Stochastic Differential Equations and Related Topics

（随机微分方程及相关领域）

Edited by
Liangjian Hu
Zengjing Chen
Xuerong Mao

Science Press
Beijing

Edited by Liangjian Hu Zengjing Chen Xuerong Mao

Stochastic Differential Equations and Related Topics

(随机微分方程及相关领域)

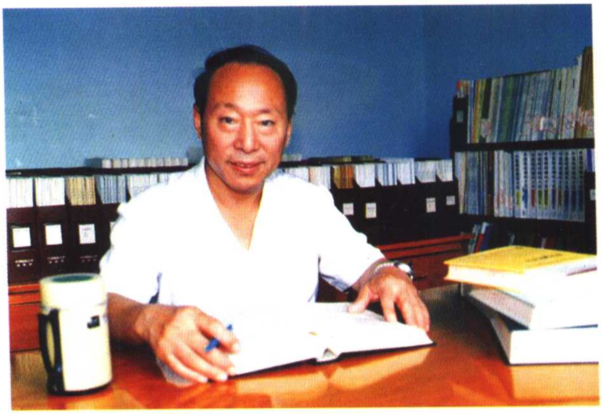
SCIENCE PRESS
Beijing

Responsible Editors: Chen Yuzhuo   Mo Danyu

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

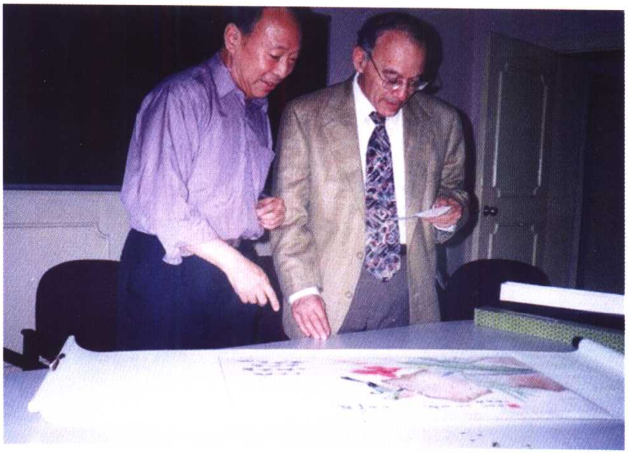
Printed in Beijing

ISBN 978-7-03-020193-5/O.2943 (Sciencep Press, Beijing)

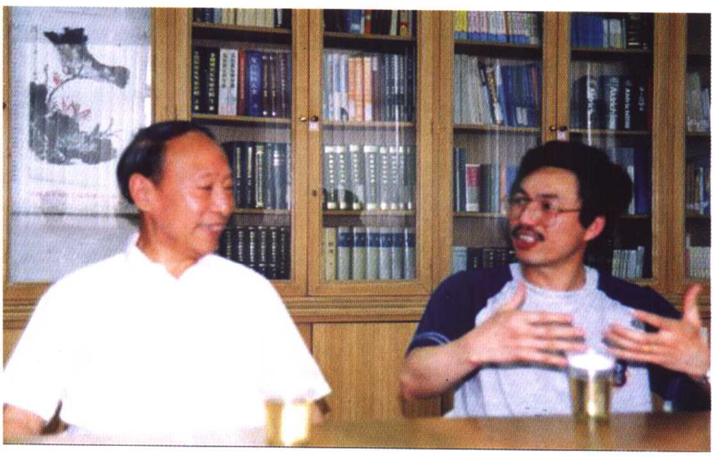


Rangquan Wu, Professor of Mathematics, at Donghua University (the former China Textile University and East China Institute of Textile Science and Technology), Shanghai, China. He was born in 1936, in Kaifeng, Henan, China. After graduating from the Department of Mathematics, Nankai University, Tianjing, China, in 1955, he worked in Harbin Institute of Technology, Harbin, Heilongjiang, China, during 1956~1975. In 1976, he moved to Donghua University. He also attended graduate studies at Naikai University during 1963~1965, and was a visit professor at Northwestern University, Evanston, Illinois, USA, during 1981~1983. Since 1960's, Professor Wu has been studying in the field of Stochastic Differential Equations and their applications (Such as Stochastic Control, Stochastic Finance and related topics). So far, He has published 7 research monographs and over 100 peer-reviewed journal papers, and supervised 50 graduate students.

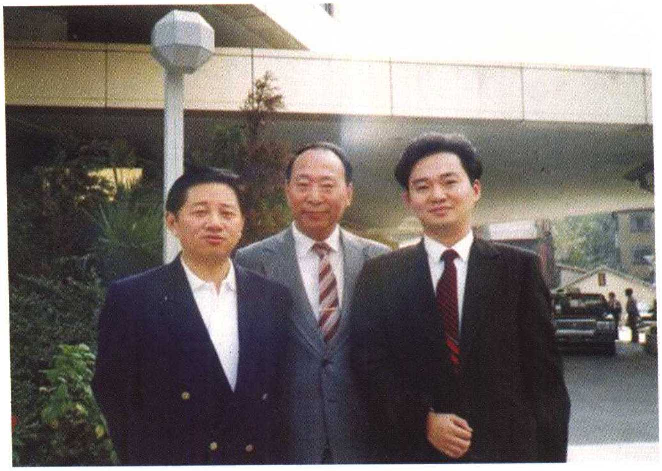
Rangquan Wu (extreme right, top row) with his graduate supervisor, Professor Zikun Wang (2nd from right, front row), an academician of the Chinese Academy of Sciences, and Professor Chengxi Zhu (extreme right, front row), as well as his classmates, Zhishan Li (extreme left), Xiangqun Yang (1st from left, top row), Rong Wu (2nd from left, top row), Wenchuan Mo (3rd from right, front row) etc. in Nankai University, Tianjing, China.



Rangquan Wu (left) with his co-worker Professsor Avner Friedman Director of Mathematical Biosciences Institute, and Distinguished Professor at The Ohio State University, Columbus, Ohio, USA.



Professor Wu (left) with his graduate student, Xuerong Mao a Professor at Strathclyde University, Glasgow, Scotland, UK.



Professor Wu (middle) with his graduate student, Zhaoxin Ye (left), the Deputy Director General of China National Petroleum Corporation (HongKong) Limited.

Professor Wu (left) with his graduate student, Zengjing Chen a Distinguished Professor at Shandong University, Jinan, Shandong, China.

Professor Wu (right) with his graduate student, Professor Zidong Wang Brunel University, West London, UK.

Professor Wu middle talked with his graduate students in 1985.

Professor Rangquan Wu (left), Professor Junyan Qian, and their son Dr. Yu Wu authored a book "Chinese Mathematical Thought" in 1993.

Professor Wu with his graduate students. From left to right: Kewei Huang, Xiaodong Ding, Rangquan Wu, Huisheng Shu, Weiguo Zhao.

A meeting in honor of Professor Wu's 20th Year as a supervisor of graduate students held in 1998. From left to right: Chengmao Sheng, Zhaoxin Ye, Shihuang Shao, Mingfang Xia, Rangquan Wu, Junyan Qian, Ruibao Zhang, Youyi Xue, Yuhu Feng.

Professor Wu's graduate students reunited in Donghua University in honor of Professor Wu's 70th birthday in 2006.

Preface

On the occasion of the 70th birthday of Professor Rangquan Wu, we edit this book for an obvious reason — to celebrate his birthday. We sincerely wish Professor Wu a very happy birthday!

There are however several other reasons for us to edit this book. One is that this book will show clearly the contributions of Professor Wu to the field of stochastic differential equations (SDEs) and their applications. Professor Wu is one of the pioneers in China in the development of the study of SDEs and has published 7 research monographs and over 100 peer-reviewed journal papers in this area. Another reason, perhaps more importantly, is to show how Professor Wu has devoted his nearly 50-year university career to Chinese education. Professor Wu has supervised over 50 graduate students and many of them have become leading experts both in home and abroad.

This book will therefore emphasize the joint contributions of Professor Wu and his students to the field of SDEs. SDEs arise in mathematical models of physical systems which possess inherent noise and uncertainty. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics and finance. This book will not only cover the theoretical research of SDEs from the fundamental existence-and-uniqueness theory through much modern SDEs driven by semimartingales to the more recent fuzzy SDEs; but will also highlight the important applications of SDEs to finance, control and biology which are currently very hot topics.

The book will have a direct impact on those who work in the field of SDEs, in particular, academic including senior undergraduates and graduates, as well as professional researchers in biology, epidemiology, mechanics, economics and finance.

Liangjian Hu, Donghua University
Zengjing Chen, Shandong University
Xuerong Mao, Strathclyde University
In November 2006

Contents

Part 1 Stochastic Differential Equations

Existence and Uniqueness of the Solutions of Stochastic Differential Equations

Rangquan Wu[a] Xuerong Mao[b]

[a] *Dept. of Mathematics, Northwestern University, Evanston. Illinois 60201 USA*

[b] *East China Institute of Textile Science and Technology, Shanghai, P.R. China*

Abstract. The standard existence and uniqueness theorem for stochastic differential equations requires Lipschitz condition of the coefficients. In this paper, we extend these results to the case in which the coefficients are not required to be Lipschitz continuous, instead they only satisfy a "weak" type of Lipschitz condition.

1. Introduction

In[1], K.A. Yan studied the following typical stochastic equation

$$X = \Phi(X) + F(X).M \tag{$*$}$$

where $X \in \mathscr{X}$, $\mathscr{X}$ is the set of all cadlag adapted process, M is a semimartingale with $M_0 = 0$, F is a map from $\mathscr{X}$ into $\mathscr{P}$ such that $F(X)$ is integrable with respect to M, $\mathscr{P}$ is the set of all predictable process, Φ is a map from $\mathscr{X}$ into $\mathscr{X}$.

Yan proved the following existence and uniqueness theorem. Let M be a semimartingale with $M_0 = 0$. $F \in \mathscr{L}_M^p(a)$, $\Phi \in \mathscr{L}^p(\beta)$, where $1 \leqslant p < \infty, a > 0, 0 \leqslant \beta < 1$, then Eq. $(*)$ has a unique solution in $\mathscr{X}$.

$$\mathscr{L}_M^p(a) \equiv \left\{ F : \begin{array}{l} \text{(i)}\forall X \in \mathscr{X}, \text{ and stopping time } T, F(X)I_{]0,T]} = F(X^{T-})I_{]0,T]}, \\ \text{(ii)}\forall X, Y \in \mathscr{X}, a > 0, \|F(X) - F(Y)\|_{\mathscr{S}^p} \leqslant a\|X - Y\|_{\mathscr{S}^p}, \\ \quad 1 \leqslant p < \infty, \\ \text{(iii)}F(0) \in L(M)(L(M) \text{ denotes the set of all predictable} \\ \quad \text{and integrable processes with respect to } M. \end{array} \right\}$$

$$\mathscr{C}^p(\beta) \equiv \left\{ \Phi : \begin{array}{l} \text{(i)}\forall X \in \mathscr{X}, \text{and stopping time } T, \Phi(X)I_{[0,T[} = \Phi(X^{T-})I_{[0,T[}, \\ \text{(ii)}\forall X, Y \in \mathscr{X}, 0 \leqslant \beta < 1, \|\Phi(X) - (Y)\|_{\mathscr{S}_p} \leqslant \beta\|X - Y\|_{\mathscr{S}_p}, \end{array} \right\}$$

In this paper we will give the proof of another existence and uniqueness theorem for Eq.(*). This new theorem does not require that F and Φ satisfy the Lipschitz conditions

$$\begin{aligned} ||F(X) - F(Y)||_{\mathscr{S}^p} &\leqslant a||X - Y||_{\mathscr{S}^p}, \\ ||\Phi(X) - \Phi(Y)||_{\mathscr{S}^p} &\leqslant \beta||X - Y||_{\mathscr{S}^p}. \end{aligned} \tag{1.1}$$

We only need the weaker local Lipschitz conditions

$$\begin{aligned} ||F(X^{[n]}) - F(Y^{[n]})||_{\mathscr{S}^p} &\leqslant a_n||X^{[n]} - Y^{[n]}||_{\mathscr{S}^p}, \\ ||\Phi(X^{[n]}) - \Phi(Y^{[n]})||_{\mathscr{S}^p} &\leqslant \beta_n||X^{[n]} - Y^{[n]}||_{\mathscr{S}^p}, \end{aligned} \tag{1.2}$$

where

$$X^{[n]} = X1_{[-n,n]}(X) + n1_{[n,\infty)} - n1_{(-\infty,-n)}(X), \quad a_n > 0, \quad 0 \leqslant \beta_n < 1,$$

and we add the growth conditions:

$$\begin{aligned} ||F(X)||_{\mathscr{S}^p} &\leqslant K_1(1 + ||X||_{\mathscr{S}^p}), \\ ||\Phi(X)||_{\mathscr{S}^p} &\leqslant K_2 + \alpha||X||_{\mathscr{S}^p}. \end{aligned} \tag{1.3}$$

It is easy to deduce from (1.1) that F and Φ satisfy (1.3), if $F \in L^p_M(a)$, $\Phi \in \mathscr{S}^p(\beta)$ and $F(0)$, $\Phi(0) \in \mathscr{S}^p$, In the above,

$$\mathscr{S}^p = \left\{ \begin{array}{c} X : X \text{ an optional process with finite norm } ||X||_{\mathscr{S}_p} < \infty, \\ ||X||_{\mathscr{S}^p} \triangleq ||X^*||_{L^p}, \quad X^* \triangleq \sup_{t\in\mathbf{R}_+} |X_t|. \end{array} \right\}$$

(See [4] p.305-308, or[8]).

This paper is separated into two parts, Sections 2 and 3, containing respectively the statements and proofs of the main theorems.

Finally, we point out that our results can be generalized to a stochastic equation

$$X = \Phi(X) + \sum_{i=1}^{m} F^i(X).M^i.$$

2. Main theorem

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, $(\mathscr{F}_t)_{t\in\mathbf{R}_+}$ be a family of σ-algebras satisfying the usual conditions and let $\mathscr{P}$ represent the set of all predictable processes. All predictable and integrable processes with respect to M will be denoted by $L(M)$.

Definition 2.1 *Let $1 \leqslant p < \infty$, $\{a_n\}$ be a sequence of positive real numbers, $K_1 > 0$, then $\mathscr{L}^p(\{a_n\}, K_1)$ is the set of all maps F from $\mathscr{X}$ into $\mathscr{P}$ satisfying the following conditions:*

i) For all $X \in \mathscr{X}$ and all stopping time T,

$$F(X)1_{]0,T]} = F(X^{T-})I_{]0,T]}. \tag{2.1}$$

ii) For all $X, Y \in \mathscr{X}, n = 1, 2, \cdots$,

$$||F(X^{[n]}) - F(Y^{[n]})||_{\mathscr{S}_p} \leqslant a_n ||X^{[n]} - Y^{[n]}||_{\mathscr{S}_p}. \tag{2.2}$$

iii) For all $X \in \mathscr{X}$,

$$||F(X)||_{\mathscr{S}p} \leqslant K_1(1 + ||X||_{\mathscr{S}p}). \tag{2.3}$$

Definition 2.2 *Let $1 \leqslant p < \infty, 0 \leqslant \alpha < 1, K_2 \geqslant 0, \{\beta_n\}$ be a sequence of real numbers satisfying $0 \leqslant \beta_n < 1$, then $\mathscr{C}^p(\{\beta_n\}, \alpha, K_2)$ denotes the set of all maps Φ from $\mathscr{X}$ into $\mathscr{X}$ satisfying:*
i) For all $X \in \mathscr{X}$ and all stopping times T,

$$\Phi(X)1_{[0,T[} = \Phi(X^{T-})1_{[0,T[}. \tag{2.4}$$

ii) For all $X, Y \in \mathscr{X}, n = 1, 2, \cdots$,

$$||\Phi(X^{[n]}) - \Phi(Y^{[n]})||_{\mathscr{S}p} \leqslant \beta_n ||X^{[n]} - Y^{[n]}||_{\mathscr{S}p}. \tag{2.5}$$

iii) For all $X \in \mathscr{X}$,

$$||\Phi(X)||_{\mathscr{S}_p} \leqslant k_2 + \alpha ||X||_{\mathscr{S}_p}. \tag{2.6}$$

Theorem 2.3 *Let M be a semimartingale with $M_0 = 0, F \in \mathscr{L}^p(\{a_n\}, K_1)$, $\Phi \in \mathscr{C}^p(\{\beta_n\}, \alpha, K_2)$, where $1 \leqslant p < \infty$, and let $\{a_n\}$ and $\{\beta_n\}$ be two sequences of nonnegative real numbers and $0 \leqslant \beta_n < 1, 0 \leqslant \alpha < 1, K_1 > 0, K_2 \geqslant 0$.*

Then the equation

$$X = \Phi(X) + F(X).M.$$

has a unique solution in $\mathscr{X}$.

Theorem 2.4 *Under the conditions of Theorem 2.3, if $0 < b < \dfrac{1-\alpha}{C_p K_1}$, then there exists a stopping time sequence $S_k \uparrow \infty$(a.s.) such that $M^{S_k -} \in \mathscr{D}(b)$[1] and the unique*

1 Let b be a constant, M be a semimartingale, $M \in \mathscr{H}^\infty$ and there exist stopping times $(T_j)_{0 \leqslant j \leqslant k}(0 = T_0, T_t, \cdots, T_k)$ such that $M = M^T k^-$ and

$$||(M - M^{T_{i-1}})^{T_i -}||_{\mathscr{H}^\infty} \leqslant b, \quad \text{for } i = 1, \cdots, k.$$

We say that M belongs to $\mathscr{D}(b)$(i.e.$M \in \mathscr{D}$(b)), and we say $(T_0, \cdots, T_k)$ partition M into sections with lengths less than b.