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N. Woodhouse

GEOMETRIC QUANTIZATION

BY

NICHOLAS WOODHOUSE

Fellow of Wadham College, Oxford

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PREFACE

ACCORDING to the Copenhagen philosophy, the physical predictions of a quantum theory must be formulated in terms of classical concepts. Thus, in addition to the traditional structure of Hilbert space, unitary transformations, and self-adjoint operators, a sensible quantum theory must contain a prescription for going over to the classical limit and for relating the quantum mechanical observables to those of the corresponding classical system.

However, as Dirac stressed in the early days of quantum theory (Dirac 1926), the correspondence between a classical theory and a quantum theory should be based not so much on a coincidence between their predictions in the limit $\hbar \to 0$, as on an analogy between their mathematical structures: the primary role of the classical theory is not in approximating the quantum theory, but in providing a framework for its interpretation.

In the simple systems that one first meets in elementary quantum mechanics, the correspondence is based on canonical quantization: a classical observable—represented by some function $f(p_a, q^b)$ of the canonical coordinates—is associated with the quantum mechanical observable represented by the operator

$$f\left(-i\hbar\frac{\partial}{\partial q^a}, q^b\right).$$

This formal substitution raises many problems: for example, except in the simplest cases, the quantum observable depends on the ordering of the p_as and q^bs in the classical expression for f; the quantization depends critically on the initial choice of coordinates and it is not invariant under general canonical transformations; and the domain of the operator is left undetermined by such a formal expression. Nonetheless, judiciously supplemented with physical intuition, canonical quantization and its various generalizations have been remarkably successful.

The mathematical questions remain, however, and, although they are relatively easy to answer for simple systems in Euclidean space, they are very much harder when the classical system involves constraints or contains particles with internal degrees of freedom.

The particular question that geometric quantization attempts to answer is: to what extent is canonical quantization a well defined mathematical procedure and to what extent does it depend on the choice of canonical coordinates?

At one level, one can regard geometric quantization as a straightforward analysis of the various structures needed for the quantization of a classical system; for example, these might be preferred symmetries or special classes of coordinate systems. The aim is not to introduce new physical ideas, but to unify and clarify the various forms of canonical quantization and to make precise the analogies between the structures of classical and quantum theories. Starting with a classical phase space, represented by a symplectic manifold, one looks for a geometric, coordinate-free construction for the Hilbert space and observables of the underlying quantum theory: with no explicit dependence on a particular coordinate system, such a construction can be expected to give a very clear insight into the ambiguities involved in passing from the classical to the quantum domain.

At a more ambitious level, on the other hand, one can apply geometric quantization to systems with no special symmetries, in which the traditional forms of canonical quantization cannot be used in the obvious way; for example, to systems in curved space–time. Here a slightly different interpretation is needed: there is no one preferred quantization, but a whole family; and these agree with each other, and, presumably, with the underlying true theory, only in the semiclassical limit. In other words, here one must think of quantization as an approximation, giving only an incomplete picture of the real physics.

This book is a survey of the constructions and applications of geometric quantization, beginning with an account of symplectic geometry and the geometric formulation of Hamiltonian mechanics. (I assume that my reader has some familiarity with coordinate-free differential geometry and at least a passing acquaintance with quantum theory and Hamiltonian mechanics; the notation and some less familiar mathematical ideas are explained in the Appendix.)

I should like to stress three points: first that this is a book about quantization and not about quantum theory. The geometric method is developed not as a substitute for the normal analytic method of quantum theory, but a means of understanding more clearly the relationship between classical and quantum mechanics. Secondly, this is a work of applied, not pure mathematics. Although the mathematical ideas are fairly sophisticated, I have made no attempt to develop them beyond their immediate applications and, without, I hope, glossing over any essential difficulties, I have made no attempt to be completely rigorous. Also, I have tried not to use abstract geometric arguments in places where coordinate-based calculations are simpler and quicker. Thirdly, I should stress that the ideas presented here are not original. However, they seem to me to be sufficiently important to justify this account of them, which, I hope, will complement the work of others. The principal published sources are: Blattner (1973), Guillemin and Sternberg (1977), Kirillov (1976), Kostant (1970), Mackey (1963), Onofri and Pauri (1972), Segal (1960), Simms (1968), Souriau (1970), and Weinstein (1977). Also, I have been heavily influenced by unpublished lectures and notes by F. A. E. Pirani (in Chapter 2), D. J. Simms (in Chapter 3), B. Kostant (in Chapter 4), J. H. Rawnsley (in Chapter 6), and R. Geroch (in Chapter 7).

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Wadham College, Oxford N.M.J.W.
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CONTENTS

CHAPTER 1. SYMPLECTIC GEOMETRY

1.1. Symplectic manifolds 1
1.2. Symplectic vector spaces 1
1.3. Darboux's theorem 7
1.4. Observables and Hamiltonian vector fields 10
1.5. Poisson brackets 11
1.6. Submanifolds and reduction 12

CHAPTER 2. LAGRANGIAN AND HAMILTONIAN MECHANICS

2.1. Hamilton's principle 14
2.2. Holonomic and semi-holonomic constraints 20
2.3. Irregular Lagrangians 24
2.4. Constraints in Hamiltonian systems 26
2.5. Momentum phase space 28
2.6. Time-dependent systems 30
2.7. Relativistic systems 35
2.8. Hamilton's principle and quantization 36

CHAPTER 3. SYMMETRY

3.1. Symmetry principles in classical and quantum physics 41
3.2. Hamiltonians and comoments 42
3.3. Examples 46
3.4. Elementary systems 52
3.5. Examples 59
3.6. Group actions 61
3.7. Reduction of symmetric Hamiltonian systems 63

CHAPTER 4. POLARIZATIONS AND HAMILTON–JACOBI THEORY

4.1. The Hamilton–Jacobi equation 66
4.2. Generating functions for Lagrangian submanifolds 67
4.3. Canonical and anti-canonical transformations 69
4.4. Real polarizations: local and global geometry 73
4.5. Polynomial observables 84
4.6. Generating functions for real polarizations 86
4.7. The Hamilton–Jacobi condition 88
4.8. Separation of the Hamilton–Jacobi equation 93
4.9. Complex Lagrangian subspaces 99
4.10. Complex polarizations 101
4.11. Construction of polarizations 109

CHAPTER 5. QUANTIZATION

5.1. The quantum condition 113
5.2. Prequantization 114
5.3. The integrality condition 116
5.4. Quantizable phase spaces 121
5.5. Examples 122
5.6. Finite transformations 129
5.7. Quantization 133
5.8. Pairing 142
5.9. Densities 150
5.10. Half-density quantization 153
5.11. Examples 164

CHAPTER 6. THE METAPLECTIC CORRECTION

6.1. The corrected quantization 188
6.2. Half-forms 188
6.3. Applications 202
6.4. Cohomological wave functions 216

CHAPTER 7. SPINORS AND RELATIVISTIC SYSTEMS

7.1. Spin 225
7.2. Spinor algebra 226
7.3. Spin structures 231
7.4. Relativistic wave equations 238
7.5. Elementary relativistic systems 245
7.6. Relativistic quantization 256
7.7. Massless particles 262
7.8. Spin and gravity 269
7.9. Symplectic field theory 273
7.10. Quantum field theory 280

APPENDIX

A.1. Notation and definitions 288
A.2. Useful formulae 292
A.3. Vector bundles and line bundles 294
A.4. Cohomology 301

NOTES 305

REFERENCES 309

INDEX 313

1

SYMPLECTIC GEOMETRY

1.1. Symplectic manifolds

THE basic object in the geometric formulation of Hamiltonian mechanics is the *symplectic manifold* (1).This is a pair (M, ω) in which M is a smooth manifold and ω is a closed, nondegenerate 2-form defined everywhere on M. In other words,

$$d\omega = 0 \tag{1.1.1}$$

and the map

$$T_m M \to T_m^* M : X \mapsto X \lrcorner\, \omega \tag{1.1.2}$$

is a linear isomorphism at each $m \in M$. It is sometimes helpful to think of ω as an antisymmetric metric on M; then (1.1.2) corresponds to 'lowering the index' on X.

To begin with, we shall deal only with finite-dimensional symplectic manifolds. In physical terms, this means restricting attention to systems with only a finite number of degrees of freedom. Infinite-dimensional symplectic manifolds (representing, for example, the phase spaces of systems of classical fields) are much less straightforward objects and their study is complicated by some subtle technical questions. For example, there are a number of different forms of the nondegeneracy condition (1.1.2) and for infinite-dimensional manifolds these are not all equivalent; in field theory, this is related to the analytic problem of exactly which set of solutions of the field equations should be used to construct the classical phase space and of what topology this set should be given. A discussion of these and other matters will be postponed until Chapter 7.

I shall begin the formal development of finite-dimensional symplectic geometry by recalling some elementary facts about symplectic vector spaces and their symmetry groups.

1.2. Symplectic vector spaces

Let (V, ω) be a symplectic vector space: in other words, V is a finite-dimensional real vector space and ω is an antisymmetric, nondegenerate bilinear form on V. Thus,

$$\omega(X, Y) = -\omega(Y, X); \qquad X, Y \in V \tag{1.2.1}$$

and

$$X \lrcorner \omega = 0 \quad \text{if and only if} \quad X = 0. \tag{1.2.2}$$

(The example to keep in mind is the tangent space at some point of a symplectic manifold.)

The first fact that we shall need is that it is always possible to find a *symplectic frame* in V. This is a basis $\{X^1, X^2, \dots, X^n, Y_1, Y_2, \dots, Y_n\}$ (where $n = \frac{1}{2}\dim V$) with the property that

$$2\omega(X^a, Y_b) = \delta^a_b; \qquad a = 1, 2, \dots, n. \tag{1.2.3}$$

The proof is a straightforward modification of the familiar Gram–Schmidt construction: we start by introducing an arbitrary basis $\{Z_1, Z_2, \dots, Z_N\}$ ($N = \dim V$) and putting

$$X^1 = Z_1 \quad \text{and} \quad Y_1 = \tfrac{1}{2}(\omega(Z_1, Z_a))^{-1} Z_a \quad (\not\Sigma) \tag{1.2.4}$$

where a is the least index such that $\omega(Z_1, Z_a) \neq 0$ (a exists since ω is nondegenerate). Next, we define $\tilde{Z}_1, \tilde{Z}_2, \dots, \tilde{Z}_{N-2}$ by

$$\begin{aligned} \tilde{Z}_b &= Z_{b+1} + 2\omega(Y_1, Z_{b+1})X^1 \\ &\quad - 2\omega(X^1, Z_{b+1})Y_1; \qquad b = 1, 2, \dots, a-2 \\ &= Z_{b+2} + 2\omega(Y_1, Z_{b+2})X^1 \\ &\quad - 2\omega(X^1, Z_{b+2})Y_1; \qquad b = a-1, \dots, N-2 \end{aligned} \tag{1.2.5}$$

Then $\{X^1, Y_1, \tilde{Z}_1, \tilde{Z}_2, \dots, \tilde{Z}_{N-2}\}$ is again a basis and

$$\omega(X^1, \tilde{Z}_b) = \omega(Y_1, \tilde{Z}_b) = 0; \qquad b = 1, 2, \dots, N-2. \tag{1.2.6}$$

Repeating the process, putting $X^2 = \tilde{Z}_1$ and so on, we arrive after $n = \frac{1}{2}N$ steps at a symplectic frame. (This argument also shows that every symplectic vector space, and hence every finite-dimensional symplectic manifold, is even-dimensional.)

The set of all symplectic frames in V is acted upon transitively (on the right) by the *symplectic group* $SP(n, \mathbb{R})$, which is the $(2n^2+n)$-dimensional subgroup of $GL(2n, \mathbb{R})$ of matrices of the form

$$M = \begin{bmatrix} C_a{}^b & D_{ab} \\ E^{ab} & F^a{}_b \end{bmatrix} \tag{1.2.7}$$

where $(C_a{}^b)$, (D_{ab}), (E^{ab}), and $(F^a{}_b)$ are $n \times n$ matrices satisfying

$$C_a{}^b F^a{}_c - E^{ab} D_{ac} = \delta^b_c, \qquad C_a{}^b E^{ac} = E^{ab} C_a{}^c, \qquad D_{ab} F^a{}_c = F^a{}_b D_{ac}. \tag{1.2.8}$$

(Here, and below, I am using the range and summation conventions for the indices $a, b, c, \dots$ running over the values $1, 2, \dots, n$; the significance of the positioning of the indices will emerge shortly.) Explicitly, the action is given by

$$M : \{X^a, Y_b\} \mapsto \{\tilde{X}^a, \tilde{Y}_b\} = \{X^c C_c{}^a + Y_c E^{ca}, X^c D_{cb} + Y_c F^c{}_b\}. \tag{1.2.9}$$

ferometer shown in Fig.1.2, in which M_1 and M_2 are mirrors, and L a half-silvered plate (a more detailed description of the equipment is found in [1.1]). Let **v** be the earth's translational velocity, oriented as shown. According to the aether model, rays 1 and 1' have velocities $(c-v)$ and $(c+v)$ with respect to the interferometer. The total "time of flight", from O to B and back to O, is therefore

$$t_1 = \frac{OB}{c-v} + \frac{OB}{c+v} = 2\,\frac{OB}{c}\,\frac{1}{1-v^2/c^2} \ . \tag{1.5}$$

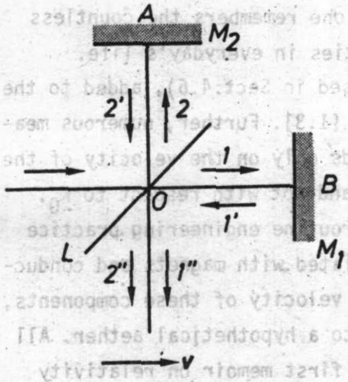

Fig.1.2. Sketch of the Michelson-Morley experiment

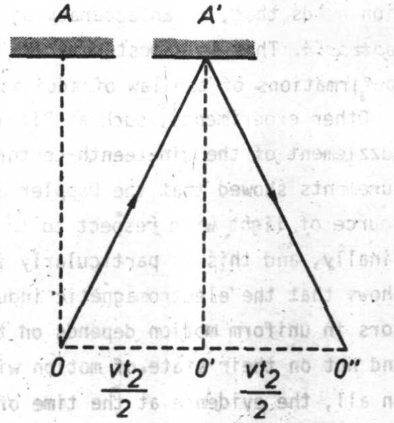

Fig.1.3. Relevant to the estimation of a travel time interval

To determine the time of flight of rays 2 and 2', we must take into account the perpendicular displacement of the interferometer during the time of flight. A look at Fig.1.3 shows that the travel time from O to A and back to O satisfies

$$t_2 = \frac{OA'}{c} + \frac{A'O''}{c} = \frac{2}{c}\,\sqrt{(O'A')^2 + (vt_2/2)^2} \ . \tag{1.6}$$

The travel time t_2 is therefore

$$t_2 = \frac{2}{c}\,\frac{OA}{\sqrt{1-v^2/c^2}} \ . \tag{1.7}$$

As the translational velocity of the earth at the location of the experiment is of the order of $300\ \mathrm{m\,s^{-1}}$, the ratio v/c is exceedingly small, and the difference of time intervals of flight can be written, to a good approximation, as

2. Two Lagrangian subspaces $H, K \subset V$ are said to be *transverse* whenever $H \cap K = \{0\}$. The subgroup of $SP(V, \omega)$ of linear canonical transformations $\rho: V \to V$ with the property that

$$\rho(H) = H \quad \text{and} \quad \rho(K) = K \tag{1.2.16}$$

is then denoted $GL(H, K)$.

Given two transverse Lagrangian subspaces H and K, we can use the bilinear form ω to identify H with K^* by associating $X \in H$ with $\alpha_X \in K^*$ where (2)

$$\alpha_X(Y) = 2\omega(X, Y); \qquad Y \in K. \tag{1.2.17}$$

The map $X \mapsto \alpha_X$ is clearly linear; it is also bijective since we have

LEMMA 1.2.1. *Let $K \subset V$ be a Lagrangian subspace. Then $X \in K$ if and only if $\omega(X, Y) = 0 \; \forall Y \in K$.*

(The proof is a straightforward dimensional argument.) Thus if $X \in H$ and $\alpha_X = 0$, then $X \in K$, which implies that $X = 0$ since H and K are transverse.

If $\{Y_a\}$ is a basis in K and $\{X^a\}$ is the dual basis in H, then $\{X^a, Y_b\}$ is a symplectic frame. Using this to identify $SP(V, \omega)$ with $SP(n, \mathbb{R})$, $GL(H, K)$ becomes the group of matrices of the form

$$M = \begin{bmatrix} \hat{L}_a{}^b & 0 \\ 0 & L^a{}_b \end{bmatrix} \quad \text{where} \quad \hat{L}_a{}^b L^a{}_c = \delta^b_c. \tag{1.2.18}$$

This in turn is isomorphic with $GL(n, \mathbb{R})$.

3. A linear canonical transformation $J: V \to V$ that satisfies $J^2 = -1$ (where 1 is the identity) is called a *compatible complex structure* on (V, ω). Associated with J there is the bilinear form

$$g(X, Y) = 2\omega(X, JY); \qquad X, Y \in V, \tag{1.2.19}$$

which is symmetric since

$$\omega(X, JY) = \omega(JX, J^2Y) = -\omega(JX, Y) = \omega(Y, JX); \tag{1.2.20}$$

J is said to be *positive* whenever g is positive definite.

The subgroup of $SP(V, \omega)$ of linear canonical transformations $\rho: V \to V$ that satisfy

$$\rho J = J\rho \tag{1.2.21}$$

is denoted $U(J)$.

If $\{X^a, Y_b\}$ is a symplectic frame and (g_{ab}) is any symmetric nonsingular $n \times n$ matrix, then we can define a compatible complex structure J by

$$JX^a = g^{ab} Y_b, \qquad JY_a = -g_{ab}X^b \tag{1.2.22}$$

where $g^{ab}g_{bc} = \delta^a_c$; J is positive whenever (g_{ab}) is positive definite.

Conversely, given a compatible complex structure, J, it is possible to find a symplectic frame $\{X^a, Y_b\}$ such that J is determined by (1.2.22). To prove this, we first note that J gives V the structure of a complex n-dimensional vector space with an indefinite inner product: scalar multiplication by $z=x+iy\in \mathbb{C}$ is defined by

$$zX=xX+yJX; \qquad X\in V \tag{1.2.23}$$

and the inner product is

$$\langle X, Y\rangle=2[\omega(X,JY)-i\omega(X,Y)]; \qquad X, Y\in V \tag{1.2.24}$$

(this is linear in the first entry and antilinear in the second; it is always nondegenerate and it is positive definite whenever J is positive). It is possible, therefore, to find n vectors $Y_1, Y_2,\ldots, Y_n$ that form a basis for V as a complex vector space and which satisfy

$$\langle Y_a, Y_b\rangle=g_{ab} \tag{1.2.25}$$

where

$$(g_{ab})=\mathrm{diag}(1, 1,\ldots, 1, -1, -1,\ldots, -1), \tag{1.2.26}$$

with k ones and $n-k$ minus ones, according to the signature of $\langle .,.\rangle$. Thus if the X^as are defined by

$$X^a=-g^{ab}JY_b \quad \text{where} \quad g^{ab}g_{bc}=\delta^a_c, \tag{1.2.27}$$

then $\{X^a, Y_b\}$ is a symplectic frame with the stated property.

When this is used to identify $SP(V,\omega)$ with $SP(n,\mathbb{R})$, $U(J)$ becomes the subgroup of $SP(n,\mathbb{R})$ of matrices of the form

$$M=\begin{bmatrix} g_{ac}A^c{}_d g^{bd} & -g_{ac}B^c{}_b \\ B^a{}_d g^{db} & A^a{}_b \end{bmatrix} \tag{1.2.28}$$

where $(A^a{}_b+iB^a{}_b)$ is a pseudo-unitary matrix; thus $U(J)$ is isomorphic with the pseudo-unitary group $U(k, n-k)$. In particular, when J is positive, $g_{ab}=\delta_{ab}$ and $(A^a{}_b+iB^a{}_b)$ is unitary. □

Thirdly, we need some facts about various special subspaces of V. Given a subspace $F\subset V$, the *annihilator* of F is defined to be the subspace

$$F^0=\{X\in V; \omega(X,Y)=0\ \forall Y\in F\}. \tag{1.2.29}$$

The basic properties of the annihilator are contained in the two lemmas:

LEMMA 1.2.2. *Let F, $G\subset V$ be subspaces. Then*
(*a*) $F^0\supset G^0$ *whenever* $G\supset F$
(*b*) $(F^0)^0=F$
(*c*) $(F+G)^0=F^0\cap G^0$
(*d*) $(F\cap G)^0=F^0+G^0$.

Proof. The first statement is obvious and the second follows from the fact that $F\subset(F^0)^0$ and $\dim F^0+\dim F=2n$. To prove statement (*c*), note that if

$X \in F^0 \cap G^0$, then

$$\omega(X, Y)=0 \ \forall Y \in F+G \tag{1.2.30}$$

so that $X \in (F+G)^0$. Thus $F^0 \cap G^0 \subset (F+G)^0$. Conversely, since $F \subset F+G$ and $G \subset F+G$, we have $F^0 \supset (F+G)^0$ and $G^0 \supset (F+G)^0$. Therefore, $F^0 \cap G^0 \supset (F+G)^0$ and hence $F^0 \cap G^0 = (F+G)^0$. Finally, using (*b*) and (*c*),

$$(F^0+G^0)^0=(F^0)^0 \cap (G^0)^0=F \cap G. \tag{1.2.31}$$

Hence, using (*b*) again, $(F \cap G)^0=F^0+G^0$, which proves (*c*). □

LEMMA 1.2.3. *Let $F \subset V$ be a subspace and let $V'=F/(F \cap F^0)$. Then ω projects onto a bilinear form ω' on V' and (V', ω') is a symplectic vector space.*

Proof. Let $\rho: F \to V'$ denote the projection. Then ω' is defined by

$$\omega'(X', Y')=\omega(X, Y) \tag{1.2.32}$$

where $X, Y \in F$ and $X'=\rho(X)$ and $Y'=\rho(Y)$. This is clearly well defined, skew-symmetric, and nondegenerate. □

A subspace $F \subset V$ is said to be

(a) *Isotropic* whenever $F \subset F^0$;
(b) *Coisotropic* whenever $F^0 \subset F$;
(c) *Symplectic* whenever $F \cap F^0=\{0\}$.

In the first case, $\dim F \leqslant n$; in the second, $\dim F \geqslant n$; and, in the third, $\dim F$ is even. If F is isotropic, then F^0 is coisotropic, and conversely. Also, F is Lagrangian if and only if it is both isotropic and coisotropic (and therefore n-dimensional). Note, however, that these categories are not exhaustive: it is possible for a subspace to be neither isotropic, nor coisotropic, nor symplectic.

Finally, we note that these ideas have obvious extensions to the complex case: a *complex symplectic vector space* is a finite-dimensional complex vector space W on which there is given a complex valued nondegenerate bilinear form σ. (In the example that we shall meet most frequently, W is the complexification of a real symplectic vector space V, and σ is defined by linearly extending the symplectic form on V.) The annihilator F^0 of a complex subspace $F \subset W$ is defined as before and we say that F is *complex isotropic*, *complex coisotropic*, *complex Lagrangian*, or *complex symplectic* as $F \subset F^0$, $F^0 \subset F$, $F^0=F$, or $F \cap F^0=\{0\}$ (but again the categories are not exhaustive). Lemmas 1.2.2 and 1.2.3 also hold for complex subspaces.

More details about symplectic linear algebra can be found in Souriau (1970), Kirillov (1976), Dixmier (1974), Weinstein (1977), and Guillemin and Sternberg (1977).

1.3. Darboux's theorem

One of the most important examples of a symplectic manifold is the cotangent bundle $M = T^*Q$ of an n-dimensional manifold Q. This is the set of pairs (p, q) where $q \in Q$ and p is a covector at q. It is made into a manifold by using as coordinates the set of $2n$ functions $\{p_a, q^b\}$ where the q^bs are coordinates on Q and the p_as are the corresponding components of the covectors p. ($\{p_a, q^b\}$ is called the *extension* to T^*Q of the coordinate system $\{q^b\}$.)
The symplectic structure on M is the *canonical 2-form* ω defined by

$$\omega = \mathrm{d}p_a \wedge \mathrm{d}q^a. \tag{1.3.1}$$

There is also an intrinsic construction for ω which gives a simple way of seeing that it is, in fact, independent of the coordinates q^a: let $\pi: M \to Q$ denote the projection map $(p, q) \mapsto q$ and, for each $m = (p, q) \in M$, define $\theta_m \in T_m^*M$ by

$$X \lrcorner \theta_m = (\pi_* X) \lrcorner p; \qquad X \in T_m M. \tag{1.3.2}$$

As m varies, θ becomes a smooth 1-form on M (called the *canonical 1-form*), which is given in the coordinates $\{p_a, q^b\}$ by

$$\theta = p_a \, \mathrm{d}q^a. \tag{1.3.3}$$

Thus ω is also determined by

$$\omega = \mathrm{d}\theta \tag{1.3.4}$$

without reference to any particular coordinate system on Q. Clearly ω is closed (since it is also exact) and nondegenerate.

In the typical physical applications, Q is the configuration space of some mechanical system (such as a collection of particles subject to holonomic constraints) and T^*Q is the phase space of the system.

The cotangent bundle is a fundamental example since all symplectic manifolds have this form locally, as follows from Darboux's theorem (3):

THEOREM 1.3.1. *Let (M, ω) be a $2n$-dimensional symplectic manifold and let $m \in M$. Then there is a neighbourhood U of m and a coordinate system $\{p_a, q^b\}$ $(a, b = 1, 2, \ldots, n)$ on U such that $\omega|_U = \mathrm{d}p_a \wedge \mathrm{d}q^a$.*

The following proof is due to Weinstein (1971) and it is based on a suggestion of Moser's. The key step is a lemma of Moser's (1965).

LEMMA 1.3.1. *Let ω and σ be two closed, nondegenerate 2-forms on a manifold M and let $m \in M$. If $\omega_m = \sigma_m$, then there are neighbourhoods U and V of m and a diffeomorphism $\rho: U \to V$ such that $\rho(m) = m$ and $\rho^*(\sigma) = \omega$.*

Proof. Since $\mathrm{d}(\sigma - \omega) = 0$, there is a 1-form α defined on some neighbourhood W of m such that

$$\mathrm{d}\alpha = \sigma - \omega. \tag{1.3.5}$$

By adding the gradient of a scalar to α (if necessary), we can ensure that $\alpha_m = 0$.

Put

$$N = W \times [0, 1] = \{(x, t); x \in W, t \in [0, 1]\} \tag{1.3.6}$$

and define a closed 2-form Ω on N by

$$\Omega = \mathrm{pr}^*(\omega) + t\, \mathrm{pr}^*(\sigma - \omega) + \mathrm{d}t \wedge \mathrm{pr}^*(\alpha) \tag{1.3.7}$$

where $\mathrm{pr}: N \to W$ is the projection onto the first factor. For each $t \in [0, 1]$, put

$$\Omega_t = \Omega|_{W \times \{t\}} = \omega + t(\sigma - \omega) \tag{1.3.8}$$

(there is a minor abuse of notation here: we are identifying $W_t = W \times \{t\}$ with W). Then, at $(m, t) \in W_t$, $\Omega_t = \omega$. Hence, provided that W has been sufficiently restricted, each Ω_t is nondegenerate. It follows that there is a unique vector field X on N such that

$$X \lrcorner \mathrm{d}t = 1 \quad \text{and} \quad X \lrcorner \Omega = 0. \tag{1.3.9}$$

For each $x \in W$, let $t \mapsto \varphi_x(t)$ be the integral curve of X through $(x, 0)$. Then the first of the equations in (1.3.9) implies that $\varphi_x(t) \in W \times \{t\}$ for each value of t for which φ_x is defined. Also, since $\alpha_m = 0$,

$$\varphi_m(t) = (m, t) \tag{1.3.10}$$

for every t in $[0, 1]$. Hence there is a neighbourhood U of m such that $\varphi_x(t)$ is defined for each $x \in U$ and for each $t \in [0, 1]$.

Now

$$\mathscr{L}_X \Omega = X \lrcorner \mathrm{d}\Omega + \mathrm{d}(X \lrcorner \Omega) = 0. \tag{1.3.11}$$

Thus if $\rho: U \to V = \rho(U) \subset W$ is the diffeomorphism determined by

$$(\rho(x), 1) = \varphi_x(1); \qquad x \in U \tag{1.3.12}$$

then $\rho^*(\Omega_1) = \Omega_0$. Also, from (1.3.8), $\Omega_1 = \sigma$ and $\Omega_0 = \omega$ and, from (1.3.10), $\rho(m) = m$. □

Proof of Theorem 1.3.1. Let $\{z^1, z^2, \ldots, z^{2n}\}$ be a coordinate system in some neighbourhood of m. Then the coordinate vectors form a basis for $T_m M$ and it follows from the discussion in §1.2 that we can find a linear transformation $\{z^1, \ldots, z^{2n}\} \mapsto \{r_1, r_2, \ldots, r_n, s^2, \ldots, s^n\}$ of the coordinates such that the new coordinate vectors

$$X^a = \frac{\partial}{\partial r_a}, \qquad Y_b = \frac{\partial}{\partial s^b} \tag{1.3.13}$$

form a symplectic frame at m. Thus if we take

$$\sigma = \mathrm{d}s_a \wedge \mathrm{d}r^a \tag{1.3.14}$$