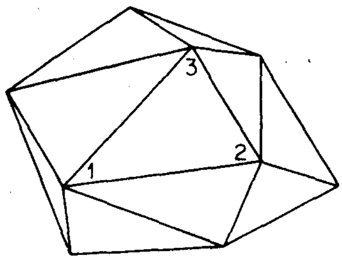


Figure 4

piecewise polynomials $g(x,y)$ defined on R such that $g(x,y) \in C^{l-1,\, l-1}(R)$ and $g(x,y)$ is a polynomial of degree $2l-1$ in each variable x and y on each rectangular element $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ $(0 \leqslant i \leqslant m-1;$ $0 \leqslant j \leqslant n-1)$ of R. Given any real-valued function $f(x,y) \in C^{l-1,\, l-1}(R)$, then its unique piecewise Hermite interpolant is the element $p_{2l-1}(x,y) \in H$ such that

$$D^{(p,q)}f(x_i,y_j) = D^{(p,q)}p_{2l-1}(x_i,y_j)$$

for all $0 \leqslant p,q \leqslant l-1$, $0 \leqslant i \leqslant m-1$, $0 \leqslant j \leqslant n-1$. The particular case $l = 1$ has already been dealt with and leads to bilinear basis functions of the type shown in Exercise 6. The case $l = 2$ is covered in Exercise 7. The interested reader is again referred to Birkhoff, Schultz and Varga (1968) for error estimates of bivariate Hermite interpolation.

(2) *Polygonal region*. This can either be a region in its own right or an approximation to a region of any shape. The polygon is subdivided in an arbitrary manner into triangular elements. In a typical triangular element with vertices (x_i,y_i) $(i = 1,2,3)$ (see Figure 4), the linear form which interpolates $f(x,y)$ over the triangular element is

$$p_1(x,y) = \sum_{i=1}^{3} \alpha_i(x,y) f_i, \tag{1.19}$$

where $f_i = f(x_i,y_i)$ $(i = 1,2,3)$. The coefficients $\alpha_i(x,y)$ $(i = 1,2,3)$ are given by

$$\alpha_1(x,y) = \frac{1}{C_{123}}(\tau_{23} + \eta_{23}x - \xi_{23}y),$$

$$\alpha_2(x,y) = \frac{1}{C_{123}}(\tau_{31} + \eta_{31}x - \xi_{31}y) \tag{1.20}$$

and

$$\alpha_3(x,y) = \frac{1}{C_{123}}(\tau_{12} + \eta_{12}x - \xi_{12}y),$$

where $|C_{123}|$ is twice the area of the triangle, and

$$\tau_{ij} = x_i y_j - x_j y_i,$$
$$\xi_{ij} = x_i - x_j \quad (i,j = 1,2,3)$$

and,

$$\eta_{ij} = y_i - y_j.$$

The functions given by (1.20) are of course only parts of the complete basis functions associated with vertices of a triangular network. The complete basis function with respect to any vertex is obtained by summing the appropriate parts associated with the triangles adjacent to the vertex. For example, the vertex 1 in Figure 4 has five adjacent triangles and so the basis function associated with this vertex has five parts. The complete basis function is known as a pyramid function.

Exercise 1 Show that the cubic polynomial $p_3(x)$ which takes the values

$$p_3(0) = f_0, p_3(1) = f_1, p_3'(0) = f_0', p_3'(1) = f_1',$$

is given by

$$p_3(x) = (1-x)^2(1+2x)f_0 + x(1-x)^2 f_0' + x^2(3-2x)f_1 + x^2(x-1)f_1'.$$

Exercise 2 Use the result of Exercise 1 to obtain the coefficients in equation (1.5), and hence obtain the basis functions in equation (1.6).

Exercise 3 Using the method outlined in the text, obtain the equation of the cubic spline in the form (1.10), where the coefficients are given by (1.11).

Exercise 4 Applying the condition (1.13) to the spline given by (1.12), show that the system of equations for the coefficients in (1.12) reduces to

$$\beta_{i-1} + 4\beta_i + \beta_{i+1} = \delta^4 f_i \quad (i = 2,3,\ldots,n-2),$$
$$6\alpha_3 + 5\beta_1 + \beta_2 = \delta^3 f_{3/2},$$
$$2\alpha_2 + 6\alpha_3 + \beta_2 = \delta^2 f_1,$$
$$\alpha_1 + \alpha_2 + \alpha_3 = \delta f_{1/2},$$
$$\alpha_0 = f_0,$$

where δ is the usual central difference operator.

Exercise 5 Solve the set of equations in Exercise 4 for $\alpha_1 = \alpha_2 = 0$ and $n = 4$. Show that in this case, using (1.14), that

$$C_2\left(\frac{x}{h}\right) = \left(\frac{x}{h}-1\right)_+^3 - 8\left(\frac{x}{h}-2\right)_+^3 + 37\left(\frac{x}{h}-3\right)_+^3.$$