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（影印版）43

I. R. Shafarevich (Ed.)

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## Algebraic Curves, Algebraic Manifolds and Schemes

# 代数几何 I

## 代数曲线，代数流形与概型

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# 《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

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V. N. Shokurov †

# Contents

# I. Riemann Surfaces and Algebraic Curves

V. V. Shokurov

Translated from the Russian
by V. N. Shokurov

## Contents

# Introduction[1]

The name 'Riemann surface' is a rare case of a designation which is fully justified historically: all fundamental ideas connected with this notion belong to Riemann. Central among them is the idea that an analytic function of a complex variable defines some natural set on which it has to be studied. This need not coincide with the domain of the complex plane where the function was initially given. Usually, this natural set of definition does not fit into the complex plane $\mathbb{C}$, but is a more complicated surface, which must be specially constructed from the function: this is what we call the Riemann surface of the function. One can get a complete picture of the function only by considering it on the whole of its Riemann surface. This surface has a nontrivial geometry, which determines some of the essential characters of the function.

The extended complex plane, obtained by adjoining a point at infinity, can be perceived as an embryonic form of this approach. Topologically, the extended plane is a two-dimensional sphere, also known as the Riemann sphere. This example already displays some features which are characteristic of the general notion of a Riemann surface:

1) The Riemann sphere $\mathbb{CP}^1$ can be defined by gluing together two disks (i.e., circles) of the complex plane; for instance, the disks $|z| < 2$ and $|w| < 2$, in which the annuli $\frac{1}{2} < |z| < 2$ and $\frac{1}{2} < |w| < 2$ are identified by means of the correspondence $w = z^{-1}$. (This yields the shaded area in Fig. 1.)

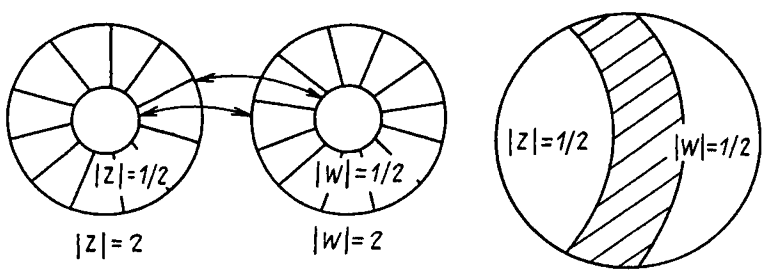


Fig. 1

2) The relation $w = z^{-1}$, which defines the gluing, is a one-to-one and analytic (conformal) correspondence of the domains it identifies. For that reason the property of being analytic at some point agrees in both circles, $|z| < 2$ and $|w| < 2$, on the identified regions. This leads to a unified notion

[1] The author expresses his profound gratitude to Professor I. R. Shafarevich for numerous remarks and suggestions, which have contributed to the improvement of the text, and for writing this introduction, which provides a fascinating bird's-eye view of the charming world of algebraic geometry.

of analytic function on the Riemann sphere glued from them. It is therefore possible to state and prove such theorems as: 'a function which is holomorphic on the whole Riemann sphere is constant', or: 'a function on the Riemann sphere which has only poles for singularities, is a rational function'.

The same principles underlie the general notion of a Riemann surface. We shall deal only with compact Riemann surfaces. By definition, this is a closed (compact) surface $S$ glued from a finite number of disks $U_1, \ldots, U_m$ in the complex plane: for any two disks, $U_i$ and $U_j$, some domains, $V_{ij} \subset U_i$ and $V_{ji} \subset U_j$, are identified by means of a correspondence $\varphi_{ij}\colon V_{ij} \to V_{ji}$, which is one-to-one and analytic.

In other words, a Riemann surface is a union of sets $U_1, \ldots, U_N$, each of which is endowed with a coordinate function $z_i$ $(i = 1, \ldots, N)$. This is a one-to-one mapping of $U_i$ onto a disk in the complex plane. Further, in an intersection $V_{ij} = U_i \cap U_j$, the coordinate $z_j$ is expressed in terms of $z_i$ as an analytic function, and similarly $z_i$ in terms of $z_j$.

Thus, just as in the case of the Riemann sphere, there is a well-defined notion of analyticity for a continuous complex-valued function, given in a neighbourhood of some point $p \in S$. Further, we can carry over to functions given on the surface $S$ such notions as a pole, the property of being meromorphic, and so forth. Hence a Riemann surface is a set on which it makes sense to say that a function is analytic, and locally (in a sufficiently small domain) this amounts to the ordinary concept of analyticity in some domain of the complex plane. This definition is explained in detail in § 1 of Chapter 1.

So, with the notion of a Riemann surface, we run into an entity of a new mathematical nature. It must be rated on a par with such notions as a Riemannian manifold in geometry, or a field in algebra. Just as some metric concepts are defined in a Riemannian manifold, and algebraic operations in a field, so is the notion of analytic function on a Riemann surface. In particular, it is now possible to formulate and prove the theorem stating that a function which is holomorphic on an entire (compact) Riemann surface is constant.

That the concept of Riemann surface is nontrivial, is manifest from its connection with the theory of multivalued analytic functions. In fact, for every such function one can construct a Riemann surface on which it becomes single-valued. We restrict ourselves to algebraic functions, so the corresponding Riemann surfaces are compact.

The simplest case, represented by the function $w = \sqrt[n]{z}$, does not yet necessitate any new type of surface. Indeed we have $z = w^n$; so, even though $w$ is a multivalued function of $z$, the function $z(w)$ is single-valued. Therefore we can regard $w$ as an independent variable, running over the Riemann sphere $S$, which is just the Riemann surface of the function $w$. The relation $z = w^n$ defines a mapping of the $w$-sphere $S$ onto the $z$-sphere $\mathbb{CP}^1$. One can think of the sphere $S$ as lying 'above' $\mathbb{CP}^1$ (in some larger space), in such a way that above each point $z = z_0$ we find the points which are mapped into it. Then for $z_0 \neq 0, \infty$ the inverse image on $S$ of a disk $U\colon |z - z_0| < \varepsilon$, for sufficiently small $\varepsilon$, is made up of $n$ disjoint domains $W_i$, $i = 1, \ldots, n$:

$$w = w_i g(t), \quad |t| < \frac{\varepsilon}{|z_0|}, \quad g(t) = \sqrt[n]{1+t}, \quad g(0) = 1, \quad t = \frac{z}{z_0} - 1,$$

where the $w_i$ are the distinct values of $\sqrt[n]{z_0}$ (Fig. 2$a$). But, in a neighbourhood of the point 0 (respectively, of $\infty$), the inverse image of a disk $|z| < \varepsilon$ (respectively, $|t| < \varepsilon$, with $t = z^{-1}$) is constituted by a single circle $W$: $|w| < \sqrt[n]{\varepsilon}$, which lies above the disk in the form of a 'helix' (see Fig. 2$b$, where $n = 2$).

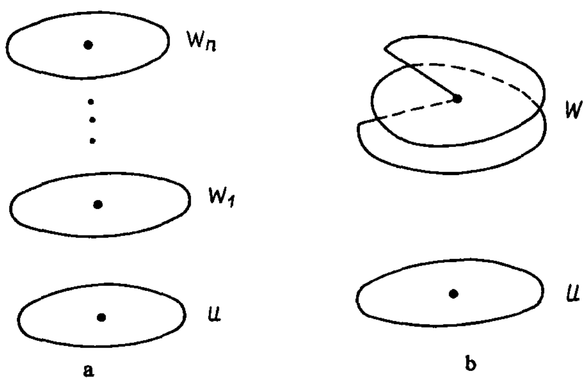


Fig. 2

In the general case, an algebraic function is defined by an equation $f(z,w) = 0$, where $f(z,w)$ is a polynomial $f(z,w) = a_0(z)\, w^n + \ldots + a_n(z)$, and the $a_i(z)$ are polynomials in $z$. As a first, rough approximation to the Riemann surface of the function $w$, we shall look at the set $\widetilde{S}$ of all solutions $(z,w)$ of $f(z,w) = 0$. On this set, $w$ is tautologically the function that takes on the value $w_0$ at $(z_0, w_0)$. However, this definition must be made more precise. We shall assume that $\widetilde{S} \subset \mathbb{C}^2$, where $\mathbb{C}^2$ is the plane of the two complex variables $z, w$, and where the topology of $\widetilde{S}$ is inherited from $\mathbb{C}^2$. In other words, $\widetilde{S}$ is a complex algebraic curve lying in the plane $\mathbb{C}^2$.

To start with, suppose $z_0$ is such that $f(z_0, w) = 0$ has $n$ distinct roots $w_1, \ldots, w_n$. This means that $a_0(z_0) \neq 0$ and $f'_w(z_0, w_i) \neq 0$. Then, by the implicit function theorem, $w$ is an analytic function $g_i(z)$ of $z$ in some neighbourhood $|z - z_0| < \varepsilon$ of $z_0$. More precisely, all solutions of $f(z,w) = 0$ close to $(z_0, w_i)$ can be represented in the form $(z, g_i(z))$, $i = 1, \ldots, n$. That is to say, the solutions with $|z - z_0| < \varepsilon$ fall into $n$ disks $W_i$, $i = 1, \ldots, n$:

$$|z - z_0| < \varepsilon, \quad w = g_i(z),$$

exactly as in Fig. 2$a$. We call them disks because the function $z$ maps them in a one-to-one manner onto the disk $U$: $|z - z_0| < \varepsilon$.

It remains to consider the cases we have omitted, in which the number of solutions of $f(z_0, w) = 0$ is less than $n$, and also the case where $z_0 = \infty$ on the Riemann sphere $\mathbb{CP}^1$. In all these cases there exists a disk $U : |z - z_0| < \varepsilon$ (respectively, $|t| < \varepsilon$, $t = z^{-1}$, if $z_0 = \infty$) with the property that, for all points $z \in U$, $z \neq z_0$, we are in the case previously considered. We denote by $\widetilde{U}$ the associated punctured disk: $|z - z_0| < \varepsilon$, $z \neq z_0$, and by $\widetilde{W}$ its inverse image in $\widetilde{S}$. The set $\widetilde{W}$ may turn out to be disconnected.

Trivially, if $f(z_0, w) = 0$ has two distinct solutions, $w_i$ and $w_j$, then two small neighbourhoods in $\widetilde{S}$ do not meet and give rise to different connected components of $\widetilde{W}$, like the sets $W_1, \ldots, W_n$ in Fig. 2*a*. But there are less trivial cases in which various connected components of $\widetilde{W}$ converge to the same point of $\widetilde{S}$. The idea is that in reality these components must define distinct points of the Riemann surface $S$ of $w$: they must be 'separated' in $S$. If, for instance, $w^2 = z^2 + z^3$ then $w = z\sqrt{1+z}$. Now the function $\sqrt{1+z}$ has two branches, $g_1(z)$ and $g_2(z) = -g_1(z)$, in a neighbourhood of $z_0 = 0$. So $\widetilde{W}$ consists of two components: $\widetilde{W}_1 = \{|z| < \varepsilon,\ z \neq 0,\ w = zg_1(z)\}$ and $\widetilde{W}_2 = \{|z| < \varepsilon,\ z \neq 0,\ w = zg_2(z)\}$, which merge as $z \to 0$ (Fig. 3*a*).

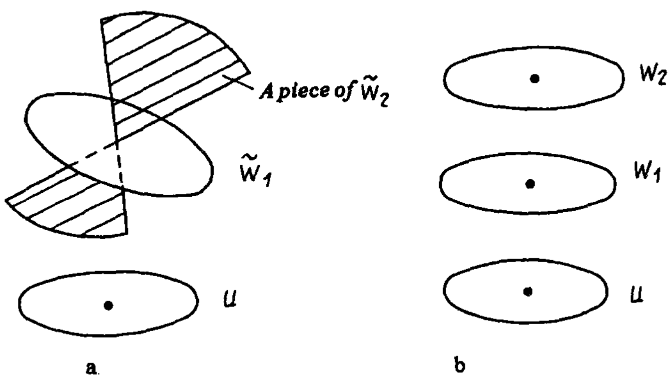


Fig. 3

In the general case, we denote by $\widetilde{W}_1, \ldots, \widetilde{W}_r$ the connected components of $\widetilde{W}$. The Riemann surface $S$ is defined in such a way that in it the $\widetilde{W}_i$ are, so to speak, 'isolated' from each other: their closures do not meet as $z \to z_0$. Set-theoretically, $S$ differs from $\widetilde{S}$ in that now there are $r$ distinct points above $z_0$, each corresponding to its own component $\widetilde{W}_i$. More precisely, each $\widetilde{W}_i$ is a connected unramified covering of the punctured disk $\widetilde{U}$: above every point $z \in \widetilde{U}$, we find the same number $n_i$ (say) of points in $\widetilde{W}_i$, and $n_1 + \ldots + n_r = n$. It is easy to prove that a function $w_i$ can be defined on each $\widetilde{W}_i$ in such a way that $\widetilde{W}_i$ is given as the punctured disk $|w_i| < \varepsilon^{1/n_i}$, $w_i \neq 0$, and the mapping $\widetilde{W}_i \to \widetilde{U}$ is defined as $z - z_0 = w_i^{n_i}$. We can then look at the unpunctured disk $W_i$: $|w_i| < \varepsilon^{1/n_i}$. The various disks $W_i$ are regarded as disjoint sets in the Riemann surface $S$ (cf. Fig. 3*b*). Each of them is mapped by the function $w_i$ onto a disk of the complex plane, and they lie above the Riemann $z$-sphere as in Fig. 2*b*.

From all the disks $W_i$ we have constructed, above the various points $z_0 \in \mathbb{CP}^1$ (including $z_0 = \infty$), we can select a finite number, $W_1, \ldots, W_N$, whose union already contains all the others. From the analyticity of all the mappings we have encountered, it is easy to deduce that the variety obtained by gluing the disks $W_1, \ldots, W_N$ verifies the condition occurring in the definition of a Riemann surface. Thus, $S$ is indeed a Riemann surface. For a detailed justification of this construction, see Chapter 1, § 2.

An arbitrary Riemann surface carries with it a large amount of geometric information. In particular, the Riemann surface of an algebraic function reveals some important characteristics of that function. Since the gluings $\varphi_{ij}$ are conformal, and hence orientation-preserving, transformations, any Riemann surface is orientable. So, from a topological point of view it has a unique invariant: the genus. In Fig. 4 are depicted surfaces of genus $g = 0, 1, 2, 3, 4$.

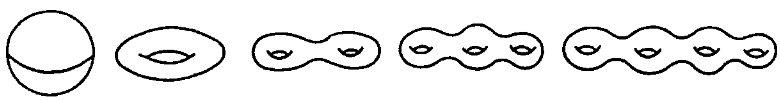

Fig. 4

If, for example, a polynomial $f(z)$ (of degree $2n$ or $2n-1$, say) has no multiple roots, then the Riemann surface of the function $w = \sqrt{f(z)}$ is of genus $n-1$. But, in addition, one can define on a Riemann surface all the notions which are invariant under conformal transformations: it has a 'conformal geometry'. Among such notions are the Laplace operator and harmonic functions. In particular, the real and imaginary parts of a function which is analytic in some domain of a Riemann surface, are harmonic. This enables us to study functions on a Riemann surface by applying the apparatus of elliptic differential operators and even some physical intuition. A harmonic function on a Riemann surface can be conceived as a description of a stationary state of some physical system: a distribution of temperatures, for instance, in case the Riemann surface is a homogeneous heat conductor. Klein (following Riemann) had a very concrete picture in his mind:

*"This is easily done by covering the Riemann surface with tin foil... Suppose the poles of a galvanic battery of a given voltage are placed at the points $A_1$ and $A_2$. A current arises, whose potential $u$ is single-valued, continuous, and satisfies the equation $\Delta u = 0$ across the entire surface, except for the points $A_1$ and $A_2$, which are discontinuity points of the function."*

[*Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, p. 260]

The existence of functions, which is suggested by such physical considerations, is established on the basis of the theory of elliptic partial differential equations. This provides an absolutely new method of constructing analytic functions on a Riemann surface: once a harmonic function $u$ has been constructed, we select its conjugate function $v$, so that $u + iv$ is analytic.

In particular, this enables one to describe the stock of all meromorphic functions on any Riemann surface $S$. If $S$ is the Riemann surface of an algebraic function $w$ given by $f(z, w) = 0$, then both $w$ and $z$ are meromorphic functions on $S$. Therefore any rational function of $w$ and $z$ is meromorphic. It can easily be proved that this is the way all meromorphic functions on $S$ are obtained. This is a generalization of the theorem saying that a meromorphic function on the Riemann sphere is a rational function of $z$. For an arbitrary Riemann surface, however, it is by no means obvious that there is even one nonconstant meromorphic function. Such a function is constructed, as we have just said, by using methods from the theory of elliptic partial differential equations. Furthermore, one can construct along the same lines two meromorphic functions $w$ and $z$ on $S$, connected by a relation of the form $f(z, w) = 0$, where $f$ is a polynomial, and with the property that $S$ is just the Riemann surface of the algebraic function $w$ defined by the equation $f = 0$. This result is known as 'Riemann's existence theorem'.

Hence the abstract notion of a (compact) Riemann surface reduces to that of Riemann surface for an algebraic function. This is a highly nontrivial result, with powerful applications. Indeed, in a number of particular situations, what arises is an 'abstract' Riemann surface. Then the preceding theorem provides a very explicit realization of such a surface. The simplest example of such a situation is when $S$ is the quotient group $\mathbb{C}/\Lambda$ of the complex plane $\mathbb{C}$ modulo a lattice $\Lambda = \{\omega_1 n_1 + \omega_2 n_2 \mid n_1, n_2 \in \mathbb{Z}\}$, spanned by two complex numbers $\omega_1$ and $\omega_2$. Let $U$ be any sufficiently small disk, so that no two of its points differ by a vector from $\Lambda$. Then the coordinate $z$ on $\mathbb{C}$ is a one-to-one mapping of $U$ onto a domain in $S = \mathbb{C}/\Lambda$ (Fig. 5). Further, these disks form a covering of $S$. Topologically $S$ is a torus: it is of genus 1. In this situation, Riemann's existence theorem shows that $S$ is the Riemann surface of an algebraic function $w = \sqrt{z^3 + az + b}$, where $a$ and $b$ are some complex numbers and the polynomial $z^3 + az + b$ has no multiple roots. It can be shown that every Riemann surface of genus 1 can be obtained in this way. The meromorphic functions on $S$ are interpreted as being all meromorphic functions of $z$ which are invariant under translations by vectors of the lattice $\Lambda$, that is, elliptic functions. In this case, Riemann's existence theorem furnishes a very explicit description of an elliptic function field.

Such a description is possible for Riemann surfaces of genus $g > 1$ as well. One has to consider discrete groups of linear fractional transformations acting in the disk $|z| < 1$. Two points are identified if they are sent to each other by an element of such a group $\Gamma$. Thus the Riemann surface is represented as a quotient $\Gamma \backslash \mathbb{D}$, where $\mathbb{D}$ is the unit disk. Just like the plane $\mathbb{C}$ (for surfaces of genus 1), the unit disk, for genus $g > 1$, is the universal covering of the Riemann surface $S$. For $g = 0$, $S$ is nothing else than the Riemann sphere and is its own universal covering. In the plane $\mathbb{C}$ the Euclidean metric

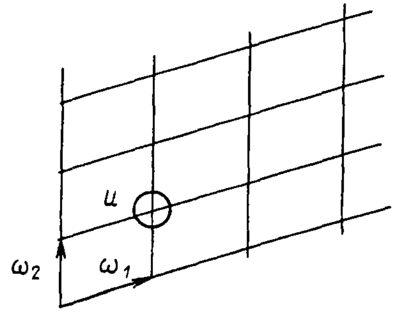


Fig. 5

$ds^2 = |dz|^2$ is invariant under transformations of the group $\Lambda$ and specifies a metric of zero-curvature on the surface $S$. Likewise, in the unit circle the metric $ds^2 = |dz|^2/(1 - |z|^2)$ defines a Lobachevskian geometry of constant negative curvature, and hence a similar metric on the surface $S = \Gamma\backslash\mathbb{D}$ as well. Finally, there is a metric of constant positive curvature on the sphere $\mathbb{CP}^1$. In all three cases, these metrics provide the Riemann surface $S$ with a 'conformal geometry'. Hence the properties of Riemann surfaces depending on their topology can be summarized in the following Table:

| Genus | Type of universal covering | Metric of constant curvature $K$ |
|---|---|---|
| 0 | Riemann sphere $\mathbb{CP}^1$ | $K > 0$ |
| 1 | $\mathbb{C}$ | $K = 0$ |
| $> 1$ | $\mathbb{D} = \{z,\ |z| < 1\}$ | $K < 0$ |

One sees from this table that on any Riemann surface $S$ one can define a metric $ds^2$ of constant curvature $K$ which provides the surface with a conformal geometry. The converse is also true: any metric $ds^2 = Edx^2 + 2Fdxdy + Gdy^2$ on a compact orientable surface $S$ defines on it a Riemann surface structure. Namely, it can be proved that, in a neighbourhood $U$ of any point on the surface, any such metric can be written in some coordinate system as $ds^2 = \lambda(dx^2 + dy^2)$ ($x$ and $y$ are called isothermal coordinates). Setting $z = x + iy$ we may write the metric as $ds^2 = \lambda\, dz\, d\bar{z}$. If, similarly, $ds^2 = \mu\, dw\, d\bar{w}$ in another domain $V$, one checks easily (in view of the orientability of the surface) that $dw = \varphi dz$. It follows that $w$ is an analytic function of $z$. Thus the domains $U$, together with their coordinates $z$, define a Riemann surface structure on $S$. Two metrics define the same Riemann surface if they differ by a factor $\psi$, where $\psi$ is an everywhere positive, real function on $S$. Multiplication by such a function is called a gauge transformation. Thus a Riemann surface is the same as a surface with a metric of differential geometry, which is considered up to gauge transformations. This is how Riemann surfaces arise in quantized field theory, in the so-called boson string theory.