GAME THEORY

Mathematical models of conflict

A.J. JONES

GAME THEORY: MATHEMATICAL MODELS OF CONFLICT

A. J. JONES, B.Sc., Ph.D.
Department of Mathematics
Royal Holloway College
University of London

ELLIS HORWOOD LIMITED
Publishers Chichester

Halsted Press: a division of
JOHN WILEY & SONS
New York - Chichester - Brisbane - Toronto

First published in 1980 by
ELLIS HORWOOD LIMITED
Market Cross House, Cooper Street, Chichester, West Sussex, PO19 1EB, England

The publisher's colophon is reproduced from James Gillison's drawing of the ancient Market Cross, Chichester.

Distributors:

Australia, New Zealand, South-east Asia:
Jacaranda-Wiley Ltd., Jacaranda Press,
JOHN WILEY & SONS INC.,
G.P.O. Box 859, Brisbane, Queensland 40001, Australia.

Canada:
JOHN WILEY & SONS CANADA LIMITED
22 Worcester Road, Rexdale, Ontario, Canada.

Europe, Africa:
JOHN WILEY & SONS LIMITED
Baffins Lane, Chichester, West Sussex, England.

North and South America and the rest of the world:
Halsted Press, a division of
JOHN WILEY & SONS
605 Third Avenue, New York, N.Y. 10016, U.S.A.

British Library Cataloguing in Publication Data
Jones, A J
Game theory. – (Mathematics and its applications).
1. Game theory; mathematical models of conflict.
I. Title II. Series
519.3 QA269 79–40972
ISBN 0–85312–147–8 (Ellis Horwood Ltd., Publishers, Library Edition)
ISBN 0–85312–154–0 (Ellis Horwood Ltd., Publishers, Student Edition)
ISBN 0–470–26870–0 (Halsted Press, Library Edition)

Typeset in Press Roman by Ellis Horwood Ltd.
Printed in Great Britain by W. & J. Mackay Ltd., Chatham

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Author's Preface

The origins of game playing are lost forever in the mists of time. Psychologists speculate that games, which enable children to practise roles and decision processes in relative security, are an essential feature of the learning process. It seems likely that many adult games were originally conceived as training aids for warfare. For games such as chess the military origins are clear, but could it be, as suggested to me by W. Haythorn, that backgammon was designed to train tribesmen in the art of filtering through a hostile tribe in ones and twos? Certainly a useful strategy in many common forms of backgammon is the 'buddy system' – never move alone. Whatever the truth of this matter, there is a rich field for investigation in the historical origins of many common games, and we can probably learn a lot about a society by studying the games it invents.

There is no mystery, however, about the origins of game *theory*. Game theory is the mathematical study of conflict situations, and it was, to all intents and purposes, born in 1944 with the publication of a single book *Theory of Games and Economic Behaviour* by J. von Neumann and O. Morgenstern. Unfortunately that book is singularly difficult to read and, apart from the simplest case of the finite two-person zero sum game, it was a long time before the contents became available to a wider audience. This process was not aided by the fact that early expositors, enthusiastically following in the footsteps of the masters, tended to create more confusion than they helped to dispel. The subtleties of game theory lie more in the modelling than in the mathematics, but it is of course essential that the mathematical reasoning be correct.

The fact is that a comprehensive course in game theory is no more mathematically demanding than a similar course in classical mechanics. An exception is the theory of *differential* games, a topic founded somewhat later by R. Isaacs and one which is only lightly touched upon in this book. Nevertheless there have been very few textbooks dealing with the mathematical development of the subject which one could recommend with a clear conscience. One such was the book by J. C. C. McKinsey (1952), which ought to have served as a model for later writers but apparently did not. More recently a welcome addition to the literature has been the Leningrad lecture notes of N. N. Vorob'ev made available in translation by Springer-Verlag (1977).

This book evolved from a series of lectures on the theory of games which I have given at Royal Holloway College for the last few years. I hope that it will serve to complement the existing literature and perhaps encourage other teachers to experiment by offering a course in game theory. My experience has been that students are keen to learn the subject and, although it is not always what they expect, they enjoy it, provided that one does not dwell on the technically more difficult proofs.

Before one is able to appreciate the virtue of proving things it is first necessary to have a firm intuitive grasp of the structure in question. With this object in view I have written Chapter 1 with virtually no proofs at all. After working through this chapter the reader should have a firm grasp of the extensive form, the idea of a pure strategy, the normal form, the notion of a mixed strategy, and the minimax theorem, and be able to solve a simple two-person zero sum game starting from the rules. These ideas, particularly that of a pure strategy, are quite subtle, and the subtleties *can only* be grasped by working through several of the problems provided and making the usual mistakes.

Beyond Chapter 1 there are just two items which should be included in any course: the *equilibrium point* as a solution concept for an *n*-person non-cooperative game, which is introduced early in Chapter 2, and the brief discussion of *utility*, which is postponed until the beginning of Chapter 4. Granted these exceptions, Chapters 2-5 are logically independent and can be taken in any order or omitted entirely! Thus a variety of different courses can be assembled from the text, depending on the time available.

The mathematical background required to read most of the book is quite modest. Apart from the four items listed below, linear algebra, calculus, and the idea of a continuous function from one Euclidean space to another will suffice. The exceptions are

(i) Section 2.4 where I have proved the results for compact topological spaces, and the teacher is expected to rewrite the proofs for closed bounded subsets of Euclidean space if this is necessary.
(ii) The proof of Theorem 2.2, which uses transfinite ordinals.
(iii) The proof of Theorem 2.4, which uses the Brouwer fixed point theorem.
(iv) The proof of Theorem 5.3, which uses the Kakutani fixed point theorem.

In the last three cases no loss of understanding results from omitting the proof in question.

Some teachers may be displeased with me for including fairly detailed solutions to the problems, but I remain unrepentant. Once the ideas have been grasped, numbers can be changed and slight variations introduced if this seems desirable. In my view any author of a mathematical textbook should be *required* by the editor to produce detailed solutions for all the problems set, and these should be included where space permits. The advantages of such a practice will be obvious to anyone who has struggled for hours with an incorrectly worded or unrealistically difficult problem from a book without solutions. Moreover, even where the author has been conscientious in solving the problems, for students working alone detailed solutions are an essential check on their own understanding.

It is too much to hope that I have managed to produce an error-free text, but several successive years of critical students have been a great help in this respect. I owe a considerable debt to the secretariat of Royal Holloway College mathematics department – Mrs B. Alderman, Mrs M. Brooker and Mrs M. Dixon – for their unflagging kindness and patience in typing successive versions of the text under pressing time constraints. I should like to thank my publisher Mr Ellis Horwood, his sons Mike and Clive and daughter Sue, Fred Sowan the house editor and his colleagues at Ellis Horwood Ltd, and Professor G. M. Bell of Chelsea College, London University, the series editor of *Mathematics and its Applications*, for their friendly cooperation and patience in seeing the book from the manuscript stage through to its printing and publication.

ACKNOWLEDGEMENTS

The author gratefully acknowledges permission to reproduce the following material:

Two passages from R. D. Luce and H. Raiffa: *Game and Decisions*, J. Wiley and Sons (1957), New York.

The section on simplified poker from H. W. Kuhn and A. W. Tucker: *Contributions to the Theory of Games*, Vol I. (Annuals of Mathematical Studies, No. 24), Princeton University Press (1950), Princeton.

Problems 8, 10, 11, 12, 13 (Chapter II) and their solutions from Robert R. Singleton and William F. Tyndall, *Games and Programs, mathematics for modeling*, W. H. Freeman and Camp. (1974), San Francisco.

Exercise 3 (Chapter II) and Exercises 1, 7 (Chapter IV) from Kathleen Trustrum, *Linear Programming*, Rauthedge and Kegan Paul Ltd. (1971), London.

A. J. Jones
Wokingham, Berkshire.
1979

Glossary of Symbols

Free use is made of the logical quantifiers $\exists$ (there exists) and $\forall$ (for every) and of basic set notation. Thus $x \in S$ means x is an element of the set S (read: x belongs to S), $x \notin S$ means x does not belong to S, and $S \subseteq T$ (S is a subset of T) means $x \in S$ implies $x \in T$. Classifier brackets $\{\ ;\ \}$ are used throughout to specify a set. For example

$$S \cup T = \{x\ ; x \in S \text{ or } x \in T\} \qquad \textbf{(union)},$$

is read as: S union T equals the set of all x such that $x \in S$ or $x \in T$. The commonly used set notations are

$$\phi = \text{the set with no elements} \qquad \text{(the }\textbf{empty set}\text{)},$$
$$S \cap T = \{x\ ;\ x \in S \text{ and } x \in T\} \qquad \textbf{(intersect)},$$
$$S_1 \times \ldots \times S_n = \{(x_1, \ldots, x_n)\ ; x_i \in S_i\ (1 \leqslant i \leqslant n)\} \qquad \textbf{(cartesian product)},$$
$$S \setminus T = \{x\ ; x \in S \text{ and } x \notin T\} \qquad \textbf{(difference)},$$
$$\mathcal{P}(S) = \{x\ ;\ x \subseteq S\} \qquad \textbf{(power set)},$$
$$|S| = \text{the number of elements in } S \qquad \textbf{(cardinal)}.$$

For sets of numbers we use

$$\mathbb{N} = \{1,2,3, \ldots\} \qquad \text{(the set of }\textbf{natural numbers}\text{)},$$
$$\mathbb{R} = \text{the set of real numbers},$$
$$|\mathbb{N}| = \aleph_0 \qquad \text{(aleph zero, the }\textbf{first transfinite cardinal}\text{)},$$

and for $x \in \mathbb{R}$

$$|x| = \max\{x, -x\} \qquad \textbf{(absolute value)}.$$

The set of ordered n-tuples $(x_1, \ldots, x_n)$, $x_i \in \mathbb{R}$ $(1 \leqslant i \leqslant n)$, the cartesian product of n copies of $\mathbb{R}$, is denoted by $\mathbb{R}^n$. Elements of this set are n-dimensional vectors and are denoted by bold faced type

$$\mathbf{x} = (x_1, \ldots, x_n)\ .$$

The **inner product** of two vectors $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is written as

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \ldots + x_n y_n \ ,$$

or as $\mathbf{x}\,\mathbf{y}^T$ using the transpose (defined on page 119) and matrix multiplication. Other vector notation includes the use of

$$\mathbf{J}_n = (1, 1, \ldots, 1)$$

for the n-dimensional vector with all components equal to one, and $\mathbf{x} \geqslant \mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, for $x_i \geqslant y_i$ for all i, $1 \leqslant i \leqslant n$.

We also use the sum and product notation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n \ ,$$

$$\prod_{i=1}^{n} x_i = x_1 x_2 \ldots x_n \ ,$$

and

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{(read : } n \text{ choose } r\text{)}$$

for the number of ways of choosing r objects from n objects not counting different orderings of the r objects.

The following game theoretic notations are frequently employed

2-person zero sum games

σ, τ pure strategies,
$\mathbf{x}$ a mixed strategy for player 1,
X or $\Sigma^{(1)}$ set of all mixed strategies for player 1,
$\mathbf{y}$ a mixed strategy for player 2,
Y or $\Sigma^{(2)}$ set of all mixed strategies for player 2,
$P(\mathbf{x},\mathbf{y})$ payoff to player 1.

n-person non-cooperative games

σ_i a pure strategy for player i,
S_i set of all pure strategies for player i,
$\mathbf{x}_i$ a mixed strategy for player i,
X_i set of all mixed strategies for player i,
$P_i(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ payoff to player i.
If $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, $\mathbf{x}_i \in X_i$ $(1 \leqslant i \leqslant n)$, then
$\mathbf{x} \| \mathbf{x}_i' = (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_i', \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n)$.

n-person cooperative games

$I = \{1,2, \ldots ,n\}$ the set of players ,
$S, T \subseteq I$ coalitions S, T.
S_T set of all coordinated pure strategies for coalition T.
X_T set of all coordinated mixed strategies for coalition T.
$\mathbf{x} \prec \mathbf{y}$ imputation **x** dominates imputation **y**(defined on page 199).

Chapter 1

The Name of the Game

He thinks that I think that he thinks . . . . . .

1.1 INTRODUCTION

Our aim is to study various problems of conflict by abstracting common strategic features for study in theoretical 'models' that are termed 'games' because they are patterned on actual games such as bridge and poker. By stressing strategic aspects, that is aspects controlled by the participants, the theory of games goes beyond the classical theory of probability, in which the treatment of games is limited to aspects of pure chance. First broached in 1921 by E. Borel [1] the theory was established in 1928 by J. von Neumann who went on with O. Morgenstern [2] to lay the foundations for virtually the whole of what is now called the mathematical theory of games.

One does not expect to see a completely developed theory leap into existence with the publication of a single book, and with the hindsight of a further thirty odd years of development it is now clear that this did not in fact happen; von Neumann and Morgenstern made a monumental start, but only a start. Nevertheless to the economists and social scientists of the time it must have seemed that the answer to their prayers had magically appeared overnight, and game theory was hailed with great enthusiasm at its birth [3]. It has opened up to mathematical attack a wide range of gamelike problems not only in economics but also in sociology, psychology, politics and war. A striking conceptual achievement of game theory has been to clarify the logical difficulties inherent in any rational analysis of many conflict situations.

Although most actual games elude full-scale analysis, because of the amount of computation involved, many can be anaylsed in miniature or simplified form, and the results often throw light onto the original game. For example an exciting game of poker usually involves at least four or five players; but pot-limit, 2-person, straight poker can be convincingly analysed in game-theoretic terms [4].† The optimal strategy for the opening player involves always betting on the first

† See Appendix 2 for poker terminology.

round with J9632 or *worse* but checking (and folding if bet into) with hands in the range J9632–22J109; that is, one should *always* bluff with a bad enough hand but *never* with a slightly better one. Equally suprising is the fact that, with an eight high straight flush (a remarkably rare hand) one should bet, make three re-raises, and then *fold* if the opponent persists. Before you rush out to make your fortune I should stress that this game is "about as interesting as watching paint dry", and that no real poker player could be persuaded to play for more than five minutes. Still, it is interesting how different the strategies described are from those practised by the average player, who typically seldom bluffs enough and *never* believes that a big hand can be beaten. Against such an opponent, who is consistently rash or unduly cautious, one can of course improve one's gains by deviating from optimal play in order to exploit the opponent's 'bad play'.

The counter-intuitive nature of the optimal strategy in this example is not unusual. People seem to have little natural feel for these problems and, as in learning to ski or to helm a sailing dinghy, one's natural tendencies are often quite different from what is optimal.

Social psychologists and political scientists are concerned for the most part with what individuals *usually* do when they find themselves in a gamelike situation. This is a question which can be answered only by experiment. For example it has been found if a subject is allowed to observe a large number of independent tosses of a coin weighted 75% in favour of heads and is then invited to predict subsequent tosses, he or she will guess heads 75% of the time (roughly speaking) and tails 25% of the time. Precisely the same behaviour has been observed in the case of laboratory rats, suitably rewarded for correct guesses, placed in an analogous situation. To maximise the proportion of successful guesses one should, of course, guess heads *every* time.

A management consultant or an economist might very well take a less passive view of the situation and ask what is optimal for one of the players, given that the others play as usual. This is the question professional gamblers ask themselves when deciding what strategy to adopt if dealing blackjack (pontoon). What to do in these situations is by no means transparent. See, for example, the highly enjoyable book *Beat the Dealer* by E. Thorp [5]. For practical decision making, it is clearly this approach which is important.

The question asked by a game theorist is very different. It is: what would each player do, if all the players were doing as well for themselves as they possibly could? The extent to which this question has been answered in practical applications is, we must frankly admit, sharply limited. For example a convincing analysis of Monopoly would be of serious interest to economists. The fact is, as we shall see later chapters, in many contexts the very meaning of the question poses considerable philosophical difficulties and, even where its meaning can be clarified the technical problems of seeking an answer are seldom easy. Thus the difficulties we can expect to encounter range from the practical impossibility of performing staggeringly vast computations to basic questions of philosophy. Nevertheless the question must be grappled with directly if progress is to be made.

1.2 EXTENSIVE FORMS AND PURE STRATEGIES

From the rules of any game one can proceed to several levels of abstraction. The first level, the extensive form of a game, eliminates all features which refer specifically to the means of playing it (for example, cards or dice). Thus the **extensive form** amounts to a literal translation of the rules into the technical terms of a formal system designed to describe all games. For the present we consider only 2-person games, but the ideas of this section apply quite naturally to any game.

Traditionally the game of *Nim* is played as follows. There are several piles of matches and two players who takes alternate turns. At each turn the player selects a pile and removes at least one match from this pile. The players may take as many matches as they wish but only from the one pile. When all the matches have been removed there are no more turns; the player who removes the last match loses. This game is capable of complete anaylsis, and interested readers who wish to baffle their friends can find the details in Hardy and Wright [6]. Consider the following simplified version.

Example 1.1 (2-2 Nim). The game of 2-2 Nim is adequately described by the following rules:

Initial state: Four matches are set out in 2 piles of 2 matches each.

Admissible transformations: Two players take alternate turns. At each turn the player selects a pile that has at least one match and removes at least one match from this pile. He or she may take several matches, but only from one pile.

Terminal state: When both piles have no matches there are no more turns and the game is ended.

Outcome: The player who removes the last match loses.

The term **state** is used here in a precise technical sense, and a few words of explanation are in order. One is well aware that in a game such as chess the same position can be reached in a variety of different ways. In game theory we do not regard these same positions as identical states. That is to say, a state is determined not merely by the current position but also by the unique history of play which led to that position.

The conventional way of representing a game in extensive form is to use a 'game tree'. The states and their admissible transformations are represented in a decision tree. If the game tree faithfully reproduces each possible state, together with the possible decisions leading from it, and each possible outcome, it is said to represent the game in extensive form. Fig. 1.1 represents 2-2 Nim in extensive form as a game tree.†

† Following established tradition all trees are drawn with their roots at the top.