Problems in Perturbation

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Preface

Many of the problems facing physicists, engineers, and applied mathematicians involve difficulties such as nonlinear governing equations, nonlinear boundary conditions at complex known or unknown boundaries, and variable coefficients that preclude exact solutions. Consequently, solutions are approximated using numerical techniques, analytic techniques, and combinations of both. For given initial and boundary conditions and specified parameters, one can use modern computers to integrate linear and nonlinear differential equations fairly accurately. However, if one needs to obtain some insight into the character of the solutions of nonlinear problems and their dependence on certain parameters, one may need to repeat the calculations for many different values of the parameters and initial conditions. Even for simple nonlinear problems the output may be so large that it is difficult to recognize even simple general phenomena. On the other hand, analytic methods often easily delineate general phenomena, yielding useful results in closed form. In the case of nonlinear partial differential equations with variable coefficients and complicated boundaries, the combination of an analytic and a numerical method often provides an optimum procedure. The linear problem is solved using the numerical method and the Ritz–Galerkin procedure can be used to reduce the problem to solving an infinite number of coupled nonlinear ordinary differential equations, which are solved using the analytic method.

Often one is interested in a situation in which one or more of the parameters become either very large or very small. Typically these are difficult situations to treat by straightforward numerical procedures. In these situations, analytic methods can often provide an accurate approximation and even suggest a way to improve the numerical procedure.

Foremost among the analytic techniques are the systematic methods of perturbation (asymptotic expansions) in terms of a small or a large parameter or coordinate. The book *Perturbation Methods* presents in a unified way an account of most of the perturbation techniques devised by physicists, engineers, and applied mathematicians, pointing out their similarities, differences, advantages, and limitations. However, the material is concise and advanced and, therefore, is intended for researchers and advanced graduate students. In *Introduction to Perturbation Techniques* the material is presented in an elementary way, making it easily accessible to advanced undergraduates and first-year graduate students in a wide variety of scientific and engineering fields. In both books the material is presented using examples; the second volume contains more than 360 problems.

Problems in Perturbation contains detailed solutions of all the problems in *Introduction to Perturbation Techniques* and about an equal number of unsolved supplementary problems. Each chapter begins with a short introduction that gives a summary of the definitions, basic theory, and available methods. The material is self-contained for the reader who has a background in calculus and elementary ordinary-differential equations. Although the solved problems are the exercises in *Introduction to Perturbation Techniques*, the material is general and could be used to accompany any of the existing books on perturbations, as well as those on nonlinear oscillations and applied mathematics that include asymptotics and perturbations. Since perturbation techniques are best explained using examples, this book is ideal for self-study.

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CHAPTER 1

Introduction

Most of the physical problems facing physicists, engineers, and applied mathematicians today exhibit certain essential features that preclude exact analytical solutions. These features include nonlinear governing equations, nonlinear boundary conditions at known or, in some cases, unknown boundaries, variable coefficients, and complex boundary shapes. Hence physicists, engineers, and applied mathematicians are forced to determine approximate solutions of the problems they are facing. The approximations may be purely numerical, purely analytical, or a combination of numerical and analytical techniques. In this book we concentrate on analytical techniques, which, when combined with a numerical method, yield very powerful and versatile techniques.

The analytical approximations can be broadly divided into rational and irrational. An irrational approximation is usually obtained by an ad hoc, mathematical-modeling process that involves keeping certain elements, neglecting some, and approximating yet others. Thus it represents a dead end, because the resulting accuracy cannot be improved by successive approximations. A rational approximation represents a systematic expansion, called asymptotic or perturbation, that can in principle be continued indefinitely.

1.1. Parameter Perturbations

The key to solving modern problems is mathematical modeling that involves deriving the governing equations and boundary and initial conditions. Then the mathematical problem should always be expressed in nondimensional or dimensionless variables before any approximations are attempted.

If the physical problem involving the dimensionless scalar or vector variable $u(x, \varepsilon)$ can be represented mathematically by the differential equation $L(y, x, \varepsilon) = 0$ and the boundary condition $B(x, \varepsilon) = 0$, where x is a scalar or vector-independent dimensionless variable and ε is a dimensionless parameter, it cannot, in general, be solved exactly. However, if there exists an $\varepsilon = \varepsilon_0$ (ε can be scaled so that $\varepsilon_0 = 0$) for which the problem given can be solved exactly, more readily, or numerically, one seeks to approximate the solution $u(x, \varepsilon)$ for small values of ε, say in an expansion in powers of ε in the form

$$u(x,\varepsilon)=u_0(x)+\varepsilon u_1(x)+\varepsilon^2 u_2(x)+\cdots$$

where the $u_n(x)$ are independent of ε and $u_0(x)$ is the solution of the problem when $\varepsilon=0$. Such expansions are called parameter perturbations.

1.2. Coordinate Perturbations

If the physical problem is represented mathematically by the differential equation $L(u,x)=0$ and the boundary condition $B(x)=0$, and if $u(x)$ takes a known form u_0 as $x\to x_0$ (x_0 can be scaled to either 0 or ∞), in a coordinate perturbation, one determines the deviation of u from u_0 for x near x_0 in terms of powers of x for $x_0=0$, or in terms of powers of x^{-1} for $x_0=\infty$. Examples of coordinate perturbations are

$$u=x^{\sigma}\sum_{n=1}^{\infty}a_n x^n$$

$$u=x^{\sigma}\sum_{n=1}^{\infty}a_n x^{-n}$$

$$u=e^{-2x}x^{\sigma}\sum_{n=1}^{\infty}a_n x^n$$

1.3. Gauge Functions

In both parameter and coordinate perturbations, one is interested in the behavior of functions such as $f(\varepsilon)$ as the dimensionless parameter or coordinate ε tends to a specific value ε_0 (ε can always be normalized so that $\varepsilon_0=0$). One way of classifying the function $f(\varepsilon)$ is based on its limit as $\varepsilon\to 0$; if this limit exists, there are three possibilities:

$$\left.\begin{aligned}f(\varepsilon)&\to 0\\ f(\varepsilon)&\to A\\ f(\varepsilon)&\to\infty\end{aligned}\right\}\quad \text{as } \varepsilon\to 0,\ 0<A<\infty$$

The classification given above based on the limit is not very useful because there are an infinite number of functions that tend to zero or infinity as $\varepsilon\to 0$. Therefore, to narrow down this classification, we subdivide the first and third classes according to the rate at which they tend to zero or infinity. To accomplish this, we compare the rate at which these functions tend to zero and infinity with the rate at which a set of gauge functions tends to zero and infinity. These gauge functions are so familiar that their limiting behavior is known intuitively. The simplest possible examples of gauge functions are the powers of ε.

1.4. Order Symbols

If

$$\lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{g(\varepsilon)}=A$$

where $0<A<\infty$, we write

$$f(\varepsilon)=O[g(\varepsilon)] \quad \text{as } \varepsilon\to 0$$

and say that $f(\varepsilon)$ is order $g(\varepsilon)$ as $\varepsilon\to 0$ or $f(\varepsilon)$ is big "oh" of $g(\varepsilon)$ as $\varepsilon\to 0$.

If

$$\lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{g(\varepsilon)}=0$$

we write

$$f(\varepsilon)=o[g(\varepsilon)] \quad \text{as } \varepsilon\to 0$$

and say that $f(\varepsilon)$ is little "oh" of $g(\varepsilon)$ as $\varepsilon\to 0$.

1.5. Asymptotic Series

Given a series $\sum_{n=0}^{\infty}(a_n/x^n)$, where the a_n are independent of x, we say that the series is an asymptotic series and write

$$f(x)\sim\sum_{n=0}^{\infty}\frac{a_n}{x^n} \quad \text{as } |x|\to\infty$$

if and only if

$$f(x)=\sum_{n=0}^{N}\frac{a_n}{x^n}+o\left(\frac{1}{|x|^N}\right) \quad \text{as } |x|\to\infty$$

which is equivalent to

$$f(x)=\sum_{n=0}^{N-1}\frac{a_n}{x^n}+O\left(\frac{1}{|x|^N}\right) \quad \text{as } |x|\to\infty$$

1.6. Asymptotic Sequences and Expansions

A sequence of functions $\delta_n(\varepsilon)$ is called an asymptotic sequence as $\varepsilon \to 0$ if

$$\delta_n(\varepsilon) = o[\delta_{n-1}(\varepsilon)] \quad \text{as } \varepsilon \to 0$$

Given an expansion $\sum_{n=0}^{\infty} a_n \delta_n(\varepsilon)$, where the a_n are independent of ε and $\delta_n(\varepsilon)$ is an asymptotic sequence, we say that this is an asymptotic expansion and write

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon) \quad \text{as } \varepsilon \to 0$$

if and only if

$$f(\varepsilon) = \sum_{n=0}^{N} a_n \delta_n(\varepsilon) + o[\delta_N(\varepsilon)] \quad \text{as } \varepsilon \to 0$$

which is equivalent to

$$f(\varepsilon) = \sum_{n=0}^{N-1} a_n \delta_n(\varepsilon) + O[\delta_N(\varepsilon)] \quad \text{as } \varepsilon \to 0$$

Clearly, an asymptotic series is a special case of an asymptotic expansion.

1.7. Convergent Versus Asymptotic Series

Let a function $f(x)$ be represented by the first N terms of a series in inverse powers of x plus a remainder $R_N(x)$ as

$$f(x) = \sum_{n=0}^{N} \frac{a_n}{x^n} + R_N(x)$$

where the a_n are independent of x. This series converges if and only if

$$\lim_{\substack{N \to \infty \\ x \text{ fixed}}} R_N(x) = 0$$

This series is an asymptotic series as $|x| \to \infty$ if and only if

$$R_N(x) = o(|x|^{-N}) \quad \text{as } |x| \to \infty$$

Clearly, a convergent series is an asymptotic series; however, an asymptotic series need not converge.

1.8. Nonuniform Expansions

An asymptotic expansion of more than one variable, such as

$$f(x,\varepsilon) \sim \sum_{n=0}^{\infty} a_n(x)\delta_n(\varepsilon) \quad \text{as } \varepsilon \to 0$$

is uniform if and only if

$$f(x,\varepsilon) = \sum_{n=0}^{N} a_n(x)\delta_n(\varepsilon) + R_N(x,\varepsilon) \quad \text{as } \varepsilon \to 0$$

where $R_N(x,\varepsilon) = o[\delta_N(\varepsilon)]$ as $\varepsilon \to 0$ for all x in the domain of interest.

Solved Exercises

1.1. For small ε determine three terms in the expansions of

(a) $(1-\frac{3}{8}a^2\varepsilon+\frac{51}{256}a^4\varepsilon^2)^{-1}=1+(\frac{3}{8}a^2\varepsilon-\frac{51}{256}a^4\varepsilon^2)+(\frac{3}{8}a^2\varepsilon-\frac{51}{256}a^4\varepsilon^2)^2$
$+\cdots=1+\frac{3}{8}a^2\varepsilon-\frac{51}{256}a^4\varepsilon^2+\frac{9}{64}a^4\varepsilon^2+\cdots=1+\frac{3}{8}a^2\varepsilon-\frac{15}{256}a^4\varepsilon^2+\cdots$

(b) $\cos(\sqrt{1-\varepsilon t})=\cos\left[\left(1-\frac{1}{2}\varepsilon t+\frac{\frac{1}{2}\times-\frac{1}{2}}{2!}\varepsilon^2t^2+\cdots\right)\right]$
$=\cos[(1-\frac{1}{2}\varepsilon t-\frac{1}{8}\varepsilon^2t^2)+\cdots]$
$=\cos 1-\sin 1(-\frac{1}{2}\varepsilon t-\frac{1}{8}\varepsilon^2t^2)-\frac{1}{2!}\cos 1(-\frac{1}{2}\varepsilon t-\frac{1}{8}\varepsilon^2t^2)^2+\cdots$
$=\cos 1+(\frac{1}{2}\varepsilon t+\frac{1}{8}\varepsilon^2t^2)\sin 1-\frac{1}{8}\varepsilon^2t^2\cos 1+\cdots$
$=\cos 1+\frac{1}{2}\varepsilon t\sin 1+\frac{1}{8}\varepsilon^2t^2(\sin 1-\cos 1)+\cdots$

(c) $\sqrt{1-\frac{1}{2}\varepsilon+2\varepsilon^2}=[1-(\frac{1}{2}\varepsilon-2\varepsilon^2)]^{1/2}=1-\frac{1}{2}(\frac{1}{2}\varepsilon-2\varepsilon^2)$
$+\frac{\frac{1}{2}\times-\frac{1}{2}}{2!}(\frac{1}{2}\varepsilon-2\varepsilon^2)^2+\cdots=1-\frac{1}{4}\varepsilon+\varepsilon^2-\frac{1}{32}\varepsilon^2+\cdots=1-\frac{1}{4}\varepsilon+\frac{31}{32}\varepsilon^2+\cdots$

(d) $\sin(1+\varepsilon-\varepsilon^2)=\sin 1+(\varepsilon-\varepsilon^2)\cos 1-\frac{1}{2!}(\varepsilon-\varepsilon^2)^2\sin 1+\cdots$
$=\sin 1+(\varepsilon-\varepsilon^2)\cos 1-\frac{1}{2}\varepsilon^2\sin 1+\cdots$
$=\sin 1+\varepsilon\cos 1-\varepsilon^2(\cos 1+\frac{1}{2}\sin 1)+\cdots$

1.2. Expand each of the following expressions for small ε and keep three terms:

(a) $\sqrt{1-\frac{1}{2}\varepsilon^2t-\frac{1}{8}\varepsilon^4t}=[1-(\frac{1}{2}\varepsilon^2t+\frac{1}{8}\varepsilon^4t)]^{1/2}$
$=1-\frac{1}{2}(\frac{1}{2}\varepsilon^2t+\frac{1}{8}\varepsilon^4t)+\frac{\frac{1}{2}\times-\frac{1}{2}}{2!}(\frac{1}{2}\varepsilon^2t+\frac{1}{8}\varepsilon^4t)^2+\cdots$
$=1-\frac{1}{4}\varepsilon^2t-\frac{1}{16}\varepsilon^4t-\frac{1}{32}\varepsilon^4t^2+\cdots=1-\frac{1}{4}\varepsilon^2t-\frac{1}{16}\varepsilon^4t(1+\frac{1}{2}t)+\cdots$

(b) $(1+\varepsilon\cos f)^{-1}=1-\varepsilon\cos f+\varepsilon^2\cos^2 f+\cdots$

(c) $(1+\varepsilon\omega_1+\varepsilon^2\omega_2)^{-2}=1-2(\varepsilon\omega_1+\varepsilon^2\omega_2)+\frac{-2\times-3}{2!}(\varepsilon\omega_1+\varepsilon^2\omega_2)^2+\cdots$
$=1-2\varepsilon\omega_1+\varepsilon^2(-2\omega_2+3\omega_1^2)+\cdots$

(d) $\sin(s+\varepsilon\omega_1 s+\varepsilon^2\omega_2 s)=\sin s+(\varepsilon\omega_1 s+\varepsilon^2\omega_2 s)\cos s-\frac{1}{2!}(\varepsilon\omega_1 s+\varepsilon^2\omega_2 s)^2\sin s$
$+\cdots=\sin s+\varepsilon\omega_1 s\cos s+\varepsilon^2 s(\omega_2\cos s-\frac{1}{2}\omega_1^2 s\sin s)+\cdots$

(e) $\sin^{-1}\left(\dfrac{\varepsilon}{\sqrt{1+\varepsilon}}\right) = \sin^{-1}[\varepsilon(1+\varepsilon)^{-1/2}]$

$= \sin^{-1}\left[\varepsilon\left(1-\tfrac{1}{2}\varepsilon+\dfrac{-\frac{1}{2}\times-\frac{3}{2}}{2!}\varepsilon^2\right)+\cdots\right] = \sin^{-1}(\varepsilon-\tfrac{1}{2}\varepsilon^2+\tfrac{3}{8}\varepsilon^3+\cdots)$

$= (\varepsilon-\tfrac{1}{2}\varepsilon^2+\tfrac{3}{8}\varepsilon^3)+\tfrac{1}{3!}(\varepsilon-\tfrac{1}{2}\varepsilon^2+\tfrac{3}{8}\varepsilon^3)^3+\cdots$

$= \varepsilon-\tfrac{1}{2}\varepsilon^2+\tfrac{3}{8}\varepsilon^3+\tfrac{1}{6}\varepsilon^3+\cdots = \varepsilon-\tfrac{1}{2}\varepsilon^2+\tfrac{13}{24}\varepsilon^3+\cdots$

(f) $\ln\dfrac{1+2\varepsilon-\varepsilon^2}{\sqrt[3]{1+2\varepsilon}} = \ln(1+2\varepsilon-\varepsilon^2)-\tfrac{1}{3}\ln(1+2\varepsilon)$

$= 2\varepsilon-\varepsilon^2-\tfrac{1}{2}(2\varepsilon-\varepsilon^2)^2+\tfrac{1}{3}(2\varepsilon-\varepsilon^2)^3$

$+\cdots-\tfrac{1}{3}[2\varepsilon-\tfrac{1}{2}(2\varepsilon)^2+\tfrac{1}{3}(2\varepsilon)^3+\cdots] = \tfrac{4}{3}\varepsilon-\tfrac{7}{3}\varepsilon^2+\tfrac{34}{9}\varepsilon^3+\cdots$

1.3. Let $\mu = \mu_0 + e\mu_1 + e^2\mu_2$ in $h = \frac{3}{2}\left[1-\sqrt{1-3\mu(1-\mu)}\right]$, expand for small e, and keep three terms.

Using a Taylor series expansion, we have

$$h(\mu_0+e\mu_1+e^2\mu_2) = h(\mu_0)+h'(\mu_0)(e\mu_1+e^2\mu_2)+\tfrac{1}{2!}h''(\mu_0)(e\mu_1+e^2\mu_2)^2+\cdots$$

But

$$h'(\mu) = -\tfrac{3}{2}\cdot\tfrac{1}{2}[1-3\mu(1-\mu)]^{-1/2}\times(-3+6\mu)$$

$$= \tfrac{9}{4}(1-2\mu)[1-3\mu(1-\mu)]^{-1/2}$$

$$h''(\mu) = -\tfrac{9}{2}[1-3\mu(1-\mu)]^{-1/2}-\tfrac{1}{2}\times\tfrac{9}{4}(1-2\mu)[1-3\mu(1-\mu)]^{-3/2}\times(-3+6\mu)$$

$$= -\tfrac{9}{2}[1-3\mu(1-\mu)]^{-1/2}+\tfrac{27}{8}(1-2\mu)^2[1-3\mu(1-\mu)]^{-3/2}$$

Therefore,

$$h = \tfrac{3}{2}\left[1-\sqrt{1-3\mu_0(1-\mu_0)}\right]+\tfrac{9}{4}(1-2\mu_0)[1-3\mu_0(1-\mu_0)]^{-1/2}\mu_1 e$$

$$+e^2\left\{\tfrac{9}{4}(1-2\mu_0)[1-3\mu_0(1-\mu_0)]^{-1/2}\mu_2\right.$$

$$-\tfrac{9}{4}[1-3\mu_0(1-\mu_0)]^{-1/2}\mu_1^2+\tfrac{27}{16}(1-2\mu_0)^2$$

$$\left.\times[1-3\mu_0(1-\mu_0)]^{-3/2}\mu_1^2\right\}+\cdots$$

1.4. For small ε, determine the order of the functions, $\sinh(1/\varepsilon)$, $\ln(1+\sin\varepsilon)$, $\ln(2+\sin\varepsilon)$, and $e^{\ln(1-\varepsilon)}$.

Since as $\varepsilon \to 0$,

$$\sinh\frac{1}{\varepsilon} = \frac{e^{1/\varepsilon}-e^{-1/\varepsilon}}{2} \approx \tfrac{1}{2}e^{1/\varepsilon}$$

$$\sinh\frac{1}{\varepsilon} = O(e^{1/\varepsilon}) \quad \text{as } \varepsilon\to 0$$

Since as $\varepsilon \to 0$,

$$\ln(1+\sin\varepsilon) \approx \ln(1+\varepsilon) \approx \varepsilon$$

$$\ln(1+\sin\varepsilon) = O(\varepsilon) \quad \text{as } \varepsilon \to 0$$

Since as $\varepsilon \to 0$,

$$\ln(2+\sin\varepsilon) \approx \ln(2+\varepsilon) \approx \ln 2 + \ln(1+\tfrac{1}{2}\varepsilon) \approx \ln 2 + \tfrac{1}{2}\varepsilon$$

$$\ln(2+\sin\varepsilon) = O(1) \quad \text{as } \varepsilon \to 0$$

Since as $\varepsilon \to 0$,

$$e^{\ln(1-\varepsilon)} \approx e^{-\varepsilon} \approx 1-\varepsilon$$

$$e^{\ln(1-\varepsilon)} = O(1) \quad \text{as } \varepsilon \to 0$$

1.5. Determine the order of the following expressions as $\varepsilon \to 0$:

(a) Since $\sqrt{\varepsilon(1-\varepsilon)} \approx \sqrt{\varepsilon}$

$$\sqrt{\varepsilon(1-\varepsilon)} = O(\varepsilon^{1/2})$$

(b) $4\pi^2\varepsilon = O(\varepsilon)$

(c) $1000\varepsilon^{1/2} = O(\varepsilon^{1/2})$

(d) Since $\ln(1+\varepsilon) \approx \varepsilon$

$$\ln(1+\varepsilon) = O(\varepsilon)$$

(e) Since $\dfrac{1-\cos\varepsilon}{1+\cos\varepsilon} \approx \dfrac{1-1+\frac{1}{2}\varepsilon^2}{1+1-\frac{1}{2}\varepsilon^2} \approx \frac{1}{4}\varepsilon^2$

$$\frac{1-\cos\varepsilon}{1+\cos\varepsilon} = O(\varepsilon^2)$$

(f) Since $\dfrac{\varepsilon^{3/2}}{1-\cos\varepsilon} \approx \dfrac{\varepsilon^{3/2}}{1-1+\frac{1}{2}\varepsilon^2} = 2\varepsilon^{-1/2}$

$$\frac{\varepsilon^{3/2}}{1-\cos\varepsilon} = O(\varepsilon^{-1/2})$$

(g) Let $\operatorname{sech}^{-1}\varepsilon = u$ so that $\operatorname{sech} u = \varepsilon$ and $\cosh u = 1/\varepsilon$. Hence

$$e^u + e^{-u} = \frac{2}{\varepsilon}$$

As $\varepsilon \to 0$, $u \to \infty$ and hence

$$e^u \approx \frac{2}{\varepsilon} \quad \text{or} \quad u \approx \ln 2 + \ln\frac{1}{\varepsilon}$$

Therefore,

$$\operatorname{sech}^{-1}\varepsilon = O\left(\ln\frac{1}{\varepsilon}\right)$$

(h) Since $e^{\tan\varepsilon} \approx e^{\varepsilon} \approx 1$

$$e^{\tan\varepsilon} = O(1)$$

(i) Since $\ln\left[1+\frac{\ln(1+2\varepsilon)}{\varepsilon(1-2\varepsilon)}\right] \approx \ln\left(1+\frac{2\varepsilon}{\varepsilon}\right) = \ln 3$

$$\ln\left[1+\frac{\ln(1+2\varepsilon)}{\varepsilon(1-2\varepsilon)}\right] = O(1)$$

(j) Since $\ln\left[1+\frac{\ln[(1+2\varepsilon)/\varepsilon]}{1-2\varepsilon}\right] \approx \ln\left[1+\frac{\ln(1/\varepsilon)}{1}\right] \approx \ln\left(\ln\frac{1}{\varepsilon}\right)$

$$\ln\left[1+\frac{\ln[(1+2\varepsilon)/\varepsilon]}{1-2\varepsilon}\right] = O\left[\ln\left(\ln\frac{1}{\varepsilon}\right)\right]$$

(k) Since $e^{-\cosh(1/\varepsilon)} \approx \exp\left(-\frac{1}{2}\exp\frac{1}{\varepsilon}\right)$

$$e^{-\cosh(1/\varepsilon)} = O\left[\exp\left(-\frac{1}{2}\exp\frac{1}{\varepsilon}\right)\right]$$

(l) Since $\int_0^{\varepsilon} e^{-s^2}\,ds \approx \int_0^{\varepsilon}(1-s^2)\,ds \approx \varepsilon$

$$\int_0^{\varepsilon} e^{-s^2}\,ds = O(\varepsilon)$$

1.6. Determine the order of the following expressions as $\varepsilon \to 0$:

(a) Since $\ln(1+5\varepsilon) \approx 5\varepsilon$

$$\ln(1+5\varepsilon) = O(\varepsilon)$$

(b) Since $\sin^{-1}\frac{\varepsilon}{\sqrt{1+\varepsilon}} \approx \sin^{-1}\varepsilon \approx \varepsilon$

$$\sin^{-1}\frac{\varepsilon}{\sqrt{1+\varepsilon}} = O(\varepsilon)$$

(c) Since $\frac{\sqrt{\varepsilon}}{\sin\varepsilon} \approx \frac{\sqrt{\varepsilon}}{\varepsilon} = \varepsilon^{-1/2}$

$$\frac{\sqrt{\varepsilon}}{\sin\varepsilon} = O(\varepsilon^{-1/2})$$

(d) Since $1-\frac{1}{2}\varepsilon^2-\cos\varepsilon \approx 1-\frac{1}{2}\varepsilon^2-(1-\frac{1}{2!}\varepsilon^2+\frac{1}{4!}\varepsilon^4) = -\frac{1}{4!}\varepsilon^4$

$$1-\frac{1}{2}\varepsilon^2-\cos\varepsilon = O(\varepsilon^4)$$

(e) Since $\ln\left(\sinh\frac{1}{\varepsilon}\right) \approx \ln\left[\frac{e^{1/\varepsilon}-e^{-1/\varepsilon}}{2}\right] \approx \ln\left(\frac{1}{2}e^{1/\varepsilon}\right) \approx \frac{1}{\varepsilon}$

$$\ln\left(\sinh\frac{1}{\varepsilon}\right) = O\left(\frac{1}{\varepsilon}\right)$$

1.7. Determine the order of the following as $\varepsilon \to 0$:

(a) Since $\ln(\cot\varepsilon) = \ln\left(\frac{\cos\varepsilon}{\sin\varepsilon}\right) \approx \ln\left(\frac{1}{\varepsilon}\right)$

$$\ln(\cot\varepsilon) = O\left(\ln\frac{1}{\varepsilon}\right)$$

(b) Since $\sinh\frac{1}{\varepsilon} = \frac{e^{1/\varepsilon}-e^{-1/\varepsilon}}{2} \approx \frac{1}{2}e^{1/\varepsilon}$

$$\sinh\frac{1}{\varepsilon} = O(e^{1/\varepsilon})$$

(c) Since $\coth\left(\frac{1}{\varepsilon}\right) = \frac{\cosh(1/\varepsilon)}{\sinh(1/\varepsilon)} = \frac{e^{1/\varepsilon}+e^{-1/\varepsilon}}{e^{1/\varepsilon}-e^{-1/\varepsilon}} \approx 1$

$$\coth\left(\frac{1}{\varepsilon}\right) = O(1)$$

(d) Since $\frac{\varepsilon^{3/4}}{1-\cos\varepsilon} \approx \frac{\varepsilon^{3/4}}{1-\left(1-\frac{1}{2}\varepsilon^2\right)} \approx \frac{\varepsilon^{3/4}}{\frac{1}{2}\varepsilon^2} = 2\varepsilon^{-5/4}$

$$\frac{\varepsilon^{3/4}}{1-\cos\varepsilon} = O(\varepsilon^{-5/4})$$

(e) Since $\ln\left[1+\ln\frac{1+2\varepsilon}{\varepsilon}\right] \approx \ln\left[1+\ln\frac{1}{\varepsilon}\right] \approx \ln\left(\ln\frac{1}{\varepsilon}\right)$

$$\ln\left[1+\ln\frac{1+2\varepsilon}{\varepsilon}\right] = O\left[\ln\left(\ln\frac{1}{\varepsilon}\right)\right]$$

1.8. Arrange the following in descending order for small ε:

$$\varepsilon^2,\ \varepsilon^{1/2},\ \ln\left(\ln\frac{1}{\varepsilon}\right),\ 1,\ \varepsilon^{1/2}\ln\frac{1}{\varepsilon},\ \varepsilon\ln\frac{1}{\varepsilon},\ e^{-1/\varepsilon},\ \ln\frac{1}{\varepsilon},\ \varepsilon^{3/2},\ \varepsilon,\ \varepsilon^2\ln\frac{1}{\varepsilon}$$

Answer:

$$\ln\frac{1}{\varepsilon} > \ln\left(\ln\frac{1}{\varepsilon}\right) > 1 > \varepsilon^{1/2}\ln\frac{1}{\varepsilon} > \varepsilon^{1/2} > \varepsilon\ln\frac{1}{\varepsilon} > \varepsilon > \varepsilon^{3/2} > \varepsilon^2\ln\frac{1}{\varepsilon} > \varepsilon^2 > e^{-1/\varepsilon}$$

1.9. Arrange the following in descending order for small ε:

$$e^{-1/\varepsilon},\ \ln\frac{1}{\varepsilon},\ \varepsilon^{-0.01},\ \cot\varepsilon, \text{ and } \sinh\frac{1}{\varepsilon}$$

Answer:

$$\sinh\frac{1}{\varepsilon} > \cot\varepsilon > \varepsilon^{-0.01} > \ln\frac{1}{\varepsilon} > e^{-1/\varepsilon}$$