ELECTROMAGNETIC THEORY

BY

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PREFACE

The pattern set nearly 70 years ago by Maxwell's *Treatise on Electricity and Magnetism* has had a dominant influence on almost every subsequent English and American text, persisting to the present day. The *Treatise* was undertaken with the intention of presenting a connected account of the entire known body of electric and magnetic phenomena from the single point of view of Faraday. Thus it contained little or no mention of the hypotheses put forward on the Continent in earlier years by Riemann, Weber, Kirchhoff, Helmholtz, and others. It is by no means clear that the complete abandonment of these older theories was fortunate for the later development of physics. So far as the purpose of the *Treatise* was to disseminate the ideas of Faraday, it was undoubtedly fulfilled; as an exposition of the author's own contributions, it proved less successful. By and large, the theories and doctrines peculiar to Maxwell—the concept of displacement current, the identity of light and electromagnetic vibrations—appeared there in scarcely greater completeness and perhaps in a less attractive form than in the original memoirs. We find that all of the first volume and a large part of the second deal with the stationary state. In fact only a dozen pages are devoted to the general equations of the electromagnetic field, 18 to the propagation of plane waves and the electromagnetic theory of light, and a score more to magnetooptics, all out of a total of 1,000. The mathematical completeness of potential theory and the practical utility of circuit theory have influenced English and American writers in very nearly the same proportion since that day. Only the original and solitary genius of Heaviside succeeded in breaking away from this course.

For an exploration of the fundamental content of Maxwell's equations one must turn again to the Continent. There the work of Hertz, Poincaré, Lorentz, Abraham, and Sommerfeld, together with their associates and successors, has led to a vastly deeper understanding of physical phenomena and to industrial developments of tremendous proportions.

The present volume attempts a more adequate treatment of variable electromagnetic fields and the theory of wave propagation. Some attention is given to the stationary state, but for the purpose of introducing fundamental concepts under simple conditions, and always with a view to later application in the general case. The reader must possess a general knowledge of electricity and magnetism such as may be acquired from an elementary course based on the experimental laws of Coulomb, Ampère, and Faraday, followed by an intermediate course dealing with the more general properties of circuits, with thermionic and electronic devices, and with the elements of electromagnetic machinery, terminating in a formulation of Maxwell's equations. This book takes up at that point. The first chapter contains a general statement of the equations governing fields and potentials, a review of the theory of units, reference material on curvilinear coordinate systems and the elements of tensor analysis, concluding with a formulation of the field equations in a space-time continuum. The second chapter is also general in character, and much of it may be omitted on a first reading. Here one will find a discussion of fundamental field properties that may be deduced without reference to particular coordinate systems. A dimensional analysis of Maxwell's equations leads to basic definitions of the vectors **E** and **B**, and an investigation of the energy relations results in expressions for the mechanical force exerted on elements of charge, current, and neutral matter. In this way a direct connection is established between observable forces and the vectors employed to describe the structure of a field.

In Chaps. III and IV stationary fields are treated as particular cases of the dynamic field equations. The subject of wave propagation is taken up first in Chap. V, which deals with homogeneous plane waves. Particular attention is given to the methods of harmonic analysis, and the problem of dispersion is considered in some detail. Chapters VI and VII treat the propagation of cylindrical and spherical waves in unbounded spaces. A necessary amount of auxiliary material on Bessel functions and spherical harmonics is provided, and consideration is given to vector solutions of the wave equation. The relation of the field to its source, the general theory of radiation, and the outlines of the Kirchhoff-Huygens diffraction theory are discussed in Chap. VIII.

Finally, in Chap. IX, we investigate the effect of plane, cylindrical, and spherical surfaces on the propagation of electromagnetic fields. This chapter illustrates, in fact, the application of the general theory established earlier to problems of practical interest. The reader will find here the more important laws of physical optics, the basic theory governing the propagation of waves along cylindrical conductors, a discussion of cavity oscillations, and an outline of the theory of wave propagation over the earth's surface.

It is regrettable that numerical solutions of special examples could not be given more frequently and in greater detail. Unfortunately the demands on space in a book covering such a broad field made this impractical. The primary objective of the book is a sound exposition of electromagnetic theory, and examples have been chosen with a view to illustrating its principles. No pretense is made of an exhaustive treatment of antenna design, transmission-line characteristics, or similar topics of engineering importance. It is the author's hope that the present volume will provide the fundamental background necessary for a critical appreciation of original contributions in special fields and satisfy the needs of those who are unwilling to accept engineering formulas without knowledge of their origin and limitations.

Each chapter, with the exception of the first two, is followed by a set of problems. There is only one satisfactory way to study a theory, and that is by application to specific examples. The problems have been chosen with this in mind, but they cover also many topics which it was necessary to eliminate from the text. This is particularly true of the later chapters. Answers or references are provided in most cases.

This book deals solely with large-scale phenomena. It is a sore temptation to extend the discussion to that fruitful field which Frenkel terms the "quasi-microscopic state," and to deal with the many beautiful results of the classical electron theory of matter. In the light of contemporary developments, anyone attempting such a program must soon be overcome with misgivings. Although many laws of classical electrodynamics apply directly to submicroscopic domains, one has no basis of selection. The author is firmly convinced that the transition must be made from quantum electrodynamics toward classical theory, rather than in the reverse direction. Whatever form the equations of quantum electrodynamics ultimately assume, their statistical average over large numbers of atoms must lead to Maxwell's equations.

The m.k.s. system of units has been employed exclusively. There is still the feeling among many physicists that this system is being forced upon them by a subversive group of engineers. Perhaps it is, although it was Maxwell himself who first had the idea. At all events, it is a good system, easily learned, and one that avoids endless confusion in practical applications. At the moment there appears to be no doubt of its universal adoption in the near future. Help for the tories among us who hold to the Gaussian system is offered on page 241.

In contrast to the stand taken on the m.k.s. system, the author has no very strong convictions on the matter of rationalized units. Rationalized units have been employed because Maxwell's equations are taken as the starting point rather than Coulomb's law, and it seems reasonable to make the point of departure as simple as possible. As a result of this choice all equations dealing with energy or wave propagation are free from the factor 4π. Such relations are becoming of far greater practical importance than those expressing the potentials and field vectors in terms of their sources.

The use of the time factor $e^{-i\omega t}$ instead of $e^{+i\omega t}$ is another point of mild controversy. This has been done because the time factor is invariably discarded, and it is somewhat more convenient to retain the positive exponent e^{+ikR} for a positive traveling wave. To reconcile any formula with its engineering counterpart, one need only replace $-i$ by $+j$.

The author has drawn upon many sources for his material and is indebted to his colleagues in both the departments of physics and of electrical engineering at the Massachusetts Institute of Technology. Thanks are expressed particularly to Professor M. F. Gardner whose advice on the practical aspects of Laplace transform theory proved invaluable, and to Dr. S. Silver who read with great care a part of the manuscript. In conclusion the author takes this occasion to express his sincere gratitude to Catherine N. Stratton for her constant encouragement during the preparation of the manuscript and untiring aid in the revision of proof.

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ELECTROMAGNETIC THEORY

CHAPTER I

THE FIELD EQUATIONS

A vast wealth of experimental evidence accumulated over the past century leads one to believe that large-scale electromagnetic phenomena are governed by Maxwell's equations. Coulomb's determination of the law of force between charges, the researches of Ampère on the interaction of current elements, and the observations of Faraday on variable fields can be woven into a plausible argument to support this view. The historical approach is recommended to the beginner, for it is the simplest and will afford him the most immediate satisfaction. In the present volume, however, we shall suppose the reader to have completed such a preliminary survey and shall credit him with a general knowledge of the experimental facts and their theoretical interpretation. Electromagnetic theory, according to the standpoint adopted in this book, is the theory of Maxwell's equations. Consequently, we shall postulate these equations at the outset and proceed to deduce the structure and properties of the field together with its relation to the source. No single experiment constitutes proof of a theory. The true test of our initial assumptions will appear in the persistent, uniform correspondence of deduction with observation.

In this first chapter we shall be occupied with the rather dry business of formulating equations and preparing the way for our investigation.

MAXWELL'S EQUATIONS

1.1. The Field Vectors.—By an electromagnetic field let us understand the domain of the four vectors **E** and **B**, **D** and **H**. These vectors are assumed to be finite throughout the entire field, and at all ordinary points to be continuous functions of position and time, with continuous derivatives. Discontinuities in the field vectors or their derivatives may occur, however, on surfaces which mark an abrupt change in the physical properties of the medium. According to the traditional usage, **E** and **H** are known as the intensities respectively of the electric and magnetic field, **D** is called the electric displacement and **B**, the magnetic induction. Eventually the field vectors must be defined in terms of the experiments by which they can be measured. Until these experiments are formulated, there is no reason to consider one vector more fundamental than another, and we shall apply the word intensity to mean indiscriminately the strength or magnitude of any of the four vectors at a point in space and time.

The source of an electromagnetic field is a distribution of electric charge and current. Since we are concerned only with its macroscopic effects, it may be assumed that this distribution is continuous rather than discrete, and specified as a function of space and time by the density of charge ρ, and by the vector current density $\mathbf{J}$.

We shall now *postulate* that at every ordinary point in space the field vectors are subject to the Maxwell equations:

(1) $$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0,$$

(2) $$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}.$$

By an ordinary point we shall mean one in whose neighborhood the physical properties of the medium are continuous. It has been noted that the transition of the field vectors and their derivatives across a surface bounding a material body may be discontinuous; such surfaces must, therefore, be excluded until the nature of these discontinuities can be investigated.

1.2. Charge and Current.—Although the corpuscular nature of electricity is well established, the size of the elementary quantum of charge is too minute to be taken into account as a distinct entity in a strictly macroscopic theory. Obviously the frontier that marks off the domain of large-scale phenomena from those which are microscopic is an arbitrary one. To be sure, a macroscopic element of volume must contain an enormous number of atoms; but that condition alone is an insufficient criterion, for many crystals, including the metals, exhibit frequently a microscopic "grain" or "mosaic" structure which will be excluded from our investigation. We are probably well on the safe side in imposing a limit of one-tenth of a millimeter as the smallest admissible element of length. There are many experiments, such as the scattering of light by particles no larger than 10^{-3} mm. in diameter, which indicate that the macroscopic theory may be pushed well beyond the limit suggested. Nonetheless, we are encroaching here on the proper domain of quantum theory, and it is the quantum theory which must eventually determine the validity of our assumptions in microscopic regions.

Let us suppose that the charge contained within a volume element Δv is Δq. The *charge density* at any point within Δv will be defined by the relation

(3) $$\Delta q = \rho\, \Delta v.$$

Thus by the charge density at a point we mean the average charge per unit volume in the neighborhood of that point. In a strict sense (3) does not define a continuous function of position, for Δv cannot approach zero without limit. Nonetheless we shall assume that ρ can be represented by a function of the coordinates and the time which at ordinary points is continuous and has continuous derivatives. The value of the total charge obtained by integrating that function over a large-scale volume will then differ from the true charge contained therein by a microscopic quantity at most.

Any ordered motion of charge constitutes a current. A current distribution is characterized by a vector field which specifies at each point not only the intensity of the flow but also its direction. As in the study of fluid motion, it is convenient to imagine streamlines traced through the distribution and everywhere tangent to the direction of flow. Consider a surface which is orthogonal to a system of streamlines. The *current density* at any point on this surface is then defined as a vector $\mathbf{J}$ directed along the streamline through the point and equal in magnitude to the charge which in unit time crosses unit area of the surface in the vicinity of the point. On the other hand the current I across *any* surface S is equal to the rate at which charge crosses that surface. If $\mathbf{n}$ is the positive unit normal to an element Δa of S, we have

$$\Delta I = \mathbf{J} \cdot \mathbf{n}\, \Delta a. \tag{4}$$

Since Δa is a macroscopic element of area, Eq. (4) does not define the current density with mathematical rigor as a continuous function of position, but again one may represent the distribution by such a function without incurring an appreciable error. The total current through S is, therefore,

$$I = \int_S \mathbf{J} \cdot \mathbf{n}\, da. \tag{5}$$

Since electrical charge may be either positive or negative, a convention must be adopted as to what constitutes a positive current. If the flow through an element of area consists of positive charges whose velocity vectors form an angle of less than 90 deg. with the positive normal $\mathbf{n}$, the current is said to be positive. If the angle is greater than 90 deg., the current is negative. Likewise if the angle is less than 90 deg. but the charges are negative, the current through the element is negative. In the case of metallic conductors the carriers of electricity are presumably negative electrons, and the direction of the current density vector is therefore opposed to the direction of electron motion.

Let us suppose now that the surface S of Eq. (5) is closed. We shall adhere to the customary convention that *the positive normal to a closed*