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A. L. Onishchik (Ed.)

Lie Groups and Lie Algebras I

Foundations of Lie Theory, Lie Transformation Groups

李群与李代数 I

李理论基础，李交换群

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

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I. Foundations of Lie Theory

A. L. Onishchik, E. B. Vinberg

Translated from the Russian
by A. Kozlowski

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Introduction

The theory of Lie groups, to which this volume is devoted, is one of the classical well established chapters of mathematics. There is a whole series of monographs devoted to it (see, for example, Pontryagin 1984, Postnikov 1982, Bourbaki 1947, Chevalley 1946, Helgason 1962, Sagle and Walde 1973, Serre 1965, Warner 1983). This theory made its first appearance at the end of the last century in the works of S. Lie, whose aim was to apply algebraic methods to the theory of differential equations and to geometry. During the past one hundred years the concepts and methods of the theory of Lie groups entered into many areas of mathematics and theoretical physics and became inseparable from them.

The first three chapters of the present work contain a systematic exposition of the foundations of the theory of Lie groups. We have tried to give here brief proofs of most of the more important theorems. Certain more complex theorems, not used in the text, are stated without proof. Chapter 4 is of a special character: it contains a survey of certain contemporary generalizations of Lie groups.

The authors deliberately have not touched upon structural questions of the theory of Lie groups and algebras, in particular, the theory of semi-simple Lie groups. To these questions will be devoted a separate study in one of the future volumes of this series.

In this entire work Lie groups, as a rule, will be denoted with capital Latin letters, and their tangent algebras with the corresponding small Gothic letters, In addition the following notation will be used:

G^0 – connected component of the identity of a Lie group (or a topological group) G;
$G' = (G, G)$ – the commutator subgroup of a group G; $G^{(p)} = (G^{(p-1)}, G^{p-1)})$;
$\operatorname{Rad} G$ – the radical of a Lie group G;
$\operatorname{rad} \mathfrak{g}$ – the radical of a Lie algebra $\mathfrak{g}$;
$\rtimes$ – the semidirect product of groups (normal subgroup on the left);
$\oplus$ – the semidirect sum of Lie algebras (ideal on the left);
$\mathbb{T}$ – the group of complex numbers of modulus 1;
exp – the exponential mapping;
Ad – the adjoint representation of a Lie group;
ad – the adjoint representation of a Lie algebra;
$\operatorname{Aut} A$ – the group of automorphisms of a group or algebra A;
$\operatorname{Int} G$ – the group of inner automorphisms of a group G;
$\operatorname{Der} A$ – the Lie algebra of derivations of an algebra A;
$\operatorname{Int} \mathfrak{g}$ – the group of inner automorphisms of a Lie algebra $\mathfrak{g}$;
$\operatorname{GL}(V)$ – the group of all automorphisms (invertible linear transformations) of a vector space V;

$L_n(K)$ – the associative algebra of all square matrices of order n over a field K;
$GL_n(K)$ – the group of all non singular matrices of order n over K;
$SL_n(K)$ – the group of all matrices of order n with determinant 1;
$PGL_n(K) = GL(K)/\{\lambda E\}$ – the projective linear group;
$GL_n^+(\mathbb{R})$ – the group of all real matrices of order n with positive determinant;
$O_n(K)$ – the group of all orthogonal matrices of order n over K;
$SO_n(K) = O_n(K) \cap SL_n(K)$;
$Sp_n(K)$ – the group of all symplectic matrices of order n over K (n even):
$O_{k,l}$ – the group of all pseudo-orthogonal real matrices of signature (k,l);
$SO_{k,l} = O_{k,l} \cap SL_n(\mathbb{R})$;
$O'_{k,l}$ – the group of pseudo-orthogonal matrices of signature (k,l) whose minor of order k at the top left corner is positive;
U_n – the group of unitary complex matrices of order n;
$U_{k,l}$ – the group of pseudo-unitary complex matrices of signature (k,l);
$SU_n = U_n \cap SL_n(\mathbb{C})$; $SU_{k,l} = U_{k,l} \cap SL_{k+l}(\mathbb{C})$.

Finally we would like to mention a piece of non-standard terminology: we use the term "the tangent algebra of a Lie group" instead of the usual "the Lie algebra of a Lie group". We do so with a view to emphasise the construction of this Lie algebra as the tangent space to the Lie group. This seems to be appropriate here since, in particular, the tangent algebra of an analytic loop is not, in general, a Lie algebra. We reserve the term "Lie algebra" for its algebraic context.

Chapter 1
Basic Notions

We will assume familiarity with the basic concepts of manifold theory. However in order to avoid misunderstandings some of them will be defined in the text. The basic field, by which we mean either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, will be denoted by K. Unless stated otherwise, differentiability of functions will be understood in the following sense: in every case there exist as many derivatives as are needed. Differentiability of manifolds and maps is understood in the same sense. The Jacobian matrix of a system of differentiable functions $f_1, \ldots, f_m$ of variables $x_1, \ldots, x_n$ will be denoted by $\frac{\partial(f_1,\ldots,f_m)}{\partial(x_1,\ldots,x_n)}$. For $m = n$ its determinant will be denoted by $\frac{D(f_1,\ldots,f_n)}{D(x_1,\ldots,x_n)}$.

The tangent space of a manifold X at a point x will be denoted by $T_x(X)$ and the differential of a map $f : X \to Y$ at a point x by $d_x f$. In many cases, when it is clear which point is being considered, the subscript will be omitted in denoting a tangent space or a differential.

All differentiable manifolds will be assumed to possess a countable base of open sets.

§1. Lie Groups, Subgroups and Homomorphisms

1.1. Definition of a Lie Group. A Lie group over the field K is a group G equipped with the structure of a differentiable manifold over K in such a way that the map

$$\mu : G \times G \to G, \ (x, y) \mapsto xy$$

is differentiable. In other words, the coordinates of the product of two elements have to be differentiable functions of the coordinates of the factors.

With the aid of the implicit function theorem it is easy to show that in any Lie group the inverse

$$\iota : G \to G, \ x \mapsto x^{-1}$$

is also a differentiable map. Lie groups over $\mathbb{C}$ are called *complex Lie groups* and Lie groups over $\mathbb{R}$ – *real Lie groups*. Any complex Lie group can be viewed as a real Lie group of twice the dimension.

One can also consider analytic groups by requiring that the manifold G and the map μ be analytic over the field K. Clearly, every complex Lie group is analytic, but even in the real case it turns out that in any Lie group there exists an atlas with analytic transition functions, in which the map μ is expressed in terms of analytic functions (see 3.3 of Chap. 3).

Examples. 1. The additive group of the field K (we will denote it also by K).

2. The multiplicative group $K^{\times}$ of the field K.

3. 'The circle' $\mathbb{T} = \{z \in \mathbb{C}^{\times} : |z| = 1\}$ is a real Lie group.

4. The group $\mathrm{GL}_n(K)$ of invertible matrices of order n over the field K, with the differentiable structure of an open subset of the vector space $\mathrm{L}_n(K)$ of all matrices, i.e. (global) coordinates are given by the matrix entries.

5. The group $\mathrm{GL}(V)$ of invertible linear transformations of an n-dimensional vector space over the field K can be regarded as a Lie group in view of the isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}_n(K)$, which assigns to each linear transformation its matrix with respect to some fixed basis.

6. The group $\mathrm{GA}(S)$ of (invertible) affine transformations of an n-dimensional affine space S over the field K possesses also a canonical differentiable structure, which turns it into a Lie group. Namely, with respect to the affine coordinate system of the space S affine transformations can be written in the form $X \mapsto AX + B$, where X is a column vector of coordinates of a point, A an invertible square matrix and B a column vector. The entries of the matrix A and the column vector B can be taken as (global) coordinates in the group $\mathrm{GA}(S)$.

7. Any finite or countable group equipped with the discrete topology and the structure of a 0-dimensional differentiable manifold.

The *direct product of Lie groups* is the direct product of the corresponding abstract groups endowed with the differentiable structure of the direct product of differentiable manifolds.

The Lie group K^n (the direct product of n copies of the additive group of the field K) is called the n-dimensional *vector Lie group*. The Lie group $\mathbb{T}^n$ (the direct product of n copies of the group $\mathbb{T}$) is called the *n-dimensional torus*.

1.2. Lie Subgroups. A subgroup H of a Lie group G is said to be a *Lie subgroup* if it is a submanifold of the underlying manifold of G.

Let us recall that by a m-dimensional submanifold of an n-dimensional manifold X we mean a subset $Y \subset X$ such that for each of its points y one of the following equivalent conditions is satisfied:

(1) in a local coordinate system in some neighbourhood U of the point y the subset $Y \cap U$ can be described parametrically in the form

$$x_i = \phi_i(t_1, \dots, t_m) \qquad (i = 1, \dots, n)$$

where $\phi_1, \dots, \phi_n$ are differentiable functions defined in some domain of the space K^m and the rank of the matrix $\frac{\partial(\phi_1,\dots,\phi_n)}{\partial(t_1,\dots,t_m)}$ at all points of this domain is equal to m.

(2) in a local coordinate system in some neighbourhood U of the point y the set $Y \cap U$ can be given by equations of the form

$$f_i(x_1, \dots, x_n) = 0 \qquad (i = 1, \dots, n-m),$$

where $f_1, \dots, f_{n-m}$ are differentiable functions and the rank of the matrix $\frac{\partial(f_1,\dots,f_{n-m})}{\partial(x_1,\dots,x_n)}$ at all points of the neighbourhood U is $n-m$.

(3) in a suitable local coordinate system in some neighbourhood U of the point y the subset $Y \cap U$ is given by equations

$$x_{m+1} = \dots = x_n = 0.$$

(Sometimes the terms 'submanifold' and correspondingly 'Lie subgroup' are understood in a wider sense. In this book this wider meaning is referred to by the term 'virtual Lie subgroup'; see 2.3 of Chap. 2. Lie subgroups in our sense are also known as 'closed Lie subgroups'.)

Every m-dimensional submanifold of a differentiable manifold carries the structure of a m-dimensional differentiable manifold, as local coordinates on which we can take, for example, the parameters $t_1, \dots, t_m$ from condition (1). Every Lie subgroup, endowed with this differentiable structure is itself a Lie group.

From the topological and the differential geometric viewpoints every subgroup H of a Lie group G looks at any point $h \in H$ the same as at the identity, since it is transformed into itself by a translation (left or right) by h, which is a diffeomorphism of the manifold G. Therefore in order to verify that a subgroup H is a Lie subgroup it suffices to establish that it is a submanifold in some neighbourhood of the identity.

Examples. 1. Any subspace of a vector space is a Lie subgroup of the corresponding Lie group.

2. The group $\mathbb{T}$ (see Example 3 of 1.1) is a Lie subgroup of the group $\mathbb{C}^\times$, viewed as a real Lie group.

3. Any discrete subgroup is a Lie subgroup.

4. The group of non-singular diagonal matrices is a Lie subgroup of the Lie group $\mathrm{GL}_n(K)$.

5. The group of non-singular triangular matrices is a Lie subgroup of the Lie group $\mathrm{GL}_n(K)$.

6. The group $\mathrm{SL}_n(K)$ of unimodular matrices is a codimension 1 Lie subgroup of the Lie group $\mathrm{GL}_n(K)$.

7. The group $\mathrm{O}_n(K)$ of orthogonal matrices is a Lie subgroup of dimension $\frac{n(n-1)}{2}$ of the Lie group $\mathrm{GL}_n(K)$.

8. The group $\mathrm{Sp}_n(K)$ (n even) of symplectic matrices is a Lie subgroup of dimension $\frac{n(n+1)}{2}$ of the Lie group $\mathrm{GL}_n(K)$.

9. The group U_n of unitary matrices is a real Lie subgroup of dimension n^2 of the Lie group $\mathrm{GL}_n(\mathbb{C})$.

A Lie subgroup of the Lie group $\mathrm{GL}_n(V)$ (and in particular of $\mathrm{GL}_n(K) = \mathrm{GL}(K^n)$) is called a *linear Lie group*.

As any submanifold, a Lie subgroup is an open subset of its closure. However, any open subgroup of a topological group is at the same time closed, since it is the complement of the union of its own cosets, which, like the subgroup itself, are open subsets. Hence any Lie subgroup is closed. For real Lie groups the converse is also valid, see Theorem 3.6 of Chap. 2.

1.3. Homomorphisms of Lie Groups. Let G and H be Lie groups. A map $f : G \to H$ is a *homomorphism* if it is simultaneously a homomorphism of abstract groups and a differentiable map. A homomorphism $f : G \to H$ is called an *isomorphism* if there exists an inverse $f^{-1} : H \to G$, i.e. if f is simultaneously an isomorphism of abstract groups and a diffeomorphism of manifolds (however, in connection with this, see the corollary to Theorem 3.4).

Examples. 1. The exponential map $x \mapsto e^x$ is a homomorphism from the additive Lie group K to the Lie group $K^\times$

2. The map $A \mapsto \det A$ is a homomorphism from the Lie group $\mathrm{GL}_n(K)$ to the Lie group $K^\times$

3. For any element g of a Lie group G the inner automorphism $a(g) : x \mapsto gxg^{-1}$ is a Lie group automorphism.

4. The map $x \mapsto e^{ix}$ is a homomorphism from the Lie group $\mathbb{R}$ to the Lie group $\mathbb{T}$.

5. The map assigning to each affine transformation of an affine space its differential (linear part) is a homomorphism from the Lie group $\mathrm{GA}(S)$ (see Example 6 of 1.1) to the Lie group $\mathrm{GL}(V)$, where V is the vector space associated with S.

6. Any homomorphism from a finite or a countable group to a Lie group is a homomorphism in the sense of the theory of Lie groups.

Obviously the composition of homomorphisms of Lie groups is also a homomorphism of Lie groups.

1.4. Linear Representations of Lie Groups. A homomorphism from a Lie group G to the Lie group $\mathrm{GL}(V)$ is called its *linear representation* in the space V.

For example, if to each matrix $A \in \mathrm{GL}_n(K)$ we assign the transformations $\mathrm{Ad}(A)$ and $\mathrm{Sq}(A)$ of the space $\mathrm{L}_n(K)$, defined by the formulas

$$\mathrm{Ad}(A)X = AXA^{-1}, \ \mathrm{Sq}(A)X = AXA^T, \tag{1}$$

then we obtain linear representations Ad and Sq of the Lie group $\mathrm{GL}_n(K)$ in the space $\mathrm{L}_n(K)$.

Sometimes one considers complex linear representations of real Lie groups or real linear representations of complex Lie groups. In the former case, it is understood that the group of linear transformations of a complex vector space is being considered as a real Lie group, in the latter – that the given complex Lie group is being considered as a real one.

Let R and S be linear representations of some group G in spaces V and U respectively. Recall that, by the sum of representations R and S, is meant the linear representation $R + S$ of the group G in the space $V \oplus U$, defined by the formula

$$(R+S)(g)(v+u) = R(g)v + S(g)u \tag{2}$$

by the product of the representations R and S the linear representation RS of the group G in the space $V \otimes U$, defined on decomposable elements by the formula

$$(RS)(g)(u \otimes v) = R(g)v \otimes S(g)u \tag{3}$$

The sum and product of an arbitrary number of representations are defined analogously.

By the dual representation of a representation R we mean the representation R^* of the group G in the space V^* – the dual of V, given by the formula

$$(R^*(g)f)(v) = f(R(g)^{-1}v) \tag{4}$$

It is easy to see that, if R and S are linear representations of a Lie group G, then the representations $R+S$, RS and R^* are also linear representations of it as a Lie group (i.e. they are differentiable).

For any integers $k, l \geq 0$ the identity linear representation Id of the group $\mathrm{GL}(V)$ in the space V generates its linear representation $T_{k,l} = Id^k (Id^*)^l$ in the space $\underbrace{V \otimes \ldots \otimes V}_{k} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{l}$ of tensors of type (k,l) on V. We will give convenient interpretations of representations $T_{k,l}$ in the two most commonly met cases: $k=0$ and $k=1$. Tensors of type $(0,l)$ can be viewed as l-linear forms on V. For any such form f we have

$$(T_{0,l}(A)f)(v_1, \ldots, v_l) = f(A^{-1}v_1 \ldots, A^{-1}v_l) \tag{5}$$

Tensors of type $(1,l)$ can be viewed as l-linear maps $V \times \ldots \times V \to V$. For any such map F we have

$$(T_{1,l}(A)F)(v_1, \ldots, v_l) = AF(A^{-1}v_1 \ldots, A^{-1}v_l) \tag{6}$$

The representations Ad and Sq of the group $\mathrm{GL}_n(K)$ considered above, are just its representations in the spaces of tensors (on K^n) of type $(1,1)$ and $(2,0)$ respectively, expressed in the matrix form.

If R is a linear representation of some group G in a space V and $U \subset V$ is an invariant subspace, there is a natural way to define the subrepresentation $R_U : G \to \mathrm{GL}(U)$ and the quotient representation $R_{V/U} : G \to \mathrm{GL}(V/U)$. Clearly, every subrepresentation and every quotient representation of a linear representation of a Lie group G are linear representations of it as a Lie group.

A special role in group theory is played by one-dimensional representations, which are precisely the homomorphisms from the given group to the multiplicative group of the base field. They are referred to as characters [1] of the group G. Characters form a group with respect to the operation of multiplication of representations; the inverse of an element in this group is its dual representation. We will denote the group of characters of a group G by

[1] Here the word character is being used in its narrower sense. In its wider sense character refers to the trace of any (not necessarily one-dimensional) linear representation.