Walter Greiner

# Relativistic Quantum Mechanics

## Wave Equations

With a Foreword by D. A. Bromley

Walter Greiner

# Relativistic Quantum Mechanics

## Wave Equations

With a Foreword by D. A. Bromley

With 62 Figures

Verlag Berlin Heidelberg New York
Paris Tokyo Hong Kong

Professor Dr. Walter Greiner
Institut für Theoretische Physik, Universität Frankfurt, Robert-Mayer-Str. 8–10
D-6000 Frankfurt, Fed. Rep. of Germany

Title of the original German edition: *Theoretische Physik*, Band 6, *Relativistische Quantenmechanik, Wellengleichungen*
2., überarbeitete und erweiterte Auflage 1987


ISBN 3-540-50986-0 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-50986-0 Springer-Verlag New York Berlin Heidelberg



Printing: Druckhaus Beltz, 6944 Hemsbach/Bergstr.
Binding: J. Schäffer GmbH & Co. KG., 6718 Grünstadt
2156/3150-543210 – Printing on acid-free paper

# Foreword

More than a generation of German-speaking students around the world have worked their way to an understanding and appreciation of the power and beauty of modern theoretical physics – with mathematics, the most fundamental of sciences – using Walter Greiner's textbooks as their guide.

The idea of developing a coherent, complete presentation of an entire field of science in a series of closely related textbooks is not a new one. Many older physicists remember with real pleasure their sense of adventure and discovery as they worked their ways through the classic series by Sommerfeld, by Planck and by Landau and Lifshitz. From the students' viewpoint, there are a great many obvious advantages to be gained through use of consistent notation, logical ordering of topics and coherence of presentation; beyond this, the complete coverage of the science provides a unique opportunity for the author to convey his personal enthusiasm and love for his subject.

The present five volume set, *Theoretical Physics*, is in fact only that part of the complete set of textbooks developed by Greiner and his students that presents the quantum theory. I have long urged him to make the remaining volumes on classical mechanics and dynamics, on electromagnetism, on nuclear and particle physics, and on special topics available to an English-speaking audience as well, and we can hope for these companion volumes covering all of theoretical physics some time in the future.

What makes Greiner's volumes of particular value to the student and professor alike is their completeness. Greiner avoids the all too common "it follows that..." which conceals several pages of mathematical manipulation and confounds the student. He does not hesitate to include experimental data to illuminate or illustrate a theoretical point and these data, like the theoretical content, have been kept up to date and topical through frequent revision and expansion of the lecture notes upon which these volumes are based.

Moreover, Greiner greatly increases the value of his presentation by including something like one hundred completely worked examples in each volume. Nothing is of greater importance to the student than seeing, in detail, how the theoretical concepts and tools under study are applied to actual problems of interest to a working physicist. And, finally, Greiner adds brief biographical sketches to each chapter covering the people responsible for the development of the theoretical ideas and/or the experimental data presented. It was Auguste Comte (1798–1857) in his *Positive Philosophy* who noted, "To understand a science it is necessary to know its history". This is all too often forgotten in modern physics teaching and the bridges that Greiner builds to the pioneering figures of our science upon whose work we build are welcome ones.

Greiner's lectures, which underlie these volumes, are internationally noted for their clarity, their completeness and for the effort that he has devoted to making physics an integral whole; his enthusiasm for his science is contagious and shines through almost every page.

These volumes represent only a part of a unique and Herculean effort to make all of theoretical physics accessible to the interested student. Beyond that, they are of enormous value to the professional physicist and to all others working with quantum phenomena. Again and again the reader will find that, after dipping into a particular volume to review a specific topic, he will end up browsing, caught up by often fascinating new insights and developments with which he had not previously been familiar.

Having used a number of Greiner's volumes in their original German in my teaching and research at Yale, I welcome these new and revised English translations and would recommend them enthusiastically to anyone searching for a coherent overview of physics.

*D. Allan Bromley*
Henry Ford II Professor of Physics
Yale University
New Haven, CT USA

# Preface

Theoretical physics has become a many-faceted science. For the young student it is difficult enough to cope with the overwhelming amount of new scientific material that has to be learned, let alone obtain an overview of the entire field, which ranges from mechanics through electrodynamics, quantum mechanics, field theory, nuclear and heavy-ion science, statistical mechanics, thermodynamics, and solid-state theory to elementary-particle physics. And this knowledge should be acquired in just 8–10 semesters, during which, in addition, a Diploma or Master's thesis has to be worked on or examinations prepared for. All this can be achieved only if the university teachers help to introduce the student to the new disciplines as early on as possible, in order to create interest and excitement that in turn set free essential, new energy. Naturally, all inessential material must simply be eliminated.

At the Johann Wolfgang Goethe University in Frankfurt we therefore confront the student with theoretical physics immediately, in the first semester. Theoretical Mechanics I and II, Electrodynamics, and Quantum Mechanics I – An Introduction are the basic courses during the first two years. These lectures are supplemented with many mathematical explanations and much support material. After the fourth semester of studies, graduate work begins, and Quantum Mechanics II – Symmetries, Statistical Mechanics and Thermodynamics, Relativistic Quantum Mechanics, Quantum Electrodynamics, the Gauge Theory of Weak Interactions, and Quantum Chromodynamics are obligatory. Apart from these, a number of supplementary courses on special topics are offered, such as Hydrodynamics, Classical Field Theory, Special and General Relativity, Many-Body Theories, Nuclear Models, Models of Elementary Particles, and Solid-State Theory. Some of them, for example the two-semester courses Theoretical Nuclear Physics and Theoretical Solid-State Physics, are also obligatory.

The form of the lectures that comprise *Relativistic Quantum Mechanics – Wave Equations* follows that of all the others: together with a broad presentation of the necessary mathematical tools, many examples and exercises are worked through. We try to offer science in as interesting a way as possible. With relativistic quantum mechanics we are dealing with a broad, yet beautiful, theme. Therefore we have had to restrict ourselves to relativistic wave equations. The selected material is perhaps unconventional, but corresponds, in our opinion, to the importance of this field in modern physics:

The Klein–Gordon equation (for spin-0 particles) and the Dirac equation (for spin-$\frac{1}{2}$ particles) and their applications constitute the backbone of these lectures. Wave equations for particles with higher spin (the Rarita–Schwinger, spin-$\frac{3}{2}$, Kemmer and Proca, spin-1, and general Bargmann–Wigner equations) are confined to the last chapters.

After introducing the Klein–Gordon equation we discuss its properties and difficulties (especially with respect to the single-particle interpretation); the Feshbach–Villars representation is given. In many worked-out exercises and examples its practical applications can be found: pionic atoms as a modern field of research and the particularly challenging examples on the effective pion–nucleus potential (the Kisslinger potential) and its improvement by Ericson and Ericson stand in the foreground.

Most of these lectures deal with Dirac's theory. The covariance properties of the Dirac equation are discussed in detail. So, for example, its free solutions are on the one hand determined directly and on the other hand through Lorentz transformations from the simple solutions in the rest frame. Here the methodical issue is emphasized: the same physical phenomenon is illuminated from different angles. We proceed in a similar manner in the discussion of single-particle operators (the odd and even parts of an operator) and the so-called *Zitterbewegung*, which is also derived from the consideration of wave packets of plane Dirac waves. In many worked-out problems and examples the new tools are exercised. Thus the whole of Chap. 9 is dedicated to the motion of Dirac particles in external potentials. It contains simple potential problems, extensively the case of the electron in a Coulomb potential (the fine-structure formula), and muonic atoms. In Chap. 10 we present the two-centre Dirac equation, which is of importance in the modern field of heavy-ion atomic physics. The fundamental problem of overcritical fields and the decay of the electron–positron vacuum is only touched upon. A full treatment is reserved for *Quantum Electrodynamics* (Vol. 4 of this series). However, we give an extended discussion of hole theory and also of Klein's paradox. The Weyl equation for the neutrino (Chap. 14) and relativistic wave equations for particles with arbitrary spin (Chap. 15) follow. Starting with the Bargmann–Wigner equations the general frame for these equations is set, and in numerous worked-out examples and exercises special cases (spin-1 particles with and without mass, and spin-$\frac{3}{2}$ particles according to Rarita and Schwinger) are considered in greater detail. In the last chapter we give an overview of relativistic symmetry principles, which we enjoy from a superior point of view, since by now we have studied *Quantum Mechanics – Symmetries* (Vol. 2 of this series).

We hope that in this way the lectures will become ever more complete and may lead to new insights.

Biographical notes help to obtain an impression, however short, of the life and work of outstanding physicists and mathematicians. We gratefully acknowledge the publishers Harri Deutsch and F.A. Brockhaus (*Brockhaus Enzyklopädie*, F.A. Brockhaus – Wiesbaden indicated by BR) for giving permission to use relevant information from their publications.

Special thanks go to Prof. Dr. Gerhard Soff, Dr. Joachim Reinhardt, and Dr. David Vasak for their critical reading of the original draft of these lectures. Many students and collaborators have helped during the years to work out examples and exercises. For this first English edition we enjoyed the help of Maria Berenguer, Christian Borchert, Snježana Butorac, Christian Derreth, Carsten Greiner, Kordt Griepenkerl, Christian Hofmann, Raffaele Mattiello, Dieter Neubauer, Jochen Rau, Wolfgang Renner, Dirk Rischke, Alexander Scherdin, Thomas Schönfeld, and Dr. Stefan Schramm. Miss Astrid Steidl drew the graphs and prepared the figures. To all of them we express our sincere thanks.

We would especially like to thank Mr. Béla Waldhauser, Dipl.-Phys., for his overall assistance. His organizational talent and his advice in technical matters are very much appreciated.

Finally, we wish to thank Springer-Verlag; in particular, Dr. H.-U. Daniel, for his encouragement and patience, Mr. Michael Edmeades for expertly copy-editing the English edition, and Mr. R. Michels and his team for the excellent layout.

Frankfurt am Main, May 1990 *Walter Greiner*

# Contents

# 1. Relativistic Wave Equation for Spin-O Particles The Klein-Gordon Equation and Its Applications

The description of phenomena at high energies requires the investigation of relativistic wave equations. This means equations which are invariant under Lorentz transformations. The transition from a nonrelativistic to a relativistic description implies that several concepts of the nonrelativisti theory have to be reinvestigated, in particular:

1) Spatial and temporal coordinates have to be treated equally within the theory.

2) Since

$$\Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{m_0 c} \quad ,$$

a relativistic particle cannot be localized more accurately than $\approx \hbar/m_0c$; otherwise pair creation occurs for $E > 2m_0c^2$. Thus, the idea of a free particle only makes sense, if the particle is not confined by external constraints to a volume which is smaller than approximately the *Compton wavelength* $\lambda_c = \hbar/m_0c$. Otherwise the particle automatically has companions due to particle-antiparticle creation.

3) If the position of the particle is uncertain, i.e. if

$$\Delta x > \frac{\hbar}{m_0 c} \quad ,$$

then the time is also uncertain, because

$$\Delta t \sim \frac{\Delta x}{c} > \frac{\hbar}{m_0 c^2} \quad .$$

In a nonrelativistic theory $\Delta t$ can become arbitrarily small, because $c \to \infty$. Thereby, we recognize the necessity to reconsider the concept of probability density

$$\varrho(x, y, z, t) \quad ,$$

which describes the probability of finding a particle at a definite place $\boldsymbol{r}$ at fixed time $t$.

4) At high (relativistic) energies pair creation and annihilation processes occur, usually in the form of creating particle-antiparticle-pairs. Thus, at relativistic energies particle conservation is no longer a valid assumption. A relativistic theory must be able to describe pair creation, vacuum polarization, particle conversion, et.

## 1.1 The Notation

First we shall remark on the notation used. Until now we have expressed four-vectors by Minkowski's notation, with an imaginary fourth component, as for example

$$
\begin{aligned}
x &= \{x,\ y,\ z,\ \mathrm{i}ct\} && \text{(world vector)}\ , \\
p &= \{p_x,\ p_y,\ p_z,\ \mathrm{i}E/c\} && \text{(four-momentum)}\ , \\
A &= \{A_x,\ A_y,\ A_z,\ \mathrm{i}A_0\} && \text{(four-potential)}\ , \\
V &= \left\{\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ \frac{\partial}{\mathrm{i}\partial(ct)}\right\} && \text{(four-gradient), etc.}\ .
\end{aligned}
\tag{1.1}
$$

The letters $x$, $p$, $A$, $V$ abbreviate the full four-vector. Sometimes we shall also denote them by $\vec{\vec{x}}$, $\vec{\vec{p}}$, $\vec{\vec{A}}$, $\vec{\vec{V}}$, etc., i.e. with a double arrow. As long as there is no confusion arising, we prefer the former notation. For the following it is useful to introduce the metric tensor (*covariant components*)

$$
g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\ . \tag{1.2}
$$

Thereby, one can denote the length of the vector $dx = \{dx^\mu\}$ as $ds^2 = dx \cdot dx = g_{\mu\nu}dx^\mu dx^\nu$. This relation is often taken as the defining relation of the metric tensor.[1] The *contravariant form* $g^{\mu\nu}$ of the metric tensor follows from the condition

$$
g^{\mu\sigma}g_{\sigma\nu} = \delta^\mu_\nu \stackrel{\text{def}}{\equiv} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ , \tag{1.3}
$$

$$
g^{\mu\sigma} = \left(g^{-1}\right)_{\mu\sigma} = \frac{\Delta_{\mu\sigma}}{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\ . \tag{1.4}
$$

Here $\Delta_{\mu\sigma}$ is the cofactor of $g_{\mu\sigma}$ [i.e. the subdeterminant, obtained by crossing out the $\mu$th row and the $\sigma$th column and multiplying it with the phase $(-1)^{\mu+\sigma}$] and $g$ is given by $g = \det(g_{\mu\nu}) = -1$. For the special Lorentz metric the contravariant and covariant metric tensor are identical:

$$
g^{\mu\nu} = g_{\mu\nu} \quad \text{[for Lorentz metric!]}\ .
$$

From now on we will use the contravariant four-vector

$$
x^\mu = \{x^0,\ x^1,\ x^2,\ x^3\} \equiv \{ct,\ x,\ y,\ z\} \tag{1.5}
$$

for the description of the space-time coordinates, where the time-like component is denoted as zero component. We get the covariant form of the four-vector by "lowering" the index $\mu$ with the help of the metric tensor, i.e.

$$
x_\mu = g_{\mu\nu}x^\nu = \{ct,\ -x,\ -y,\ -z\} = \{x_0,\ x_1,\ x_2,\ x_3\}\ . \tag{1.6}
$$

[1] We adopt the same notation as J.D. Bjorken, S.D. Drell: *Relativistic Quantum Mechanics* (McGraw Hill, New York 1964).

Similarly the indices can be "raised" to give

$$x^\mu = g^{\mu\nu} x_\nu = \{x^0, x^1, x^2, x^3\} \quad .$$

This means that one can easily transform the covariant into the contravariant form of a vector (respectively of a tensor) and vice versa. Except in special cases, where we denote it explicitly, we use *Einstein's summation convention*: We automatically add from 0 to 3 over indices occuring doubly (one upper and one lower index). So we have, for example,

$$\begin{aligned} x \cdot x = x^\mu x_\mu \equiv \sum_{\mu=0}^{3} x^\mu x_\mu &= x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 \\ &= c^2 t^2 - x^2 - y^2 - z^2 \\ &= c^2 t^2 - \boldsymbol{x}^2 \quad . \end{aligned} \tag{1.7}$$

The definition of the four-momentum vector is analogous,

$$p^\mu = \{E/c, p_x, p_y, p_z\} \quad , \tag{1.8}$$

and we write the scalar product in four dimensions (space-time) as

$$p_1 \cdot p_2 = p_1^\mu p_{2\mu} = \frac{E_1}{c}\frac{E_2}{c} - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 \quad , \tag{1.9}$$

or equally

$$x \cdot p = x^\mu p_\mu = x_\mu p^\mu = Et - \boldsymbol{x} \cdot \boldsymbol{p} \quad . \tag{1.10}$$

We identify the four-vectors by a common letter. Thus, for instance,

$$a = \{a_0, a_1, a_2, a_3\} \quad .$$

In contrast to this we denote three-vectors by bold type as in

$$\boldsymbol{a} = \{a_1, a_2, a_3\} \quad .$$

Often we write only the components. Hence,

$$a^\mu = \{a^0, a^1, a^2, a^3\}$$

means a four-vector with contravariant components. Greek indices, such as $\mu$, always run from 0 to 3. Latin indices, as for example $i$, imply values from 1 to 3. A three-vector can thus also be written in contravariant form as

$$a^i = \{a^1, a^2, a^3\} \quad \text{or as} \quad a_i = \{a_1, a_2, a_3\}$$

in covariant form. So the *four-momentum operator* is therefore denoted by

$$\begin{aligned} \hat{p}^\mu &= \mathrm{i}\hbar \frac{\partial}{\partial x_\mu} = \left\{\mathrm{i}\hbar \frac{\partial}{\partial(ct)}, +\mathrm{i}\hbar\frac{\partial}{\partial x_1}, +\mathrm{i}\hbar\frac{\partial}{\partial x_2}, +\mathrm{i}\hbar\frac{\partial}{\partial x_3}\right\} \\ &\equiv \mathrm{i}\hbar\nabla^\mu = \left\{\frac{\partial}{\partial(ct)}, -\mathrm{i}\hbar\frac{\partial}{\partial x}, -\mathrm{i}\hbar\frac{\partial}{\partial y}, -\mathrm{i}\hbar\frac{\partial}{\partial z}\right\} \\ &= \mathrm{i}\hbar\left\{\frac{\partial}{\partial(ct)}, -\nabla\right\} \quad . \end{aligned} \tag{1.11}$$

It transforms as a contravariant four-vector, so that

$$\begin{aligned}\hat{p}^{\mu}\hat{p}_{\mu} &= -\hbar^2\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} = -\hbar^2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right) \\ &\equiv -\hbar^2\Box = -\hbar^2\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) \quad . \end{aligned} \tag{1.12}$$

This equation defines both the three-dimensional delta operator ($\Delta = \nabla^2$) and the four-dimensional d'Alembertian ($\Box = (1/c^2\partial^2/\partial t^2 - \Delta)$. Finally we check the commutation relations of momentum and position by means of (1.11 and 1.5), obtaining

$$\begin{aligned}[\hat{p}^{\mu}, x^{\nu}]_- &= \mathrm{i}\hbar\left[\frac{\partial}{\partial x_\mu}, g^{\nu\sigma}x_\sigma\right]_- = \mathrm{i}\hbar g^{\nu\sigma}\frac{\partial x_\sigma}{\partial x_\mu} \\ &= \mathrm{i}\hbar g^{\nu\sigma}\delta^{\mu}_{\sigma} = \mathrm{i}\hbar g^{\nu\mu} = \mathrm{i}\hbar g^{\mu\nu} \quad . \end{aligned} \tag{1.13}$$

On the right hand side (rhs), the metric tensor $g^{\mu\nu}$ appears expressing the *covariant form of the commutation relation.*

The four-potential of the electromagnetic field is given by

$$A^{\mu} = \{A_0, \boldsymbol{A}\} = \{A_0, A_x, A_y, A_z\} = g^{\mu\nu}A_\nu \quad . \tag{1.14}$$

Here $A^\mu$ are the contravariant, and $A_\mu = \{A_0, -A_x, -A_y, -A_z\}$ the covariant components. From $A^\mu$ the electromagnetic field tensor follows in the well-known way:

$$F^{\mu\nu} = \frac{\partial A^\mu}{\partial x_\nu} - \frac{\partial A^\nu}{\partial x_\mu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \quad . \tag{1.15}$$

## 1.2 The Klein-Gordon Equation

From elementary quantum mechanics[2] we know that the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m_0}\nabla^2 + V(\boldsymbol{x})\right]\psi(\boldsymbol{x}, t) \tag{1.16}$$

corresponds to the nonrelativistic energy relation in operator form,

$$\hat{E} = \frac{\hat{\boldsymbol{p}}^2}{2m_0} + V(\boldsymbol{x}) \quad , \quad \text{where} \tag{1.17}$$

$$\hat{E} = \mathrm{i}\hbar\frac{\partial}{\partial t} \quad , \quad \hat{\boldsymbol{p}} = -\mathrm{i}\hbar\nabla \tag{1.18}$$

are the operators of energy and momentum, respectively. In order to obtain a relativistic

[2] See Vol. 1 of this series, *Quantum Mechanics – An Introduction* (Springer, Berlin, Heidelberg 1989).

wave equation we start by considering free particles with the relativistic relation

$$p^\mu p_\mu = \frac{E^2}{c^2} - \boldsymbol{p}\cdot\boldsymbol{p} = m_0^2c^2 \quad . \tag{1.19}$$

We now replace the four-momentum $p^\mu$ by the four-momentum operator

$$\begin{aligned}\hat{p}^\mu = \mathrm{i}\hbar\frac{\partial}{\partial x_\mu} &= \mathrm{i}\hbar\left\{\frac{\partial}{\partial(ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right\}\\ &= \mathrm{i}\hbar\left\{\frac{\partial}{\partial(ct)}, -\nabla\right\} = \{\hat{p}_0, \hat{\boldsymbol{p}}\} \quad .\end{aligned} \tag{1.20}$$

Following (1.6) and (1.11), the result is in accordance with (1.18). Thus, we obtain the ***Klein-Gordon*** equation for free particles,

$$\hat{p}^\mu\hat{p}_\mu\psi = m_0^2c^2\psi \quad . \tag{1.21}$$

Here $m_0$ is the rest mass of the particle and $c$ the velocity of light in vacuum. With the help of (1.12) we can write (1.21) in the form

$$\left(\Box + \frac{m_0^2c^2}{\hbar^2}\right) = \left(\frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2c^2}{\hbar^2}\right)\psi = 0 \quad . \tag{1.22}$$

We can immediately verify the Lorentz covariance of the Klein-Gordon equation, as $\hat{p}^\mu\hat{p}_\mu$ is Lorentz-invariant. We also recognize (1.22) as the classical wave equation including the *mass term* $m_0^2c^2/\hbar^2$. Free solutions are of the form

$$\psi = \exp\left(-\frac{\mathrm{i}}{\hbar}p_\mu x^\mu\right) = \exp\left[-\frac{\mathrm{i}}{\hbar}\left(p_0x^0 - \boldsymbol{p}\cdot\boldsymbol{x}\right)\right] = \exp\left[+\frac{\mathrm{i}}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{x} - Et)\right] \quad . \tag{1.23}$$

Indeed, insertion of (1.23) into (1.21) leads to the condition

$$\begin{aligned}\hat{p}_\mu\hat{p}^\mu\psi = m_0^2c^2\psi &\rightarrow p^\mu p_\mu \exp\left(-\frac{\mathrm{i}}{\hbar}p_\mu x^\mu\right) = m_0^2c^2\exp\left(-\frac{\mathrm{i}}{\hbar}p_\mu x^\mu\right)\\ &\rightarrow p^\mu p_\mu \doteq m_0^2c^2 \quad \text{or} \quad \frac{E^2}{c^2} - \boldsymbol{p}\cdot\boldsymbol{p} = m_0^2c^2 \quad ,\end{aligned}$$

which results in

$$E = \pm\sqrt{m_0^2c^2 + \boldsymbol{p}^2} \quad . \tag{1.24}$$

Thus, there exist solutions both for positive $E = +c(m_0^2c^2 + \boldsymbol{p}^2)^{1/2}$ as well as for negative $E = -c(m_0^2c^2 + \boldsymbol{p}^2)^{1/2}$ energies respectively (see Fig. 1.1). We shall see later that the solutions yielding negative energy are physically connected with antiparticles. Since antiparticles can indeed be observed in nature, we have already obtained an indication of the value of extending the nonrelativistic theory.

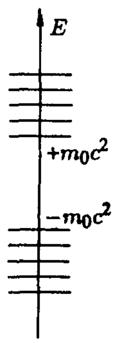

**Fig. 1.1.** Energy spectrum of the free Klein-Gordon equation

Next we construct the four-current $j_\mu$ connected with (1.21). In analogy to our considerations concerning the Schrödinger equation, we expect a conservation law for the $j_\mu$. We start from (1.22), in the form

$$\left(\hat{p}_\mu\hat{p}^\mu - m_0^2c^2\right)\psi = 0 \quad ,$$

and take the complex conjugate of this equation, i.e.

$$\left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi^* = 0 \quad .$$

Multiplying both equations from the left, the first by $\psi^*$ and the second by $\psi$, and calculating the difference of the resulting two equations yields

$$\psi^* \left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi - \psi \left(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2\right) \psi^* = 0$$

or

$$-\psi^* \left(\hbar^2 \nabla_\mu \nabla^\mu + m_0^2 c^2\right) \psi + \psi \left(\hbar^2 \nabla_\mu \nabla^\mu + m_0^2 c^2\right) \psi^* = 0$$

$$\Rightarrow \nabla_\mu (\psi^* \nabla^\mu \psi - \psi \nabla^\mu \psi^*) \equiv \nabla_\mu j^\mu = 0 \quad . \tag{1.25}$$

The four-current density is therefore

$$j_\mu = \frac{\mathrm{i}\hbar}{2m_0} (\psi^* \nabla_\mu \psi - \psi \nabla_\mu \psi^*) \quad . \tag{1.26}$$

Here we have multiplied by $\mathrm{i}\hbar/2m_0$, so that the zero component $j_0$ has the dimension of a probability density (that is $1/\mathrm{cm}^3$). Furthermore this ensures that we obtain the correct nonrelativistic limit [cf. (1.30-31)] below. In detail, (1.25) reads

$$\frac{\partial}{\partial t}\left[\frac{\mathrm{i}\hbar}{2m_0 c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right] + \mathrm{div}\left(\frac{-\mathrm{i}\hbar}{2m_0}\right)[\psi^*(\nabla\psi) - \psi(\nabla\psi^*)] = 0 \quad . \tag{1.27}$$

This expression possess the form of a continuity equation

$$\frac{\partial \varrho}{\partial t} + \mathrm{div}\, \boldsymbol{j} = 0 \quad . \tag{1.28}$$

As usual integration over the entire configuration space yields

$$\int_V \frac{\partial \varrho}{\partial t} d^3x = \frac{\partial}{\partial t} \int_V \varrho d^3x = -\int_V \mathrm{div}\, \boldsymbol{j} d^3x = -\int_F \boldsymbol{j} \cdot d\boldsymbol{F} = 0 \quad .$$

Hence,

$$\int_V \varrho d^3x = \mathrm{const.} \quad ,$$

i.e. $\int_V \varrho d^3x$ is constant in time. It would be a natural guess to interpret

$$\varrho = \frac{\mathrm{i}\hbar}{2m_0 c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \tag{1.29}$$

as a probability density. However, there is a problem with such an interpretation: At a given time $t$ both $\psi$ and $\partial\psi/\partial t$ may have arbitrary values; therefore, $\varrho(\boldsymbol{x}, t)$ in (1.29) may be either positive or negative. Hence, $\varrho(\boldsymbol{x}, t)$ is not positive definite and thus not a probability density. The deeper reason for this is that the Klein-Gordon equation is of second order in time, so that we must know both $\psi$ and $\partial\psi(\boldsymbol{x}, t)/\partial t$ for a given $t$. Furthermore there exist solutions for negative energy [see (1.24) and (1.38) below]. This and the difficulty with the probability interpretation was the reason that, for a long time, the Klein-Gordon equation was regarded to be physically senseless. One therefore looked for a relativistic wave equation of first order in time with positive definite probability, which was finally derived by Dirac (cf. Chap. 2). However, it turns out that this equation has negative energy solutions too. As we have previously remarked and as we shall discuss in greater detail later, in Chap. 2, these solutions are connected with the existence of antiparticles.

## 1.3 The Nonrelativistic Limit

We can study the nonrelativistic limit of the Klein-Gordon equation (1.21). In order to do this we make the ansatz

$$\psi(\boldsymbol{r},t)=\varphi(\boldsymbol{r},t)\exp\left(-\frac{\mathrm{i}}{\hbar}m_0c^2t\right) \quad , \tag{1.30}$$

i.e. we split the time dependence of $\psi$ into two terms, one containing the rest mass. In the nonrelativistic limit the difference of total energy $E$ of the particle and the rest mass $m_0c^2$ is small. Therefore we define

$$E' = E - m_0c^2$$

and remark that the kinematic energy $E'$ is nonrelativistic, which means $E' \ll m_0c^2$. Hence,

$$\left|\mathrm{i}\hbar\frac{\partial\varphi}{\partial t}\right| \approx E'\varphi \ll m_0c^2\varphi \tag{1.31}$$

holds also and with (1.31) we have

$$\begin{aligned}
\frac{\partial\psi}{\partial t} &= \left(\frac{\partial\varphi}{\partial t}-\mathrm{i}\frac{m_0c^2}{\hbar}\varphi\right)\exp\left(-\frac{\mathrm{i}}{\hbar}m_0c^2t\right) \approx -\mathrm{i}\frac{m_0c^2}{\hbar}\varphi\exp\left(-\frac{\mathrm{i}}{\hbar}m_0c^2t\right)\\
\frac{\partial^2\psi}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial t}-\mathrm{i}\frac{m_0c^2}{\hbar}\varphi\right)\exp\left(-\frac{\mathrm{i}}{\hbar}m_0c^2t\right)\\
&\approx \left[-\mathrm{i}\frac{m_0c^2}{\hbar}\frac{\partial\varphi}{\partial t}-\mathrm{i}\frac{m_0c^2}{\hbar}\frac{\partial\varphi}{\partial t}-\frac{m_0^2c^4}{\hbar^2}\varphi\right]\exp\left(-\frac{\mathrm{i}}{\hbar}m_0c^2t\right)\\
&= -\left[\mathrm{i}\frac{2m_0c^2}{\hbar}\frac{\partial\varphi}{\partial t}+\frac{m_0^2c^4}{\hbar^2}\varphi\right]\exp\left(-\frac{\mathrm{i}}{\hbar}m_0c^2t\right) \quad .
\end{aligned}$$

Inserting this result into (1.21) yields

$$\begin{aligned}
&-\frac{1}{c^2}\left[\mathrm{i}\frac{2m_0c^2}{\hbar}\frac{\partial\varphi}{\partial t}+\frac{m_0^2c^4}{\hbar^2}\varphi\right]\exp\left(-\frac{\mathrm{i}}{\hbar}m_0c^2t\right)\\
&= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{m_0^2c^2}{\hbar^2}\right)\varphi\exp\left(-\frac{\mathrm{i}}{\hbar}m_0c^2t\right)
\end{aligned}$$

or

$$\mathrm{i}\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m_0}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi = -\frac{\hbar^2}{2m_0}\Delta\varphi \quad . \tag{1.32}$$