Graduate Texts in Mathematics

Martin Golubitsky
Victor Guillemin

Stable Mappings and Their Singularities

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Graduate Texts in Mathematics 14

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continued after Index

PREFACE

This book aims to present to first and second year graduate students a beautiful and relatively accessible field of mathematics—the theory of singularities of stable differentiable mappings.

The study of stable singularities is based on the now classical theories of Hassler Whitney, who determined the generic singularities (or lack of them) for mappings of $R^n \to R^m$ $(m \geq 2n - 1)$ and $R^2 \to R^2$, and Marston Morse, who studied these singularities for $R^n \to R$. It was René Thom who noticed (in the late '50's) that all of these results could be incorporated into one theory. The 1960 Bonn notes of Thom and Harold Levine (reprinted in [42]) gave the first general exposition of this theory. However, these notes preceded the work of Bernard Malgrange [23] on what is now known as the Malgrange Preparation Theorem—which allows the relatively easy computation of normal forms of stable singularities as well as the proof of the main theorem in the subject—and the definitive work of John Mather. More recently, two survey articles have appeared, by Arnold [4] and Wall [53], which have done much to codify the new material; still there is no totally accessible description of this subject for the beginning student. We hope that these notes will partially fill this gap. In writing this manuscript, we have repeatedly cribbed from the sources mentioned above—in particular, the Thom-Levine notes and the six basic papers by Mather. This is one of those cases where the hackneyed phrase "if it were not for the efforts of . . . , this work would not have been possible" applies without qualification.

A few words about our approach to this material: We have avoided (although our students may not always have believed us) doing proofs in the greatest generality possible. For example, we assume in many places that certain manifolds are compact and that, in general, manifolds have no boundaries, in order to reduce the technical details. Also, we have tried to give an abundance of low-dimensional examples, particularly in the later chapters. For those topics that we do cover, we have attempted to "fill in all the details," realizing, as our personal experiences have shown, that this phrase has a different interpretation from author to author, from chapter to chapter, and—as we strongly suspect—from authors to readers. Finally, we are aware that there are blocks of material which have been included for completeness' sake and which only a diehard perfectionist would slog through —especially on the first reading although probably on the last as well. Conversely, there are sections which we consider to be right at the "heart of the matter." These considerations have led us to include a Reader's Guide to the various sections.

Chapter I: This is elementary manifold theory. The more sophisticated reader will have seen most of this material already but is advised to glance through it in order to become familiar with the notational conventions used elsewhere in the book. For the reader who has had some manifold theory before, Chapter I can be used as a source of standard facts which he may have forgotten.

Chapter II: The main results on stability proved in the later chapters depend on two deep theorems from analysis: Sard's theorem and the Malgrange preparation theorem. This chapter deals with Sard's theorem in its various forms. In §1 is proved the classical Sard's theorem. Sections 2–4 give a formulation of it which is particularly convenient for applications to differentiable maps: the Thom transversality theorem. These sections are essential for what follows, but there are technical details that the reader is well-advised to skip on the first reading. We suggest that the reader absorb the notion of k-jets in §2, look over the first part of §3 (through Proposition 3.5) but assume, without going through the proofs, the material in the last half of this section. (The results in the second half of §3 would be easier to prove if the domain X were a compact manifold. Unfortunately, even if we were only to work with compact domains, the stability problem leads us to consider certain noncompact domains like $X \times X - \Delta X$.) In §4, the reader should probably skip the details of the proof of the multijet transversality theorem (Theorem 4.13). It is here that the difficulties with $X \times X - \Delta X$ make their first appearance.

Sections 5 and 6 include typical applications of the transversality theorem. The tubular neighborhood theorem, §7, is a technical result inserted here because it is easy to deduce from the Whitney embedding theorem in §5.

Chapter III: We recommend this chapter be read carefully, as it contains in embryo the main ideas of the stability theory. The first section gives an incorrect but heuristically useful "proof" of the Mather stability theorem: the equivalence of stability and infinitesimal stability. (The theorem is actually proved in Chapter V.) For motivational reasons we discuss some facts about infinite dimensional manifolds. These facts are used nowhere in the subsequent chapters, so the reader should not be disturbed that they are only sketchily developed. In the remaining three sections, we give all the elementary examples of stable mappings. The proofs depend on the material in Chapter II and the yet to be proved Mather criterion for stability.

Chapter IV gives the second main result from analysis needed for the stability theory: the Malgrange preparation theorem. Like Chapter II, this chapter is a little technical. We have provided a way for the reader to get through it without getting bogged down in details: in the first section, we discuss the classical Weierstrass preparation theorem—the holomorphic version of the Malgrange theorem. The proof given is fairly easy to understand, and has the virtue that the adaptation of it to a proof of the Malgrange preparation theorem requires only one additional fact, namely, the Nirenberg extension lemma (Proposition 2.4). The proof of this lemma can probably be skipped by the reader on a first reading as it is hard and technical.

In the third section, the form of the preparation theorem we will be using in subsequent chapters is given. The reader should take some pains to understand it (particularly if his background in algebra is a little shaky, as it is couched in the language of rings and modules).

Chapter V contains the proof of Mather's fundamental theorem on stability. The chapter is divided into two halves; §§1–4 contain the proof that infinitesimal stability implies stability and §§5 and 6 give the converse. In the process of proving the equivalence between these two forms of stability we prove their equivalence with other types of stability as well. For the reader who is confused by the maze of implications we provide in §7 a short summary of our line of argument.

It should be noted that in these arguments we assume the domain X is compact and without boundary. These assumptions could be weakened but at the expense of making the proof more complicated. One pleasant feature of the proof given here is that it avoids Banach manifolds and the global Mather division theorem.

Chapters VI and VII provide two classification schemes for stable singularities. The one discussed in Chapter VI is due to Thom [46] and Boardman [6]. The second scheme, due to Mather and presented in the last chapter, is based on the "local ring" of a map. One of the main results of these two chapters is a complete classification of all equidimensional stable maps and their singularities in dimensions ≤ 4. (See VII, §6.) The reader should be warned that the derivation of the "normal forms" for some stable singularities (VII, §§4 and 5) tend to be tedious and repetitive.

Finally, the *Appendix* contains, for completeness, a proof of all the facts about Lie groups needed for the proofs of Theorems in Chapters V and VI.

This book is intended for first and second year graduate students who have limited—or no—experience dealing with manifolds. We have assumed throughout that the reader has a reasonable background in undergraduate linear algebra, advanced calculus, point set topology, and algebra, and some knowledge of the theory of functions of one complex variable and ordinary differential equations. Our implementation of this assumption—i.e., the decisions on which details to include in the text and which to omit—varied according to which undergraduate courses we happened to be teaching, the time of day, the tides, and possibly the economy. On the other hand, we are reasonably confident that this type of background will be sufficient for someone to read through the volume. Of course, we realize that a healthy dose of that cure-all called "mathematical sophistication" and a previous exposure to the general theory of manifolds would do wonders in helping the reader through the preliminaries and into the more interesting material of the later chapters.

Finally, we note that we have made no attempt to create an encyclopedia of known facts about stable mappings and their singularities, but rather to present what we consider to be basic to understanding the volumes of material that have been produced on the subject by many authors in the past few years. For the reader who is interested in more advanced material, we recommend perusing the volumes of the "Proceedings of Liverpool Singularities" [42, 43], Thom's basic philosophical work, "Stabilité Structurelle et Morphogenèse" [47], Tougeron's work, "Ideaux de Fonctions Differentiables" [50], Mather's forthcoming book, and the articles referred to above.

There were many people who were involved in one way or another with the writing of this book. The person to whom we are most indebted is John Mather, whose papers [26–31] contain almost all the fundamental results of stability theory, and with whom we were fortunately able to consult frequently. We are also indebted to Harold Levine for having introduced us to Mather's work, and, for support and inspiration, to Shlomo Sternberg, Dave Schaeffer, Rob Kirby, and John Guckenheimer. For help with the editing of the manuscript we are grateful to Fred Kochman and Jim Damon. For help with some of the figures we thank Molly Scheffe. Finally, our thanks to Marni Elci, Phyllis Ruby, and Kathy Ramos for typing the manuscript and, in particular, to Marni for helping to correct our execrable prose.

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TABLE OF CONTENTS

Chapter I

Preliminaries on Manifolds

§1. Manifolds

Let $\mathbf{R}$ denote the real numbers and $\mathbf{R}^n$ denote n-dimensional Euclidean space. Points of $\mathbf{R}^n$ will be denoted by n-tuples of real numbers $(x_1, \ldots, x_n)$ and $\mathbf{R}^n$ will always be topologized in the standard way.

Let U be subset of $\mathbf{R}^n$. Then denote by $\bar{U}$ the closure of U, and by Int (U) the interior of U.

Let U be an open set, $f: U \to \mathbf{R}$, and $x \in U$. Denote by $(\partial f/\partial x_i)(x)$ the partial derivative of f with respect to the ith variable x_i at x. To denote a higher order mixed partial derivative, we will use multi-indices, i.e., let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of non-negative integers. Then

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} f \quad \text{where} \quad |\alpha| = \alpha_1 + \cdots + \alpha_n$$

and $f: U \to \mathbf{r}$ is *k-times differentiable* (or of *class* C^k, or C^k) if $(\partial^{|\alpha|} f/\partial x^\alpha)(x)$ exists and is continuous for every n-tuple of non-negative integers α with $|\alpha| \le k$. (Note that when $\alpha = (0, \ldots, 0)$, $\partial^{|\alpha|} f/\partial x^\alpha$ is defined to be f.) f is *real analytic* on U if the Taylor series of f about each point in U converges to f in a neighbourhood (nbhd) of that point.

Suppose $\phi: U \to \mathbf{R}^m$ where U is an open subset of $\mathbf{R}^n$ and f is some real-valued function defined in the range of ϕ; then $\phi^* f \equiv f \cdot \phi$ (where $\cdot$ denotes composition of mappings) is called the *pull-back function* of f by ϕ.

Definition 1.1. *Let* $\phi: U \to \mathbf{R}^m$*, U an open subset of* $\mathbf{R}^n$.

(a) ϕ *is* differentiable of class C^k *if the pull-back by ϕ of any k-times differentiable real-valued function defined on the range of ϕ is k-times differentiable.*

(b) ϕ *is* smooth (*or* differentiable of class C^∞) *if for every non-negative integer k, ϕ is differentiable of class* C^k.

(c) ϕ *is* real analytic *if the pull-back by ϕ of any real analytic real-valued function defined on the range of ϕ is real analytic.*

Let $\phi: U \to \mathbf{R}^m$ be C^1 differentiable in U and x_0 a point in U. Then by Taylor's theorem there exists a unique linear map $(d\phi)_{x_0}: \mathbf{R}^n \to \mathbf{R}^m$ and a function $\rho: U \to \mathbf{R}^m$ such that

$$\phi(x) = \phi(x_0) + (d\phi)_{x_0}(x - x_0) + \rho(x)$$

for every x in a nbhd V of x_0, where

$$\lim_{x \to x_0} \frac{|\rho(x)|}{|x - x_0|} = 0.$$

Note that we will use $|x|$ to denote the Euclidean norm $(\sum x_i^2)^{1/2}$. Let $(d\phi)_{x_0}: \mathbf{R}^n \to \mathbf{R}^m$ be the *Jacobian of* ϕ *at* x_0; it is given with respect to the coordinates $x_1, \ldots, x_n$ on $\mathbf{R}^n$ and $y_1, \ldots, y_m$ on $\mathbf{R}^m$ by the $m \times n$ matrix

$$\left(\frac{\partial \phi^i}{\partial x^j}(x_0)\right)_{1 \le j \le n}^{1 \le i \le m}$$

where $\phi^i : \mathbf{R}^n \to \mathbf{R}(1 \le i \le m)$ are the m coordinate functions defining ϕ.

The chain rule holds, of course. That is, if $\phi : U \to \mathbf{R}^m$ and $\psi : V \to \mathbf{R}^p$ are both C^1 differentiable where $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$ are open and $V \supset \phi(U)$, then $d(\psi \cdot \phi)_{x_0} = (d\psi)_{\phi(x_0)} \cdot (d\phi)_{x_0}$ for every x_0 in U.

Theorem 1.2. *(Inverse Function Theorem). Let* $U \subset \mathbf{R}^n$ *be open and* p *be a point in* U. *Let* $\phi: U \to \mathbf{R}^n$ *be a* C^k *differentiable mapping. Assume that* $(d\phi)_p : \mathbf{R}^n \to \mathbf{R}^n$ *is invertible. Then there exists an open set* V *in* R^n *contained in the range of* ϕ *and a mapping* $\psi: V \to U$, *differentiable of class* C^k, *such that* $\phi \cdot \psi(x) = x$ *for every* x *in* V, *and* $\psi \cdot \phi(x) = x$ *for every* x *in* $\psi(V)$.

Proof. See appendix of Sternberg; or Lang. ▯

Definition 1.3. *A* local homeomorphism *of* $\mathbf{R}^n$ *is a homeomorphism of some open subset of* $\mathbf{R}^n$ *onto another. (So the domain of a local homeomorphism need not be all of* $\mathbf{R}^n$.)

Let ϕ be a mapping. Denote by *dom* ϕ the domain of ϕ. Also, if $U \subset$ dom ϕ denote by $\phi|U$ the restriction of ϕ to U. If X is a set, then $\mathrm{id}_X : X \to X$ denotes the identity mapping on X.

Definition 1.4. *A* pseudogroup on $\mathbf{R}^n$ *is a collection* Γ *of local homeomorphisms on* $\mathbf{R}^n$ *with the following properties:*

(a) $id_{\mathbf{R}^n}$ *is in* Γ,

(b) *if* ϕ *and* ψ *are in* Γ *with* dom ψ = *range of* ϕ *then* $\psi \cdot \phi$ *is in* Γ, *i.e.,* Γ *is closed under composition for all pairs of elements for which this operation makes sense.*

(c) *if* ϕ *is in* Γ, *then* ϕ^{-1} *is in* Γ *(where* ϕ^{-1} *denotes the inverse function of* ϕ*)*

(d) *if* ϕ *is in* Γ *and* U *is an open subset of* dom ϕ, *then* $\phi|U$ *is in* Γ, *and*

(e) *if* $\{U_\alpha\}_{\alpha \in I}$ *(I some index set) is a collection of open subsets of* $\mathbf{R}^n$, ϕ *is a local homeomorphism of* $\mathbf{R}^n$ *defined on* $U = \bigcup_{\alpha \in I} U_\alpha$, *and* $\phi|U_\alpha$ *is in* Γ *for every* α *in* I, *then* ϕ *is in* Γ.

Some examples of pseudogroups are:

(a) $(\mathrm{diff})^k$ = the set of all local diffeomorphisms on $\mathbf{R}^n$ (n fixed) which are differentiable of class C^k.

(b) $(\mathrm{diff})^\infty$ = the set of local diffeomorphisms of $\mathbf{R}^n$ (n fixed) which are smooth.

(c) $(\mathrm{diff})^\omega$ = the set of all local diffeomorphisms of $\mathbf{R}^n$ (n fixed) which are real analytic.

To show that (a) and (b) satisfy the conditions of the definition you need to use only the chain rule, the inverse function theorem, and the local character of differentiability. For (c) you need the strengthened versions of the above theorems for analytic functions.

A more general class of pseudogroups can be given as follows:

(d) Let G be a group of linear mappings of $\mathbf{R}^n \to \mathbf{R}^n$. Then the pseudogroup $\Gamma_G{}^k$ is the set

$$\{\phi \in (\text{diff})^k \mid \forall x \in \text{dom}\,\phi, (d\phi)_x \in G\}$$

(i) G = all linear maps on $\mathbf{R}^n$ with positive determinant. Then $\Gamma_G{}^k = (\text{diff})_0^k$ consists of orientation preserving C^k mappings.

(ii) G = all linear maps on $\mathbf{R}^n$ with determinant equal to 1. Then $\Gamma_G{}^k$ consists of all volume preserving C^k mappings.

(iii) Let (,) be an inner product on $\mathbf{R}^n$. Let G be the group of orthogonal matrices relative to (,); namely, $A \in G$ iff $(x, y) = (Ax, Ay)$ for every x, y in $\mathbf{R}^n$. Then $\Gamma_G{}^k$ consists of all C^k isometries in $\mathbf{R}^n$.

Definition 1.5. *Let Γ be a pseudogroup on $\mathbf{R}^n$ and X a Hausdorff topological space which satisfies the second axiom of countability. Let A be a subset of all local homeomorphisms of X into $\mathbf{R}^n$, i.e., homeomorphisms which are defined on an open subset of X and whose range is an open subset of $\mathbf{R}^n$. Then*

(i) *A is a* Γ-atlas on *X if*

(a) $X = \bigcup_{\phi \in A} \text{dom}\,\phi$

(b) *if ϕ, ψ are in A, then $\psi \cdot \phi^{-1}|\phi(\text{dom}\,\phi \cap \text{dom}\,\psi)$ is in Γ.*

(ii) *The elements of A are called* charts *on X.*

(iii) *Two Γ-atlases A_1 and A_2 on X are* compatible *if $\psi \cdot \phi^{-1}|\phi(\text{dom}\,\phi \cap \text{dom}\,\psi)$ is in Γ whenever ϕ is in A_1 and ψ is in A_2, and vice-versa.*

(iv) *A Hausdorff space X together with an equivalence class of compatible Γ-atlases is called a* Γ-structure *on X.*

Notes. (1) Recall that X satisfies the second axiom of countability if the topology on X has a countable base.

(2) If X has a Γ-structure, then X is locally compact, since it is locally Euclidean.

Definition 1.6. *Let X have a Γ-structure.*

(a) *If $\Gamma = (\text{diff})^k$ and $k > 0$, then X is a* differentiable manifold of class C^k.

(b) *If $\Gamma = (\text{diff})^0$, then X is a* topological manifold.

(c) *If $\Gamma = (\text{diff})^\infty$, then X is a* smooth manifold *or a* manifold of class C^∞.

(d) *If $\Gamma = (\text{diff})^\omega$, then X is a* real analytic manifold.

(e) *If $\Gamma = (\text{diff})_0^k$ and $k > 0$ then X is an* oriented *C^k differentiable manifold. Any differentiable manifold which has a* $(\text{diff})_0^1$ *structure in which the charts are elements of the original* $(\text{diff})^1$ *structure is* orientable.

Examples

(1) $S^{n-1} = \left\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n \,\middle|\, \sum_{i=1}^n x_i^2 = 1\right\}.$

Let $N = (1, 0, \ldots, 0)$ and $S = (-1, 0, \ldots, 0)$.

Let $\phi_N : \{S^{n-1} - \{N\}\} \to \mathbf{R}^{n-1}$ be stereographic projection via N, i.e., $\phi_N(x_1, \ldots, x_n) = (1/(1 - x_1))(x_2, \ldots, x_n)$ and $\phi_S : \{S^{n-1} - \{S\}\} \to \mathbf{R}^{n-1}$ be stereographic projection via S, i.e., $\phi_S(x_1, \ldots, x_n) = (1/(1 + x_1))(x_2, \ldots, x_n)$. Then $\phi_S \cdot \phi_N^{-1} : \mathbf{R}^{n-1} - \{0\} \to \mathbf{R}^{n-1} - \{0\}$ is given by $y \to y/|y|^2$ for all y in $\mathbf{R}^{n-1} - \{0\}$. Since $(\phi_S \cdot \phi_N^{-1}) \cdot (\phi_S \cdot \phi_N^{-1}) = id$ we see that $\det(d\phi_S \cdot \phi_N^{-1})_y = \pm 1$. Evaluate at $y = (1, 0, \ldots, 0)$ to see that, in fact, $\det(d\phi_S \cdot \phi_N^{-1}) = -1$. To show that S^{n-1} is an oriented analytic manifold we can change the last coordinate of ϕ_N to $-x_n/(1 - x_1)$ thus changing the determinant to $+1$.

(2) $\mathbf{P}^n$ = real projective n-space.

To define $\mathbf{P}^n$ we introduce the equivalence relation $\sim$ on $\mathbf{R}^{n+1} - \{0\}$:

$(x_0, \ldots, x_n) \sim (x'_0, \ldots, x'_n)$ iff there is a real constant c such that $x_i = cx'_i$ for all i.

$\mathbf{P}^n = \mathbf{R}^{n+1} - \{0\}/\sim$ is the set of these equivalence classes.

Let $\pi : \mathbf{R}^{n+1} - \{0\} \to \mathbf{P}^n$ be the canonical projection. $\mathbf{P}^n$ is given the standard quotient space topology and note that with this topology π is an open mapping. To show that $\mathbf{P}^n$ has a manifold structure it is necessary to produce local homeomorphisms of $\mathbf{P}^n$ into $\mathbf{R}^n$ which overlap properly.

Let $V_i = \mathbf{R}^{n+1} - \{\text{hyperplane } x_i = 0\}$ for $0 \le i \le n$. V_i is open in $\mathbf{R}^{n+1} - \{0\}$, hence $\pi(V_i) = U_i$ is open in $\mathbf{P}^n$. Clearly $\mathbf{P}^n = U_0 \cup \cdots \cup U_n$. Define $\phi_i : U_i \to \mathbf{R}^n$ by

$$\phi_i(p) = \frac{(-1)^i}{x_i}(x_0, \ldots, \hat{x}_i, \ldots, x_n) \quad \text{where} \quad p = \pi(x_0, \ldots, x_n)$$

and ^ indicates that coordinate is to be omitted. Using the equivalence relation defining $\mathbf{P}^n$ and the fact that p is in U_i, one sees that ϕ_i is a well-defined homeomorphism onto $\mathbf{R}^n$.

$$\phi_i(U_i \cap U_j) = \mathbf{R}^n - \{\text{hyperplane } y_j = 0\} \qquad (i > j)$$

$$\phi_i(U_i \cap U_j) = \mathbf{R}^n - \{\text{hyperplane } y_{j-1} = 0\} \qquad (i < j)$$

where we assume $y_1, \ldots, y_n$ are the coordinates on $\mathbf{R}^n$. So for $i < j$ $\phi_i \cdot \phi_j^{-1} : \mathbf{R}^n - \{\text{hyperplane } y_i = 0\} \to \mathbf{R}^n - \{\text{hyperplane } y_{j-1} = 0\}$. A computation yields for $i < j$

$$\phi_i \cdot \phi_j^{-1}(y_1, \ldots, y_n) = \frac{(-1)^{i+j}}{y_{i+1}}(y_1, \ldots, y_i, y_{i+2}, \ldots, y_j, 1, y_{j+1}, \ldots, y_n)$$

which is a real analytic mapping so $\mathbf{P}^n$ becomes a real analytic manifold. When $i < j$ another computation yields

$$\det\left(d\phi_i \cdot \phi_j^{-1}\right)_{(y_1, \ldots, y_n)} = \left(\frac{1}{y_{i+1}}\right)^{n+1} (-1)^{(n+1)(i+j)}$$

from which we see that real projective space in any odd dimension ($\mathbf{P}^{2n+1}$, $n \ge 0$) is orientable. It can be proved that $\mathbf{P}^{2n}$ is not orientable.

(3) $G_{k,n}$ = Grassmannian space of k-planes through the origin in $\mathbf{R}^n$.
= set of all k-dimensional subspaces of Euclidean n-space.

Note that $G_{1,n+1} = \mathbf{P}^n$.

We will give $G_{k,n}$ a decomposition space topology. Let W = all ordered k-tuples $P = (P_1, \ldots, P_k)$ of k linearly independent vectors in R^n. W is an open subset of

$$\underbrace{\mathbf{R}^n \oplus \cdots \oplus \mathbf{R}^n}_{k-\text{times}}.$$

Define an equivalence relation $\sim$ on W as follows:

$$P \sim Q \quad \text{if} \quad \{P_1, \ldots, P_k\} \quad \text{and} \quad \{Q_1, \ldots, Q_k\}$$

span the same k-dimensional subspace of $\mathbf{R}^n$.

Clearly $G_{k,n}$ can be identified with $W/\sim$ as sets so we may give $G_{k,n}$ the topology induced by this identification. We now give $G_{k,n}$ an analytic structure. Equip $\mathbf{R}^n$ with an inner product (,). Then given a subspace V of $\mathbf{R}^n$, there is an orthogonal projection π_V of $\mathbf{R}^n$ onto V. Suppose V is a k-dimensional subspace of R^n. Let $\pi_{U,V}$ = restriction of π_V to U. Let $W_V = \{U \in G_{k,n} \mid \pi_{U,V} \text{ is a bijection onto } V\}$.

Let $V^\perp$ = the orthogonal complement of V in $\mathbf{R}^n$. Define

$$\rho_V : W_V \to \operatorname{Hom}(V, V^\perp)$$

as follows: Let $U \in W_V$. Then $\rho_V(U) = \pi_{U,V^\perp} \cdot \pi_{U,V}^{-1} \in \operatorname{Hom}(V, V^\perp)$. We leave it to the reader to check that ρ_V is a homeomorphism. Now make the identification $\operatorname{Hom}(V, V^\perp) \cong \mathbf{R}^{k(n-k)}$, to get a chart $\phi_V : W_V \to R^{k(n-k)}$. Again it is left to the reader to check that $\rho_V \cdot \rho_{V'}^{-1} : \mathbf{R}^{k(n-k)} \to \mathbf{R}^{k(n-k)}$ is real analytic. Hence $G_{k,n}$ is a real analytic manifold of dimension $k(n-k)$. Note that for $k = 1$ this is the same atlas that we constructed for $\mathbf{P}^{n-1}$. Thus $G_{1,n} = \mathbf{P}^{n-1}$.

Definition 1.7. *Let X and Y be C^k differentiable manifolds of dimension n and m, respectively. Then $X \times Y$ can be made into a C^k differentiable manifold of dimension $n + m$ in the following natural way. Let A_X and A_Y be atlases on X and Y. Let $\phi \in A_X$, $\psi \in A_Y$. Then $\phi \times \psi : \operatorname{dom} \phi \times \operatorname{dom} \psi \to \mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^{m+n}$ is given by $\phi \times \psi(x, y) = (\phi(x), \psi(y))$ $x \in X, y \in Y$. $\phi \times \psi$ is clearly a local homeomorphism of $X \times Y \to \mathbf{R}^{n+m}$. Then $A_{X \times Y} = \{\phi \times \psi \mid \phi \in A_X, \psi \in A_Y\}$ is an atlas for $X \times Y$.*

Applications

(1) The r-Torus,

$$\underbrace{S^1 \times \cdots \times S^1}_{r-\text{times}}$$

is a smooth manifold of dimension r.

(2) If X and Y are oriented manifolds, then so is $X \times Y$.

Definition 1.8. *Let X be a topological n-manifold, and p a point in X. A set of* local coordinates on X based at p *is a collection of n real-valued functions $\{\phi_1, \ldots, \phi_n\}$ defined on an open nbhd U of p, (i.e., $\phi_i : U \to \mathbf{R}$) so that $\phi_i(p) = 0$ $(1 \leq i \leq n)$ and $\phi : U \to \mathbf{R}^n$ defined by $\phi(q) = (\phi_1(q), \ldots, \phi_n(q))$ is a chart in the manifold structure on X.*

Clearly if ϕ is a chart of X based at p (i.e., ϕ is defined on a nbhd of p and $\phi(p) = 0$) then the coordinate functions of ϕ define a system of local coordinates on X based at p.

The common domain of a set of local coordinates based at p is a *coordinate nbhd of p*.

§2. Differentiable Mappings and Submanifolds

Definition 2.1. *Let Y be a C^k-differentiable manifold of dimension m.*

(a) *Let $f: Y \to \mathbf{R}$ be a function. f is* C^k-differentiable *if for every chart ϕ : dom $\phi \to \mathbf{R}^m$, $f \cdot \phi^{-1}$: range $\phi \to \mathbf{R}$ is a C^k-differentiable mapping. f is* smooth *if f is C^k-differentiable for every k.*

(b) *Let X be a C^k-differentiable manifold. Then $\phi: X \to Y$ is* C^k-differentiable *if for every C^k-differentiable function $f: Y \to \mathbf{R}$, the pullback $f \cdot \phi$ is C^k-differentiable. ϕ is* smooth *if ϕ is C^k-differentiable for every k.*

(c) *We will use* differentiable *to mean C^k-differentiable for k at least* 1.

Remark. Suppose that $\phi: X \to Y$ is a mapping with p in X and $q = \phi(p)$ in Y. Let U and V be coordinate nbhds of X and Y based at p and q respectively, and assume that $\phi(U) \subset V$. Suppose $\rho: V \to \mathbf{R}^m$ and $\tau: U \to \mathbf{R}^n$ are charts. Then ϕ is C^k-differentiable iff $\rho \cdot \phi \cdot \tau^{-1}$: range $\tau \subset \mathbf{R}^n \to \mathbf{R}^m$ is C^k-differentiable. This shows that differentiability of a function between manifolds is a local question and is independent of the particular local representation used.

Definition 2.2. *Let X and Y be differentiable manifolds of dimension n and m, respectively. Let $\phi: X \to Y$ be differentiable. Let p be in X, ρ a chart on X with p in* dom ρ, *and τ a chart on Y with* ϕ(dom ρ) $\subset$ dom τ.

Then $(d\tau \cdot \phi \cdot \rho^{-1})_{\rho(p)}: \mathbf{R}^n \to \mathbf{R}^m$ is a linear mapping. Define rank of ϕ at p *to be rank $(d\tau \cdot \phi \cdot \rho^{-1})_{\rho(p)}$.*

Note. The definition of rank does not depend on which charts are selected. Let ρ', τ' be charts with the above properties. Then on a nbhd of p and $f(p)$,

$$\begin{aligned}\text{rank}\,(d\tau' \cdot \phi \cdot (\rho')^{-1})_{\rho'(p)} &= \text{rank}\,(d\tau' \cdot \tau^{-1} \cdot \tau \cdot \phi \cdot \rho^{-1} \cdot \rho \cdot (\rho')^{-1})_{\rho'(p)} \\ &= \text{rank}\,(d\tau \cdot \phi \cdot \rho^{-1})_{\rho(p)}\end{aligned}$$

by the chain rule and the fact that $\tau' \cdot \tau^{-1}$ and $\rho \cdot (\rho')^{-1}$ are in (diff)^1.

Definition 2.3. *Let X and Y be differentiable manifolds. Let $\phi: X \to Y$ be a differentiable mapping. Suppose that at the point p in X, ϕ has the maximum possible rank. Then*

(a) *if* dim $X \leq$ dim Y, *ϕ is an* immersion at p,
(b) *if* dim $X \geq$ dim Y, *ϕ is a* submersion at p,
(c) *if for every p in X, ϕ is an immersion* (*submersion*) *at p, then ϕ is an* immersion (submersion).