COLLEGE ALGEBRA.

BY

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PREFACE.

THIS text is intended primarily for the college and the technical school. By treating only the subjects usually given in the college course in algebra, space has been gained for a more detailed exposition of the more difficult topics. As the extent and the character of the review at the beginning of the course upon topics prescribed for entrance to college varies so widely, and as the review is usually conducted in an informal manner, it seemed best to the author to leave to the instructor the review of the elementary principles, but to give in the text review exercises.

As to the order of the subjects, the aim has been to present first those topics which are readily mastered by the student, and to reserve for the latter part of the text the questions grouped around the subjects involving infinite series. In reviewing the subject of simultaneous equations, the student is led naturally, almost unavoidably, to the determinant notation. Determinants are therefore treated early in the text, an order of presentation shown by actual experience to give very satisfactory results, especially in arousing the interest of the student.

In the chapter on graphic algebra, the first principles of coördinate geometry are introduced and applied to the study of simultaneous equations and inequalities. In this connection is presented an elementary account of the solution of numerical equations, chiefly from the graphic standpoint. The arrangement is such as to admit of a very brief course or of a fuller course involving Horner's method of synthetic division. The practice of emphasizing graphic algebra in courses for technical students commends itself also to the general student.

Attention is invited to the proofs of the fundamental theorems on logarithms, the treatment of mathematical induction and the illustrations showing the necessity of the different steps in the process, the examples from the physical sciences of the topic variation, the complete proof of the general binomial theorem independent of the principle "permanence of form," the establishment of the relations between the roots and the coefficients of the quadratic, cubic, and quartic equations prior to the proof of the general theorem, the solution of those equations before introducing the assumption that every equation has a root (here proved in the Appendix).

The attempt has been made to present the subjects limits and infinite series in as simple form as is consistent with rigor.

Forty-five sets of exercises, averaging over fifteen to a set, are given at very short intervals in the text. Some historical data have been introduced, with no attempt to give the source, the subject-matter being classical.

The author is under great obligation to Dr. Moulton, of the Department of Astronomy of the University of Chicago, who read with care the entire manuscript and offered numerous suggestions as to the form of presentation, most of which have been adopted. Likewise the thanks of the author are due Professor J. W. A. Young of the same University, who examined the more critical chapters and offered valuable suggestions.

Proof-sheets of the entire book were carefully read by Professor L. L. Conant of the Worcester Polytechnic Institute, whose corrections and suggestions have led to numerous improvements. Finally, the author is indebted to Professors Moore and Young of the University of Chicago, who examined critically the proof-sheets of certain portions of the text.

CHICAGO, January, 1902.

SYMBOLS AND ABBREVIATIONS.

$=$, equal.

$\neq$, not equal.

$\doteq$, approaches, 106.

∞, infinity, 69, 105.

$e = 2.71828\ldots$, 137.

$i = \sqrt{-1}$.

${}_nC_r$, ${}_nP_r$, 86.

A.P., G.P., H.P., 64.

sin, cos, 208.

$|t|$ = absolute value of t, 114.

$n! = 1 \times 2 \times 3 \times \ldots \times n$, 86.

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APPENDIX.

COLLEGE ALGEBRA.

CHAPTER I.

NUMBER IN ALGEBRA; SURDS AND IMAGINARIES.

1. The **natural numbers** or **positive integers** (1, 2, 3, ...) make it possible to enumerate the objects of a group considered for the time as equivalent entities. It has been established that primitive counting was done on the fingers and that in many languages the numeral 5 is merely the word for hand, 10 for both hands, and 20 for the whole man (hands and feet).* While 5, 10, 20, and 60 have been used as bases, 10 is the usual base.

If a group of objects can be partitioned into equivalent smaller groups, each smaller group or a combination of them is a **fraction** ($\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, ...) of the original group. Abstractly, a fraction is the quotient of two positive integers. Fractional results may or may not admit of an interpretation in a particular problem. A shepherd would declare it to be impossible to separate his flock of 50 sheep into three equal flocks; but would find no theoretical difficulty in dividing a 50-foot rope into three equal pieces. The algebraic statement for each problem is the same: to find x such that

* Thus in the language of the Tamanacs, the word for 6 is "one on the other hand"; the word for 12 is "two to the foot"; for 15, "a whole foot"; for 16, "one to the other foot"; the word for 20 is "one Indian"; for 40, "two Indians"; etc.

$3x = 50$. The formal solution is $x = \frac{50}{3}$; the possibility of its interpretation depends upon the character of the special problem.

The Egyptians used fractions before 1700 B.C.* and resolved them into sums of unit fractions. Thus $\frac{2}{5}$ was written $\dot{3}\ 1\dot{5}$, which meant $\frac{1}{3} + \frac{1}{15}$. By the Babylonians, $\frac{5}{8}$ was written 37 30, which meant $\frac{37}{60} + \frac{30}{60^2}$. Instead of fractions, the Greek geometers used the ratio of two numbers or two magnitudes.

For the introduction of **negative** numbers (as well as the decimal positional system and the symbol 0), algebra is indebted to the early mathematicians of India (between 500 and 600 A.D.). We now find it convenient to write $-15°$ for 15° below zero; -100 ft. for 100 feet below sea-level, thereby abbreviating our map notations. The determination of a number x such that $b + x = c$ leads to the algebraic solution $x = c - b$. If b exceeds c, the result $c - b$ is a negative number. Such a result would be excluded if the problem were to find how many feet of rope must be added to a rope b feet long to make a rope c feet long. But the negative result leads us to restate the problem so that the required c-foot rope is seen to be obtained by cutting off $b - c$ feet of rope from the b-foot rope.

In this connection, we note the Roman notations IV for 4, VI for 6, IX for 9, XI for 11, which seem to have been of Etruscan origin.

The term **rational number** includes positive and negative integers and fractions. All other numbers are called **irrational.**

The solution of equations of the form $x^n = A$, where A is a rational number and n a positive integer, introduces two classes of irrational numbers. Thus, for $n = 2$, and A a positive integer not the square of a rational number, the square root of A, denoted by the symbol $\sqrt{A}$, is an irrational number called a **quadratic surd.** Similarly, $\sqrt[3]{2}$, $\sqrt[3]{-4}$, $\sqrt[3]{A}$, A not the cube of a rational number,

* Rhind papyrus, "Directions for Attaining to the Knowledge of All Dark Things."

are **surds of the third order.** In general, $\sqrt[n]{A}$, where A is positive if n is even, is a **surd of order** n. The second class of irrational numbers are defined by the symbols $\sqrt[n]{A}$, where A is negative and n even (§ 4).

While the equation $x^2 = 80$ possesses the formal solutions $x = \pm\sqrt{80}$ (the positive root is designated $\sqrt{80}$, the negative $-\sqrt{80}$), the possibility of the interpretation of one or both results depends upon the character of the particular problem. It is possible to form a square of area 80 square feet, but impossible to arrange 80 square blocks of equal size in the form of a square and yet preserve the form of each block.

2. The fact that surds really exist as such may be illustrated by showing that $\sqrt{2}$ is not expressible as a rational number. If we take the side of a square as the unit of length, the diagonal is of length $\sqrt{2}$. But it is proved in Geometry that the side and diagonal are incommensurable (see § 55). Hence 1 and $\sqrt{2}$ have no common measure. It follows that $\sqrt{2}$ is not equal to a rational number. For, if

$$(1) \qquad \sqrt{2} = \frac{p}{q},$$

then $\frac{1}{q}$ would be contained p times in $\sqrt{2}$ and q times in 1 and hence be a common measure of 1 and $\sqrt{2}$.

To give a purely algebraic proof, suppose that equation (1) holds true, the fraction $\frac{p}{q}$ being in its lowest terms, so that p and q are integers having no common divisor. By squaring and multiplying by q^2, we get $2q^2 = p^2$, so that p^2 and therefore p is divisible by 2. Setting $p = 2r$, we get $q^2 = 2r^2$, so that q is divisible by 2. Then p and q have a common divisor 2, contrary to hypothesis.

3. But with the introduction of rational numbers and surds, we do not meet all the demands which are made upon a number system. We are led in Geometry to consider the number π which expresses the ratio of the circumference to the diameter of a circle and to approximate its value by considering the perimeters p_6 and P_6 of an inscribed and a circumscribed regular hexagon, the perimeters p_{12} and P_{12} of an inscribed and a circumscribed regular polygon of 12 sides, etc. From the results, true to four decimal places, for a circle of unit diameter:

$$p_6 = 3 \quad , \ p_{12} = 3.1058, \ \ldots, \ p_{384} = 3.1415, \ \ldots$$
$$P_6 = 3.4641, \ P_{12} = 3.2153, \ \ldots, \ P_{384} = 3.1416, \ \ldots$$

we obtain a succession of numbers between each pair of which the value of π must lie. By proving that

$$p_6 < p_{12} < p_{24} < \ldots < p_{384} < \ldots < \pi,$$
$$P_6 > P_{12} > P_{24} > \ldots > P_{384} > \ldots > \pi,$$

and that the difference $P_n - p_n$ can be made to differ from zero as little as we please by sufficiently increasing the number of sides n, we have pointed out to us, with as great a degree of precision as we may desire, a certain limit, which we take as the value of π.

In an analogous manner, we can define the number $\sqrt{2}$ by means of two sequences of *rational* numbers,

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad 1.41421, \quad \ldots$$
$$2, \quad 1.5, \quad 1.42, \quad 1.415, \quad 1.4143, \quad 1.41422, \quad \ldots$$

By the arithmetical process for the extraction of a square root, we find that the value of $\sqrt{2}$ lies between each pair of corresponding numbers in the sequences.

In general, two such sequences of rational numbers proceeding by a given law are said to define, by a limiting process, **a number**.* The value of the number may be determined to as great a degree of approximation as may be desired. All such numbers as well as all rational numbers are called **real numbers**.

* The above sequences which defined the number π can evidently be replaced by sequences of rational numbers related to the p_n and P_n.

4. An even root of a negative number is called an **imaginary quantity.** Thus $\sqrt{-1}$, $\sqrt[4]{-2}$, $\sqrt[6]{-1}$ are imaginaries. Unlike surds and other real numbers, an imaginary can not be expressed approximately in terms of rational numbers and hence has no interpretation in strictly arithmetical problems. By the introduction of imaginaries, we may give a formal solution of the equations $x^2 = -1$, $x^2 = -2$, and, indeed, of every quadratic equation. By extending the system of all real numbers by the introduction of the quantity $\sqrt{-1}$, we obtain the quantities $a + b\sqrt{-1}$, where a and b are arbitrary real numbers. We shall see that the system of these **complex quantities** $a + b\sqrt{-1}$ forms a number system within which may be performed all algebraic operations including the solution of all algebraic equations, so that a further extension is unnecessary. The employment in algebra of imaginaries has therefore a great practical value in that the operations may be effected without the limitations otherwise necessary. To further justify this extension, we recall that negative, fractional, and surd numbers were introduced to enable us to give a formal solution of many simple problems which would otherwise have remained insolvable, and that the possibility of the interpretation of negative, fractional, or irrational results depends upon the nature of the particular problem.*

5. *If $a + \sqrt{b} = c + \sqrt{d}$, where a, b, c, d are rational numbers and $\sqrt{b}$ is irrational, then $a = c$, $b = d$.*

From $a - c + \sqrt{b} = \sqrt{d}$, we derive, after squaring,

$$2(a - c)\sqrt{b} = d - b - (a - c)^2.$$

Unless the coefficient $a - c$ is zero, we could express $\sqrt{b}$ as a rational number, contrary to assumption. Hence $a = c$, so that $b = d$.

* A possible interpretation of complex quantities is given in the Appendix. The instructor may prefer the illustration by means of operations which combine a rotation with a magnification. Thus -1 rotates through 180°, $\sqrt{-1}$ through 90°, $4 + 3\sqrt{-1}$ magnifies five-fold and rotates. See Chrystal's *Algebra*, I, p. 239.

6. Let us attempt to extract the square root of $a + \sqrt{b}$, where $\sqrt{b}$ is a true surd and a is rational. We seek a result of the form $\sqrt{\alpha} + \sqrt{\beta}$ in which α and β are rational. Setting

$$\sqrt{a + \sqrt{b}} = \sqrt{\alpha} + \sqrt{\beta},$$

and squaring, we find that

$$a + \sqrt{b} = \alpha + \beta + 2\sqrt{\alpha\beta}.$$

By the above theorem, we may equate the rational parts and also the irrational (surd) parts. Hence

$$a = \alpha + \beta, \qquad b = 4\alpha\beta.$$

Then $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = a^2 - b$, so that

$$\alpha = \tfrac{1}{2}(a + \sqrt{a^2 - b}), \qquad \beta = \tfrac{1}{2}(a - \sqrt{a^2 - b}).$$

By assumption α and β are rational. *Hence the square root of $a + \sqrt{b}$ is expressible as a sum of two quadratic surds $\sqrt{\alpha} + \sqrt{\beta}$ if, and only if, $a^2 - b$ is the square of a rational number.*

For example, if $a = 6$, $b = 20$, $a^2 - b$ is the square of 4. Hence $a + \sqrt{b} = 6 + \sqrt{20}$ is the square of $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{5} + 1$.

When the problem is solvable, it may usually be done by inspection as follows. Put the expression $a + \sqrt{b}$ into the form $m + 2\sqrt{n}$ by taking $m = a$, $n = b/4$. The required root is $\sqrt{\alpha} + \sqrt{\beta}$, where $\alpha + \beta = m$, $\alpha\beta = n$.

Thus $\qquad 6 + \sqrt{20} = 6 + 2\sqrt{5} = (1 + \sqrt{5})^2,$

since $\qquad 1 + 5 = 6, 1 \cdot 5 = 5.$

Thus $\qquad 16 + 6\sqrt{7} = 16 + 2\sqrt{63} = (\sqrt{7} + \sqrt{9})^2,$

since $\qquad 7 + 9 = 16, 7 \cdot 9 = 63.$

7. Denote by i the symbol $\sqrt{-1}$. Then $+i$ and $-i$ are the roots of $x^2 = -1$. By the symbol $\sqrt{-c}$, where c is positive, we shall mean $\sqrt{-1}\sqrt{c} = i\sqrt{c}$, where $\sqrt{c}$ denotes the positive square root of c. Thus

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = +1, \quad i^{2n} = (-1)^n, \quad i^{2n+1} = i(-1)^n.$$

If c and d are any two positive real quantities,

$$\sqrt{-c}\,\sqrt{-d} = i\sqrt{c}\cdot i\sqrt{d} = -\sqrt{c}\,\sqrt{d} = -\sqrt{cd}.$$

In introducing the equation $x^2 = -c$ as an equation to be solved by algebra, we are tacitly assuming that x may be combined with itself and with real numbers according to the laws of algebra for the combination of real numbers, so that $d\sqrt{-c} = \sqrt{-c}\,d$, $d + \sqrt{-c} = \sqrt{-c} + d$, etc. In addition to these assumptions, we assume that complex quantities may be combined according to the laws holding for real numbers. Then

$$(a + bi) \pm (\alpha + \beta i) = (a \pm \alpha) + (b \pm \beta)i,$$

$$(a + bi)(\alpha + \beta i) = (a\alpha - b\beta) + (a\beta + b\alpha)i,$$

$$\frac{\alpha + \beta i}{a + bi} = \frac{(\alpha + \beta i)(a - bi)}{(a + bi)(a - bi)} = \frac{a\alpha + b\beta}{a^2 + b^2} + \frac{a\beta - b\alpha}{a^2 + b^2}i.$$

*Hence the sum, difference, product, or quotient of any two complex quantities is itself a complex quantity.**

In freeing the denominator of the above fraction from imaginaries, we used the multiplier $a - bi$, called the **conjugate** of the denominator $a + bi$. The sum and the product of two conjugate complex quantities are both real.

8. *The three cube roots of unity are*

$$1,\ \omega = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3},\quad \omega^2 = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3},$$

so that $\omega^3 = 1,\quad 1 + \omega + \omega^2 = 0.$

The roots of $x^3 = 1$ are $x = 1$ and the two roots of

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1 = 0.$$

Completing the square in the quadratic equation, we get

$$x^2 + x + \tfrac{1}{4} = (x + \tfrac{1}{2})^2 = -\tfrac{3}{4}.$$

Hence $x = -\frac{1}{2} \pm \sqrt{-\frac{3}{4}}$. Setting $-\frac{1}{2} + \frac{1}{2}\sqrt{-3} = \omega$, the second root is

$$-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3} = (-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3})^2 = \omega^2.$$

9. *In an equation between two complex quantities, the real parts are equal and also the imaginary parts.*

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* The complex quantity $a + bi$ is real if $b = 0$.

Let $a + bi = \alpha + \beta i$, where a, b, α, β are real numbers. Then

$$a - \alpha = (\beta - b)i.$$

Upon squaring, we find that the number $(a - \alpha)^2$, which is positive or zero, must equal the number $-(\beta - b)^2$, which is negative or zero, so that each must be zero. Hence $a = \alpha$, $b = \beta$.

In particular, if $a + bi = 0$, then $a = 0$, $b = 0$.

10. *The square root of any complex quantity may always be expressed as a complex quantity.**

Let the given complex quantity be $a + bi$, where a and b are real and $i = \sqrt{-1}$. We seek real numbers x and y which will make

$$\sqrt{a + bi} = x + yi.$$

Squaring, $$a + bi = x^2 - y^2 + 2xyi.$$

Equating the real parts and also the imaginary parts,

$$x^2 - y^2 = a, \quad 2xy = b.$$

Then $$(x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2 = a^2 + b^2.$$

Since x and y are to be real, $x^2 + y^2$ must be positive. Hence

$$x^2 + y^2 = \sqrt{a^2 + b^2} \qquad \text{(positive square root).}$$

Having the sum and difference of x^2 and y^2, we derive

$$x^2 = \frac{\sqrt{a^2 + b^2} + a}{2}, \quad y^2 = \frac{\sqrt{a^2 + b^2} - a}{2}.$$

Since $\sqrt{a^2 + b^2}$ is positive and greater than a, the expressions for x^2 and y^2 are positive, so that *real* values of x and y may be determined by extracting the square roots of positive quantities. Since $2xy = b$, the sign of y is determined as soon as the sign of x is chosen. Hence there are always two and only two square roots of $a + bi$.

For example, to find the square roots of $5 - 12i$, we have

$$x^2 - y^2 = 5, \quad 2xy = -12,$$

whence $x^2 + y^2 = 13$, $x^2 = 9$, $y^2 = 4$. The square roots are $\pm(3 - 2i)$.

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* Contrast with the theorem of § 6. The extraction of higher roots of complex quantities is done very simply in terms of trigonometric ratios [see Appendix].

The inspection method of § 6 may be extended to find the square roots of certain complex numbers $a+b\sqrt{-1}$. Thus for $-3+4\sqrt{-1}$, set

$$\sqrt{-3+2\sqrt{-4}}=\sqrt{x}+\sqrt{-y} \quad (x \text{ and } y \text{ positive}).$$

$$\therefore \quad -3+2\sqrt{-4}=x-y+2\sqrt{-xy}.$$

$$\therefore \quad x-y=-3, \quad xy=4 \qquad \text{(by § 9)}.$$

Hence $x=1$, $y=4$, so that one square root of $-3+4\sqrt{-1}$ is $1+2\sqrt{-1}$.

EXERCISES.

Express with rational denominators

1. $\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$. 2. $\dfrac{\sqrt{5}+3}{\sqrt{5}-2}$. 3. $\dfrac{\sqrt{3}+\sqrt{5}}{1-\sqrt{15}}$.

4. $\dfrac{\sqrt{3}+\sqrt{5}+\sqrt{10}}{\sqrt{3}-\sqrt{5}+\sqrt{10}}$. 5. $\dfrac{1+\sqrt{6}}{\sqrt{2}-\sqrt{3}+\sqrt{5}}$. 6. $\dfrac{3+\sqrt{-5}}{2+\sqrt{-1}}$.

7. $\dfrac{3+5\sqrt{-1}}{2-3\sqrt{-1}}$. 8. $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}$.

9. Approximately, $\sqrt{2}=1.4142$, $\sqrt{3}=1.7320$; find $\dfrac{1}{\sqrt{2}-\sqrt{3}}$, $\dfrac{1+\sqrt{2}}{1-\sqrt{3}}$.

10. Simplify $\sqrt{45}+\sqrt{20}+3\sqrt{5}$, $7\sqrt{24}-2\sqrt{\tfrac{1}{6}}$, $\sqrt{15}\div\sqrt[4]{25}$.

11. Simplify $7\sqrt{-12}-2\sqrt{-27}$, $\sqrt{-3}\times\sqrt{-48}$, $\sqrt{-9}\div\sqrt{-16}$

Express in terms of surds the square root of

12. $12-6\sqrt{3}$. 13. $28-\sqrt{300}$. 14. $16-8\sqrt{3}$.

15. $29+6\sqrt{22}$. 16. $75+12\sqrt{21}$. 17. $47-4\sqrt{33}$.

18. $a+b+\sqrt{2ab+b^2}$. 19. $1+a^2+\sqrt{1+a^2+a^4}$.

Express in the form $a+b\sqrt{-1}$, a and b real, the square root of

20. $-11+60\sqrt{-1}$. 21. $-47+\sqrt{-192}$.

22. $-20+48\sqrt{-1}$. 23. $-7+24\sqrt{-1}$.

24. $c^2-d^2-2\sqrt{-c^2d^2}$. 25. $4cd+2(c^2-d^2)\sqrt{-1}$.

26. Prove, as in § 2, that $\sqrt{7}$ and $\sqrt[3]{4}$ are not expressible as rational numbers.

27. Prove that $1+\sqrt{2}$ is not a surd by showing that an equation

$$(1+\sqrt{2})^n=r=\text{a rational number}$$

would require $(1-\sqrt{2})^n=r$, whereas the two equations are contradictory since $1+\sqrt{2}>1-\sqrt{2}$.

CHAPTER II.

EXPONENTS; LOGARITHMS.

11. If m is a positive integer, a^m denotes the product of m factors each of which is a. Similarly, if n is a positive integer, $a^n = a \,.\, a \ldots a$, to n factors. Hence $a^m \,.\, a^n = a \,.\, a \ldots a$ to $m + n$ factors $= a^{m+n}$. We may therefore state, for the case of positive integral indices m, n,

The First Law of Indices. *The index of the product of two powers of the same quantity is the sum of the indices of the factors:*

(1) $$a^m \,.\, a^n = a^{m+n}.$$

As a corollary, we derive the formula

$$a^m \,.\, a^n \,.\, a^p \ldots . \, a^q = a^{m+n+p+\cdots+q}.$$

12. For the division of two positive integral powers of a, $a \neq 0$,

$$\frac{a^m}{a^n} = \frac{a \,.\, a \,.\, a \ldots a \quad \text{(to } m \text{ factors)}}{a \,.\, a \,.\, a \ldots a \quad \text{(to } n \text{ factors)}}$$

$$= a \,.\, a \ldots a \quad \text{(to } m - n \text{ factors)}$$

if $m > n$. We may state, for m and n positive integers, $m > n$,

The Second Law of Indices. *The index of the quotient of two powers of the same quantity is the excess of the index of the numerator over the index of the denominator:*

(2) $$a^m/a^n = a^{m-n} \qquad (a \neq 0,\ m > n).$$

13. If m and n are positive integers, we have, by definition,

$$(a^m)^n = a^m \,.\, a^m \ldots a^m \quad \text{(to } n \text{ factors)}$$

$$= a^{m+m+\cdots+m} = a^{mn} \quad \text{(by § 11)}.$$

Hence, for positive integral indices, we may state

The Third Law of Indices. *The index of the nth power of a^m is the product of the indices m and n:*

$$(3)\qquad (a^m)^n = a^{mn}.$$

14. We next extend the use of the symbol a^n to cases in which n is negative or fractional, assuming that such new symbols a^n will satisfy certain cases of the above *first* law of indices, and proceed to determine what meaning, if any, may be attached to the generalized symbols. It is later shown (§ 15) that the symbols, with the meanings thus obtained, satisfy the three laws of indices for m and n any rational numbers. For this reason the interpretations of the symbols are justified and the desired permanence of the algebraic laws is attained.

Consider the symbol $a^{\frac{1}{q}}$, where q is any positive integer. Since the symbol shall satisfy the first law of indices,

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \ldots \text{(to } q \text{ factors)} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\ldots\text{(to } q \text{ terms)}} = a^1 = a.$$

Hence $a^{\frac{1}{q}}$ must be such that its qth power is a, that is,*

$$a^{\frac{1}{q}} = \sqrt[q]{a}.$$

Similarly, the symbol $a^{p/q}$, where p and q are positive integers, must be such that

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \ldots \text{(to } q \text{ factors)} = a^{\frac{p}{q}+\frac{p}{q}+\ldots\text{(to } q \text{ terms)}} = a^p,$$

so that $a^{p/q}$ must be a qth root of a^p. Hence

$$a^{p/q} = \sqrt[q]{a^p}.$$

Since the symbol $a^{\frac{1}{q}}$ obeys the first law of indices, we find that

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \ldots \text{(to } p \text{ factors)} = a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p.$$

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* The radical sign is used to denote a particular root. Thus

$$\sqrt[2]{4} = +2, \ -\sqrt{4} = -2; \text{ hence } 4^{\frac{1}{2}} = +2.$$