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A. N. Parshin I. R. Shafarevich (Eds.)

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Complex Algebraic Varieties, Algebraic Curves and Their Jacobians

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

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A. N. Parshin, I. R. Shafarevich, Steklov Mathematical Institute, ul. Gubkina 8, 117966 Moscow, Russia

Authors

Viktor S. Kulikov, Chair of Applied Mathematics II, Moscow State University of Transport Communications (MIIT), ul. Obraztcova 15, 101475 Moscow, Russia; e-mail: kulikov@alg.mian.su and victor@olya.ips.ras.ru

P.F. Kurchanov, Chair of Applied Mathematics II, Moscow State University of Transport Communications (MIIT), ul. Obraztcova 15, 101475 Moscow, Russia

V.V. Shokurov, Department of Mathematics, The Johns Hopkins University, Baltimore, MA 21218–2689, USA; e-mail: shokurov@chow.mat.jhu.edu

Translator

I. Rivin, Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom, e-mail: igor@maths.warwick.ac.uk; Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA, e-mail: rivin@caltech.edu

Contents

I. Complex Algebraic Varieties: Periods of Integrals and Hodge Structures

Vik. S. Kulikov, P. F. Kurchanov

Translated from the Russian
by Igor Rivin

Contents

Introduction

Starting with the end of the seventeenth century, one of the most interesting directions in mathematics (attracting the attention as J. Bernoulli, Euler, Jacobi, Legendre, Abel, among others) has been the study of integrals of the form

$$A_w(\tau) = \int_{\tau_0}^{\tau} \frac{dz}{w},$$

where w is an algebraic function of z. Such integrals are now called *abelian.*

Let us examine the simplest instance of an abelian integral, one where w is defined by the polynomial equation

$$w^2 = z^3 + pz + q, \tag{1}$$

where the polynomial on the right hand side has no multiple roots. In this case the function A_w is called an *elliptic integral.* The value of A_w is determined up to $m\nu_1 + n\nu_2$, where ν_1 and ν_2 are complex numbers, and m and n are integers. The set of linear combinations $m\nu_1 + n\nu_2$ forms a lattice $H \subset \mathbb{C}$, and so to each elliptic integral A_w we can associate the torus $\mathbb{C}/H$.

On the other hand, equation (1) defines a curve in the affine plane $\mathbb{C}^2 = \{(z, w)\}$. Let us complete $\mathbb{C}^2$ to the projective plane $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ by the addition of the "line at infinity", and let us also complete the curve defined by equation (1). The result will be a nonsingular closed curve $E \subset \mathbb{P}^2$ (which can also be viewed as a Riemann surface). Such a curve is called an *elliptic curve.*

It is a remarkable fact that the curve E and the torus $\mathbb{C}/H$ are isomorphic Riemann surfaces. The isomorphism can be given explicitly as follows.

Let $\wp(z)$ be the Weierstrass function associated to the lattice $H \subset \mathbb{C}$.

$$\wp = \frac{1}{z^2} + \sum_{h \in H, h \neq 0} \left[\frac{1}{(z-2h)^2} - \frac{1}{(2h)^2} \right].$$

It is known that $\wp(z)$ is a doubly periodic meromorphic function with the period lattice H. Further, the function $\wp(z)$ and its derivative $\wp'(z)$ are related as follows:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{2}$$

for certain constants g_2 and g_3 which depend on the lattice H. Therefore, the mapping $z \to (\wp(z), \wp'(z))$ is a meromorphic function of $\mathbb{C}/H$ onto the compactification $E' \subset \mathbb{P}^2$ of the curve defined by equation (2) in the affine plane. It turns out that this mapping is an isomorphism, and furthermore, the projective curves E and E' are isomorphic!

Let us explain this phenomenon in a more invariant fashion. The projection $(z, w) \to z$ of the affine curve defined by the equation (1) gives a double covering $\pi : E \to \mathbb{P}^1$, branched over the three roots z_1, z_2, z_3 of the polynomial $z^3 + pz + q$ and the point ∞.

The differential $\omega = dz/2w$, restricted to E is a holomorphic 1-form (and there is only one such form on an elliptic curve, up to multiplication by constants). Viewed as a C^∞ manifold, the elliptic curve E is homeomorphic to the product of two circles $S^1 \times S^1$, and hence the first homology group $H_1(E, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Let the generators of $H_1(E, \mathbb{Z})$ be γ_1 and γ_2. The lattice H is the same as the lattice $\left\{m \int_{\gamma_1} \omega + n \int_{\gamma_2} \omega\right\}$. Indeed, the elliptic integral A_w is determined up to numbers of the form $\int_l \frac{1}{\sqrt{z^3+pz+q}}$, where l is a closed path in $\mathbb{C} \backslash \{z_1, z_2, z_3\}$. On the other hand

$$\int_l \frac{1}{\sqrt{z^3 + pz + q}} = \int_\gamma \omega,$$

where γ is the closed path in E covering l twice.

The integrals $\int_{\gamma_i} \omega$ are called periods of the curve E. The lattice H is called the *period lattice*. The discussion above indicates that the curve E is uniquely determined by its period lattice.

This theory can be extended from elliptic curves (curves of genus 1) to curves of higher genus, and even to higher dimensional varieties.

Let X be a compact Riemann surface of genus g (which is the same as a nonsingular complex projective curve of genus g). It is well known that all Riemann surfaces of genus g are topologically the same, being homeomorphic to the sphere with g handles. They may differ, however, when viewed as complex analytic manifolds. In his treatise on abelian functions (see de Rham [1955]), Riemann constructed surfaces (complex curves) of genus g by cutting and pasting in the complex plane. When doing this he was concerned about the periods of abelian integrals over various closed paths. Riemann called those periods (there are $3g - 3$) *moduli*. These are continuous complex parameters which determine the complex structure on a curve of genus g.

One of the main goals of the present survey is to introduce the reader to the ideas involved in obtaining these kinds of parametrizations for algebraic varieties. Let us explain this in greater detail.

On a Riemann surface X of genus g there are exactly g holomorphic 1-forms linearly independent over $\mathbb{C}$. Denote the space of holomorphic 1-forms on X by $H^{1,0}$, and choose a basis $\omega = (\omega_1, \ldots, \omega_g)$ for $H^{1,0}$. Also choose a basis $\gamma = (\gamma_1, \ldots, \gamma_{2g})$ for the first homology group $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Then the numbers

$$\Omega_{ij} = \int_{\gamma_j} \omega_i$$

are called the *periods* of X. They form the *period matrix* $\Omega = (\Omega_{ij})$. This matrix obviously depends on the choice of bases for $H^{1,0}$ and $H_1(X, \mathbb{Z})$. It turns out (see Chapter 3, Section 1), that the periods uniquely determine the curve X. More precisely, let X and X' be two curves of genus g. Suppose

ω and ω' are bases for the spaces of holomorphic differentials on X and X', respectively, and γ and γ' be are bases for $H_1(X,\mathbb{Z})$ and $H_1(X',\mathbb{Z})$ such that there are equalities

$$(\gamma_i.\gamma_j)_X = (\gamma_i'.\gamma_j')_{X'}$$

between the intersection numbers of γ and γ'. Then, if the period matrices of X and X' with respect to the chosen bases are the same, then the curves themselves are isomorphic. This is the classical theorem of Torelli.

Now, let X be a non-singular complex manifold of dimension $d > 1$. The complex structure on X allows us to decompose any complex-valued C^∞ differential n-form ω into a sum

$$\omega = \sum_{p+q=n} \omega^{p,q}$$

of components of type (p,q). A form of type (p,q) can be written as

$$\omega^{p,q} = \sum_{(I,J)=(i_1,\ldots,i_p,j_1,\ldots,j_q)} h_{I,J} dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}.$$

If X is a projective variety (and hence a Kähler manifold; see Chapter 1, Section 7) , then this decomposition transfers to cohomology:

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}, \quad H^{p,q} = \bar{H}^{q,p}. \tag{3}$$

This is the famous Hodge decomposition (Hodge structure of weight n on $H^n(X)$, see Chapter 2, Section 1). It allows us to define the periods of a variety X analogously to those for a curve. Namely, let X_0 be some fixed non-singular projective variety, and $H = H^n(X_0,\mathbb{Z})$. Let X be some other projective variety, diffeomorphic to X_0, and having the same Hodge numbers $h^{p,q} = \dim H^{p,q}(X_0)$. Fix a $\mathbb{Z}$-module isomorphism

$$\phi : H^n(X,\mathbb{Z}) \simeq H.$$

This isomorphism transfers the Hodge structure (3) from $H^n(X,\mathbb{C})$ onto $H_\mathbb{C} = H \otimes_\mathbb{Z} \mathbb{C}$. We obtain the Hodge filtration

$$\{0\} = F^{n+1} \subseteq F^n \subseteq \ldots \subseteq F^0 = H_\mathbb{C}$$

of the space $H_\mathbb{C}$, where

$$F^p = H^{n,0} \oplus \ldots \oplus H^{p,n-p}, F^{n+1} = \{0\}.$$

This filtration is determined by the variety X up to a $GL(H,\mathbb{Z})$ action, due to the freedom in the choice of the map ϕ. The set of filtrations of a linear space $H_\mathbb{C}$ by subspaces F^p of a fixed dimension f^p is classified by the points of the complex projective variety (the *flag manifold*) $F = F(f^n,\ldots,f^1;H_\mathbb{C})$. The simplest flag manifold is the Grassmanian $G(k,n)$ of k-dimensional linear subspaces in $\mathbb{C}^n$. The conditions which must be satisfied by the subspaces $H^{p,q}$ forming a Hodge structure (see Chapter 2, Section 1) define a complex submanifold D of F, which is known as the *classifying space* or the *space of period matrices.*

This terminology is easily explained. Let $h^{p,q} = \dim H^{p,q}$. Further, let the basis of $H^{p,q}$ be $\{\omega_j^{p,q}\}$, for $j = 1, \dots, h^{p,q}$, and let the basis modulo torsion of $H_n(X, \mathbb{Z})$ be $\gamma_1, \dots, \gamma_b$. Consider the matrix whose rows are

$$I_j^{p,q} = \left(\int_{\gamma_1} \omega_j^{p,q}, \dots, \int_{\gamma_b} \omega_j p, q \right).$$

This is the period matrix of X. There is some freedom in the choice of the basis elements $\omega_j^{p,q}$, but, in any event, the Hodge structure is determined uniquely if the basis of H is fixed, and in general the Hodge structure is determined up to the action of the group Γ of automorphisms of the $\mathbb{Z}$-module H. Thus, if $\{X_i\}$, $i \in A$ is a family of complex manifolds diffeomorphic to X_0 and whose Hodge numbers are the same, we can define the *period mapping*

$$\Phi : A \to \Gamma \backslash D.$$

We see that we can associate to each manifold X a point of the classifying space D, defined up to the action of a certain discrete group. One of the fundamental issues considered in the present survey is the inverse problem — to what extent can we reconstruct a complex manifold X from the point in classifying space. This issue is addressed by a number of theorems of Torelli type (see Chapter 2, Section 5 for further details).

A positive result of Torelli type allows us, generally speaking, to construct a complete set of continuous invariants, uniquely specifying a manifold with the given set of discrete invariants. Let us look at the simplest example – that of an elliptic curve E. The two-dimensional vector space $H_{\mathbb{C}} = H^1(E, \mathbb{C})$ is equipped with the non-degenerate pairing

$$(\mu, \eta) = \int_E \mu \wedge \eta.$$

Restricting this pairing to $H = H^1(E, \mathbb{Z})$ gives a bilinear form

$$Q_H : H \times H \to \mathbb{Z},$$

dual to the intersection form of 1-cycles on E. We can, furthermore, pick a basis in H, so that

$$Q_H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$H_{\mathbb{C}}$ is also equipped with the Hodge decomposition

$$H_{\mathbb{C}} = \mathbb{C}\omega + \mathbb{C}\bar{\omega},$$

where ω is a non-zero holomorphic differential on E. It is easy to see that

$$\sqrt{-1}(\omega, \bar{\omega}) > 0,$$

and so in the chosen basis $\omega = (\alpha, \beta)$, where

$$\sqrt{-1}(\beta\bar{\alpha} - \alpha\bar{\beta}) > 0. \tag{4}$$

The form ω is determined up to constant multiple. If we pick $\omega = (\lambda, 1)$, then condition (4) means that $\operatorname{Im}\lambda > 0$, and so the space of period matrices D is simply the complex upper half-plane:

$$D = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

Now let us consider the family of elliptic curves

$$E_\lambda = \mathbb{C}/\{\mathbb{Z}\lambda + \mathbb{Z}\}, \quad \lambda \in D.$$

This family contains all the isomorphism classes of elliptic curves, and two curves E_λ and $E_{\lambda'}$ are isomorphic if and only if

$$\lambda' = \frac{a\lambda + b}{c\lambda + d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Thus, the set of isomorphism classes of elliptic curves is in one-to-one correspondence with the points of the the set $A = \Gamma\backslash D$. The period mapping

$$\Phi : A \to \Gamma\backslash D$$

is then the identity mapping. Indeed, the differential dz defines a holomorphic 1-form in each E_λ.

If γ_1, γ_2 is the basis of $H_1(E_\lambda, \mathbb{Z})$ generated by the elements $\lambda, 1$ generating the lattice $\{\mathbb{Z}\lambda + \mathbb{Z}\}$ then the periods are simply

$$\left(\int_{\gamma_1} \omega, \int_{\gamma_2} \omega\right) = (\lambda, 1).$$

The existence of Hodge structures on the cohomology of non-singular projective varieties gives a lot of topological information (see Chapter 1, Section 7). However, it is often necessary to study singular and non-compact varieties, which lack a classical Hodge structure. Nonetheless, Hodge structures can be generalized to those situations also. These are the so-called *mixed* Hodge structures, invented by Deligne in 1971. We will define mixed Hodge structures precisely in Chapter 4, Section 1, but now we shall give the simplest example leading to the concept of a mixed Hodge structure.

Let X be a complete algebraic curve with singularities. Let S be the set of singularities on X and for simplicity let us assume that all points of S are simple singularities, with distinct tangents. The singularities of X can be resolved by a normalization $\pi : \bar{X} \to X$. Then, for each point $s \in S$ the pre-image $\pi^{-1}(s)$ consists of two points x_1 and x_2, and outside the singular set the morphism

$$\pi : \bar{X}\backslash\pi^{-1}(S) \to X \setminus S$$

is an isomorphism.

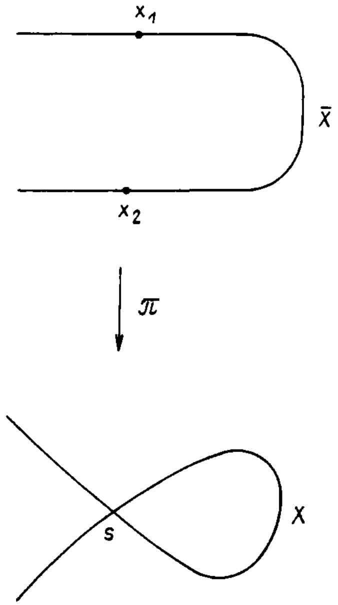

Fig. 1

For a locally constant sheaf $\mathbf{C}_X$ on X we have the exact sequence

$$0 \to \mathbf{C}_X \to \pi_*\mathbf{C}_{\bar{X}} \to \mathbf{C}_S \to 0,$$

which induces a cohomology exact sequence

$$\begin{array}{ccccccc} 0 & \to & H^0(X, \mathbf{C}_S) & \to H^1(X, \mathbf{C}_X) & \to & H^1(X, \pi_*\mathbf{C}_{\bar{X}}) & \to & 0 \\ & & \|\wr & & & \|\wr & & \\ & & H^0(S, \mathbf{C}_S) & & & H^1(\bar{X}, \mathbf{C}_{\bar{X}}) & & \end{array}$$

This sequence makes it clear that $H^1(\bar{X}, \mathbf{C}_X)$ is equipped with the filtration $0 \subset H^0(S, \mathbf{C}_S) = W_0 \subset H^1(X, \mathbf{C}_X) = W_1$. The factors of this filtration are equipped with Hodge structures in a canonical way – W_0 with a Hodge structure of weight 0, and W_1/W_0 with a Hodge structure of weight 1, induced by the inclusion of W_1/W_0 into $H^1(\bar{X}, \mathbf{C}_X)$.

Even though mixed Hodge structures have been introduced quite recently, they helped solve a number of difficult problems in algebraic geometry – the problem of invariant cycles (see Chapter 4, Section 3) and the description of degenerate fibers of families of of algebraic varieties being but two of the examples. More beautiful and interesting results will surely come.

Here is a brief summary of the rest of this survey.

In the first Chapter we attempt to give a brief survey of classical results and ideas of algebraic geometry and the theory of complex manifolds, necessary for the understanding of the main body of the survey. In particular, the first three sections give the definitions of classical algebraic and complex analytic geometry and give the results *GAGA (Géometrie algébrique et géométrie analytique)* on the comparison of algebraic and complex analytic manifolds.

In Sections 4, 5, and 6 we recall some complex analytic analogues of some standard differential-geometric constructions (bundles, metrics, connections).

Section 7 is devoted to classical Hodge theory.

Sections 8, 9, and 10 contain further standard material of classical algebraic geometry (divisors and line bundles, characteristic classes, extension formulas, Kodaira's vanishing theorem, Lefschetz' theorem on hyperplane section, monodromy, Lefschetz families).

Chapter 2 covers fundamental concepts and basic facts to do with the period mapping, to wit:

Section 1 introduces the classifying space D of polarized Hodge structures and explains the correspondence between this classifying space and a polarized algebraic variety. We study in some depth examples of classifying spaces associated to algebraic curves, abelian varieties and Kähler surfaces. We also define certain naturally arising sheaves on D.

In Section 2 we introduce the complex tori of Griffiths and Weil associated to a polarized Hodge structure. We also define the Abel-Jacobi mapping, and study in detail the special case of the Albanese mapping.

In Section 3 we define the period mapping for projective families of complex manifolds. We show that this mapping is holomorphic and horizontal.

In Section 4 we introduce the concept of variation of Hodge structure, which is a generalization of the period mapping.

In Section 5 we study four kinds of Torelli problems for algebraic varieties. We study the infinitesimal Torelli problem in detail, and give Griffiths' criterion for its solvability.

In Section 6 we study infinitesimal variation of Hodge structure and explain its connection with the global Torelli problem.

In Chapter 3 we study some especially interesting concrete results having to do with the period mapping and Torelli-type results.

In Section 1 we construct the classifying space of Hodge structures for smooth projective curves. We prove the infinitesimal Torelli theorem for non-hyperelliptic curves and we sketch the proof of the global Torelli theorem for curves.

In Section 2 we sketch the proof of the global Torelli theorem for a cubic threefold.

In Section 3 we study the period mapping for K3 surfaces. We prove the infinitesimal Torelli theorem. We construct the modular space of marked K3 surfaces. We also sketch the proof of the global Torelli theorem for K3 surfaces. We study elliptic pencil, and we sketch the proof of the global Torelli theorem for them.

In Section 4 we study hypersurfaces in $\mathbb{P}^n$. We prove the local Torelli theorem, and sketch the proof of the global Torelli theorem for a large class of hypersurfaces.

Chapter 4 is devoted to mixed Hodge structures and their applications.

Section 1 gives the basic definitions and survey the fundamental properties of mixed Hodge structures.

Sections 2 and 3 are devoted to the proof of Deligne's theorem on the existence of mixed Hodge structures on the cohomology of an arbitrary complex algebraic variety in the two special cases: for varieties with normal crossings and for non-singular incomplete varieties.

Section 4 gives a sketch of the proof of the invariant cycle theorem.

Section 5 computes Hodge structure on the cohomology of smooth hypersurfaces in $\mathbb{P}^n$.

Finally, in Section 5 we give a quick survey of some further developments of the theory of mixed Hodge structures, to wit, the period mapping for mixed Hodge structures, and mixed Hodge structures on the homotopy groups of algebraic varieties.

In Chapter 5 we study the theory of degenerations of families of algebraic varieties.

Section 1 contains the basic concepts of the theory of degenerations.

Section 2 gives the definition of the limiting mixed Hodge structure on the cohomology of the degenerate fiber (introduced by Schmid).

In Section 3 we construct the exact sequence of Clemens-Schmid, relating the cohomology of degenerate and non-degenerate fibers of a one-parameter family of Kähler manifolds.

Sections 4 and 5 are devoted to the applications of the Clemens-Schmid exact sequence to the degenerations of curves and surfaces.

In Section 6 we study the degeneration of K3 surfaces. We conclude that the period mapping is an epimorphism for K3 surfaces.

In conclusion, a few words about the prerequisites necessary to understand this survey. Aside from the standard university courses in algebra and differential geometry it helps to be familiar with the basic concepts of algebraic topology (Poincaré duality, intersection theory), homological algebra, sheaf theory (sheaf cohomology and hypercohomology, spectral sequences – see references Cartan–Eilenberg [1956], Godement [1958], Grothendieck [1957], Griffiths–Harris [1978]), theory of Lie groups and Lie algebras (see Serre [1965]), and Riemannian geometry (Postnikov [1971]).

We have tried to either define or give a reference for all the terms and results used in this survey, in an attempt to keep it as self-contained as possible.

Chapter 1
Classical Hodge Theory

§1. Algebraic Varieties

Let us recall some definitions of algebraic geometry.

1.1. Let $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) | z_i \in \mathbb{C}\}$ be the n-dimensional affine space over the complex numbers. *An algebraic set* in $\mathbb{C}^n$ is a set of the form

$$V(f_1, \ldots, f_m) = \{z \in \mathbb{C}^n | f_1(z) = \ldots = f_m(z) = 0\}.$$

where $f_i(z)$ lie in the ring $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_n]$ of polynomials in n variables over $\mathbb{C}$. An algebraic set of the form $V(f_1)$ is a *hypersurface* in $\mathbb{C}^n$, assuming that $f_1(z)$ is not a constant.

It is clear that if $f(z)$ lies in the ideal $I = (f_1, \ldots, f_m)$ of $\mathbb{C}[z]$ generated by $f_1(z), \ldots, f_m(z)$ then $f(a) = 0$ for all $a \in V(f_1, \ldots, f_m)$. Thus, to each algebraic set $V = V(f_1, \ldots, f_m)$ we can associate an ideal $I(V) \subset \mathbb{C}[z]$, defined by

$$I(V) = \{f \in \mathbb{C}[z] | f(a) = 0, a \in V\}.$$

The ideal $I(V)$ is a finitely generated ideal, and so by Hilbert's *Nullstellensatz* (Van der Waerden [1971]) $I(V) = \sqrt{(f_1, \ldots, f_m)}$,, where $\sqrt{J} = \{f \in \mathbb{C}[z] | f^k \in J$ for some $k \in \mathbb{N}\}$ is the radical of J.

The ring $\mathbb{C}[V] = \mathbb{C}[z]/I(V)$ is the *ring of regular functions* over the algebraic set V. This ring coincides with the ring of functions on V which are restrictions of polynomials over $\mathbb{C}^n$.

1.2. It is easy to see that the union of any finite number of algebraic sets and the intersection of any number of algebraic sets is again an algebraic set, and so the collection of algebraic sets in $\mathbb{C}^n$ satisfies the axioms of the collection of closed sets of some topology. This is the so-called *Zariski topology.* The Zariski topology in $\mathbb{C}^n$ induces a topology on algebraic sets $V \subset \mathbb{C}^n$, and this is also called the Zariski topology. The neighborhood basis of the Zariski topology on V is the set of open sets of the form $U_{f_1, \ldots, f_k} = \{a \in V | f_1(a) \neq 0, \ldots, f_k(a) \neq 0, f_1, \ldots, f_k \in \mathbb{C}[V]\}$.

Let $V_1 \subset \mathbb{C}^n$ and $V_2 \subset \mathbb{C}^m$ be two algebraic sets. A map $f : V_1 \to V_2$ is called a *regular mapping* or a *morphism* if there exists a set of m regular functions $f_1, \ldots, f_m \in \mathbb{C}[V_1]$ such that $f(a) = (f_1(a), \ldots, f_m(a))$ for all $a \in V_1$. Obviously a regular mapping is continuous with respect to the Zariski topology. It is also easy to check that defining a regular mapping $f : V_1 \to V_2$ is equivalent to defining a homomorphism of rings $f^* : \mathbb{C}[V_1] \to \mathbb{C}[V_2]$, which transforms the coordinate functions $z_i \in \mathbb{C}[V_2]$ into $f_i \in \mathbb{C}[V_1]$.

Two algebraic sets V_1 and V_2 are called *isomorphic* if there exists a regular mapping $f : V_1 \to V_2$ which possesses a regular inverse $f^{-1} : V_2 \to V_1$.