

INTRODUCTION to
AERODYNAMICS of a
COMPRESSIBLE FLUID

BY

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EDITORS' PREFACE

The present volume is the second of the GALCIT* series to appear after World War II. Like its predecessors in the series its subject matter was originally collected in response to the need for training the large numbers of aeronautical engineers required by the national emergency. Since peace-time interest in the compressibility phenomena of fluid motions appeared to be even greater than that during the war, it seemed useful to rework the earlier material into a systematic introduction to the field. In this process advantage was taken of the information which became available only after the war. The resulting treatment is believed to be as up to date as possible in a subject which is developing as rapidly as compressible fluid hydrodynamics, and in which military classification often remains in effect.

It is hoped that the volume may prove useful both to students who are for the first time meeting the problems of compressible flow, and to engineers and scientists who are actively working with such problems.

THEODORE VON KÁRMÁN
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January, 1947

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AUTHORS' PREFACE

During the last few years compressibility effects have become more and more important in aeronautical engineering as well as in related fields such as turbine design. The subject, Aerodynamics of Compressible Fluids, has been studied at great length by both pure mathematicians and "practical" engineers, and the literature on the subject has become extensive. The material presented in this book is designed to furnish the reader a background of fundamentals sufficient to enable him to understand or at least to systematize the observed compressibility effects, and also to enable him to approach the more mathematical literature of the subject. This double purpose not only determined the material to be included but also dictated the division essentially into two parts: a first part dealing with the basic compressibility phenomena under simple geometrical conditions, which thus involves relatively little mathematics, and a second part which is more mathematical.

This book originated in a series of lectures first given in 1942, under the sponsorship of the ESMWT, to members of the engineering staffs of the various Southern California aircraft companies. War research and the frequent absence of the authors from Pasadena delayed the completion of the manuscript, but the material has been expanded and brought up to date. In its present form, the book is the basis for a fifty-hour graduate course at the California Institute of Technology.

Emphasis is placed almost exclusively upon the derivation and the meaning of the fundamental aerodynamics relations. In other words, there has been no attempt to produce a standard classroom textbook, so no problems are given, and few practical examples are worked out. If the book is used as a text, it is assumed that the instructor will choose examples best suited to his particular group of students.

Of course, an essential difficulty was encountered in the wartime restrictions which made a presentation of recent test data impossible. In order to avoid presentation of obsolete or unreliable data, it was decided to omit reference to specific sets of test data almost entirely and to discuss experimental results only in general terms. It is thus left to the reader who has access to the confidential literature to compare the theoretical results with experimental data. Since the technique of testing at high airspeeds is still rather unsettled, it is felt that the omission of a presentation of large sets of test data is not a serious drawback at the present time.

We are indebted to many members of the Galcit staff. In particular, Messrs. S. Corrsin, J. Sternberg, H. Ashkenas, R. Schamberg, and J. Laufer; and Mrs. Kate Liepmann assisted in the preparation of the book. We also wish to thank Miss M. Ormesher for her patience in typing the manuscript, as well as Drs. W. R. Sears of the Northrop Aircraft Co. and H. W. Emmons of Harvard University for their valuable criticism.

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Part I

ONE-DIMENSIONAL MOTION OF A COMPRESSIBLE FLUID

In a large part of aeronautics, the flow of air around bodies may be studied with sufficient accuracy by assuming the air density to be constant. The major part of existing aerodynamic theory concerns computations where this assumption is made. Under certain conditions, however, the air density may vary sufficiently to cause the flow behavior to depart appreciably from that predicted by the incompressible fluid theory. It is said then that compressibility effects occur. The new flow behavior may be computed in some cases in terms of corrections or alterations to known incompressible fluid flow solutions; in other cases entirely new types of flow solutions are necessary. In aerodynamics, compressibility effects generally become of engineering importance when speed changes (i.e., relative speeds) in the fluid, or of bodies relative to the fluid, become appreciably large compared to the speed of sound in the fluid (of the order of one half or more). The reasons for this will appear later.

The introduction of this new variable, the density, into the hydrodynamics problem requires a study of its relation to the other parameters defining the condition of the air. These relations are essentially thermodynamic, so that an examination of some basic thermodynamic concepts is necessary. It will be recalled that in the solution of incompressible fluid flow problems, the density could be eliminated in such a way that the computation of the flow pattern became purely a kinematic problem. This is, of course, no longer the case.

In general, the solution of a compressible fluid flow problem will consist in finding the three unknown velocity components, the density, and the pressure as functions of three space coordinates, say (x, y, z) . We have available for this purpose the five basic equations: three equations for conservation of momentum in three directions, the equation requiring conservation of mass, and the equation requiring conservation of energy. Thermodynamic considerations introduce an additional parameter, the temperature, and one additional equation, the equation of state, which furnishes a relation between the three gas parameters.

Many of the characteristic features of the flow of a compressible fluid may be studied, however, by investigating motion in one dimension. This means that all quantities are assumed to vary extremely slowly in all directions but

one; or, $\partial(\)/\partial y$ and $\partial(\)/\partial z$ are much smaller than $\partial(\)/\partial x$. In this way, all parameters become functions of only one space variable, x . The results of such a one-dimensional study are also applicable to more general flows, because the flow within any one stream filament or stream tube of sufficiently small diameter must correspond to a one-dimensional flow. The general problem consists in finding the boundaries of these stream tubes.

The one-dimensional flow of a fluid is essentially that in a tube whose radius of curvature is large, whose cross section changes only very slowly, and across any section of which all parameters are essentially constant. There are, as a matter of fact, many problems of engineering importance in which these conditions are very nearly satisfied, so that the results of this simple theory may be applied directly. However, they must be applied with great caution in cases where the basic phenomena are actually associated with non-uniformities of velocity or pressure across a section of the tube. In such cases it is sometimes possible to apply the one-dimensional theory results to properly defined averages of the velocity, density, and so on, across the tube, but care must be exercised to avoid misinterpretation.

All the essential relations, including the flow changes through a shock wave, between quantities in a three-dimensional stream tube—velocity, pressure, and density—may be illustrated in the one-dimensional study. This study may also include two-dimensional phenomena to the extent that the direction of the flow may be changed abruptly through an oblique shock wave, while the flow on either side remains essentially one-dimensional. In Part I will be discussed all flows which may be treated under these simplifying assumptions, and indications will be given of their application to certain engineering problems, and also to the solution of more general flows.

CHAPTER 1

Basic Thermodynamics

1.1 Equation of State

As mentioned earlier, the addition of the density to the parameters defining the motion of a fluid requires investigations of its relation to the other gas parameters. In this chapter will be reviewed briefly those elementary thermodynamic concepts necessary to this investigation in the form most useful to the hydrodynamicist.

The condition or "state" of a fluid may be defined by its evident and measurable properties: pressure p , density ρ , and temperature T . Experiment shows that only two of these are independent; this means that in general an equation of the sort

$$F(p, \rho, T) = 0 \quad (1.1)$$

must exist. This is known as the equation of state of the fluid. The exact form of this equation has been determined for various gases by experiment, and also theoretically.

The elementary kinetic theory of gases supposes a gas to consist of small perfectly elastic particles, representing the gas molecules, moving in random paths at very high speeds and colliding at intervals with each other and with the walls of their container. The force they exert through these collisions against any solid surface is observed as the pressure of the gas. The kinetic energy of their random motion is proportional to the temperature. With this idealized model of a gas, an equation of state may be deduced. In other words, by applying Newton's laws to the motion of the particles, it is possible to find a relation between the number of particles in a given volume (density), their velocity and hence kinetic energy (which is proportional to temperature), and the frequency and force of their collision with a wall (pressure). It is apparent that when two of these data are given, the third must be determined. If the molecules are far enough apart so that intermolecular forces are negligible and their own volume is negligible compared to the space they occupy, then the law is found to be

$$p = \rho RT \quad (1.2)$$

where R is a constant for the gas. If these requirements are met and the gas obeys this law, it is said to be a perfect gas. It is evident that the require-

ments will be met only if the gas density is sufficiently low for large distances to exist between molecules.

Charles and Boyle arrived at this law experimentally, so it is often known as the Charles-Boyle law. For air the gas constant is

$$R = 1715 \text{ ft}^2/\text{sec}^2 \text{ } ^\circ\text{F}$$

where p is pressure in pounds per square foot, ρ is mass (slugs, or $\text{lb sec}^2 \text{ ft}^{-1}$) per cubic foot, and T is the temperature in degrees Rankine. The absolute zero of this temperature scale is located at -460° on the usual Fahrenheit scale. This value of R is equal to the conventional engineering value times the acceleration of gravity, g , since the density appears in units of mass, i.e., force divided by g .

When the density of a gas becomes high, more complicated equations of state can be devised to predict the variation of pressure with temperature and density. In general, for air at ordinary atmospheric densities, the difference between these more exact laws and the perfect gas law is insignificant for engineering purposes.

It is important to note that the equation of state is a general relation between three of the parameters of the gas; any two of them are, as yet, completely independent. It should also be pointed out that the condition or state of a fluid may be defined by many other parameters besides temperature, pressure, or density. The equation of state will again provide a relation between any three of them, so that any two may be regarded as independent.

1.2 First Law of Thermodynamics

The first law of thermodynamics is essentially a statement of the law of conservation of energy. It states the fact that *heat energy and mechanical energy are equivalent and interchangeable*. In fact, the kinetic theory of gases indicates heat energy to be really mechanical energy on a molecular scale.

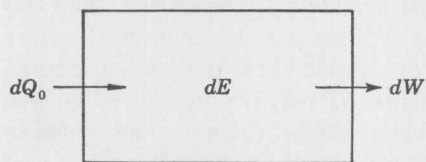


FIG. 1.1. Heat-work exchange.

According to this law, if a quantity of heat is introduced into a closed system containing a gas, this heat must either remain in the gas as internal (heat) energy, or must reappear externally as work done on its surroundings. In Fig. 1.1 let dQ_0 be heat introduced into the container from an outside source, let E be the internal energy of the gas in the container, and dW be the work done by the gas on its surroundings. Then the first law states that

$$dQ_0 = dE + dW \quad (1.3)$$

where dE is the increase in internal energy of the gas.

The mechanical work done by the gas must result from a change in volume. Consider the closed cylinder of Fig. 1.2 with a movable piston of area A at one end, containing a gas at pressure p and volume v . Suppose the piston is balanced by a force F , so that

$$F = pA$$

If now the piston moves to the right through a distance dx , the gas will do work against the force F of amount

$$dW = F dx = pA dx$$

or

$$dW = p dv \quad (1.4)$$

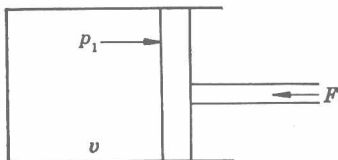


FIG. 1.2. Reversible work.

where dv is the change in volume of the gas. The first law then states

$$dQ_0 = dE + p dv \quad (1.5)$$

A process such as this, in which the gas is very near equilibrium at all times, is called a reversible process. The reversibility stems from the fact that, being very near equilibrium, the process could proceed equally well in either direction. It is in this case only that the external work done is $p dv$. This force F could be utilized to lift a weight or do other useful work of exactly the amount $p dv$. This would be the case only if F were infinitely near to pA .

A process, no matter how slow in point of time, becomes irreversible as soon as the gas is out of equilibrium with its surroundings by a finite amount.

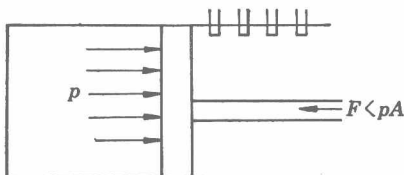


FIG. 1.3. Irreversible work.

Suppose, for example, that the piston in Fig. 1.3 is restrained from moving by a row of pegs while a force F , less than pA , acts on it. These pegs may be very close together, and may be pulled out one at a time very slowly, so that the actual motion of the piston outward is very slow. However, in the very short

time interval after the pulling of a peg, the piston must be subject to a finite acceleration. If the piston moves a distance dx , the only work which appears externally is $dW = F dx$, so that the first law must state

$$dQ_0 = dE + F dx \quad (1.6)$$

This could also be written

$$dQ_0 = dE + p dv - \left(p - \frac{F}{A} \right) dv$$

or

$$dQ_0 + \left(p - \frac{F}{A} \right) dv = dE + p dv \quad (1.7)$$

In this equation the last term on the right is not useful work done, but really internal work done by the gas. Not all of this appeared as useful work; in fact an amount $[p - (F/A)] dv$ was wasted. This amount of work went into accelerating the gas molecules uniformly during the brief instant that the piston was out of equilibrium. In order for the gas to come to equilibrium again, this energy of motion must be dissipated by internal friction into heat.

Let the left-hand side of equation 1.7 be considered as consisting of a quantity of heat dQ_0 introduced from the outside, plus a quantity of wasted energy coming from the inside of the gas itself. This energy supply is then, as in equation 1.5, equal to $dE + p dv$. Apparently, the first law for both irreversible and reversible processes can be written as

$$dQ = dE + p dv \quad (1.8)$$

where now dQ is understood to consist of two parts, dQ_0 and the heat resulting from energy dissipation within the fluid. Only in the case of a reversible process is $dQ = dQ_0$. The first law has now been expressed in two forms, as shown in equations 1.3 and 1.8.

1.3 Internal Energy

The internal energy of a gas is a function of its state, and therefore of any two of the gas parameters. Hence,

$$dE = \left(\frac{\partial E}{\partial v} \right)_T dv + \left(\frac{\partial E}{\partial T} \right)_v dT \quad (1.9)$$

A method of investigating the relative size of these derivatives was devised by Joule. Using the apparatus shown in Fig. 1.4 he filled one of the

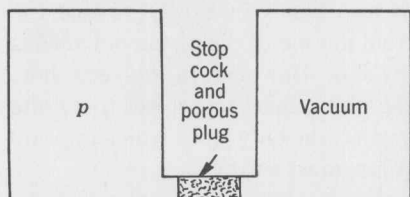


FIG. 1.4. Joule-Thompson experiment.

containers with a gas at high pressure p and evacuated the other. The stop-cock connecting the two was opened, and the system was allowed to come to equilibrium. From the first form of the first law, it is obvious that, since no heat was added from outside and no external work was done by the system,

the change in internal energy must be zero. Joule observed that in this experiment the change in temperature was practically zero. Since the dv in equation 1.9 is certainly not zero, it may be concluded that $(\partial E/\partial v)_T$ is

nearly zero; that is, the internal energy of a gas is nearly independent of its volume. Actually this is true exactly for a perfect gas, but not for any other; the value of $(\partial E/\partial v)_T$ is related to the Joule-Thompson coefficient, which gives the rate of temperature drop with pressure drop during this process.

Joule actually carried out his final experiment in a somewhat revised form, using the steady-state flow in a pipe, but it is the same in principle.

The kinetic theory shows this to mean that the internal energy of a perfect gas is contained entirely within the molecule itself, as energy of translation, rotation, or vibration, and that the intermolecular forces are not important.

1.4 Specific Heats

If heat is added to a gas and the temperature rise is observed, the rate of heat addition per degree temperature rise per unit mass of gas is almost a constant. This rate, $\partial Q/\partial T$, is called specific heat. Its value will depend on the type of the process, i.e., its path in the p - v diagram. In particular, it will have one value if the heating is carried out at constant pressure, another at constant volume. We can find a relation between the specific heat at constant volume, called C_v , and the internal energy, by writing the first law as

$$dQ = p dv + \left(\frac{\partial E}{\partial T}\right)_v dT + \left(\frac{\partial E}{\partial v}\right)_T dv \quad (1.10)$$

This relation is valid for any gas. If we heat with no volume change,

$$\left(\frac{\partial Q}{\partial T}\right)_v = \left(\frac{\partial E}{\partial T}\right)_v = C_v \quad (1.11)$$

Therefore, the internal energy for a perfect gas in which $(\partial E/\partial v)_T = 0$ can be written as

$$E = \int C_v dT + E_0 = C_v T + E_0 \quad (1.12)$$

The specific heat at constant pressure, C_p , is obtained by writing the first law as

$$dQ = d(pv) - v dp + C_v dT$$

This is valid only for a perfect gas. Thus

$$C_p = \left(\frac{\partial Q}{\partial T}\right)_p = \frac{d(pv)}{dT} + C_v \quad (1.13)$$

For a perfect gas, $pv = RT$, so that

$$C_p = R + C_v \quad (1.14)$$

The ratio of the specific heats is given a special name,

$$\gamma = \frac{C_p}{C_v} \quad (1.15)$$

Throughout the following chapters, the gas constants will be in mechanical, rather than in thermal, units; thus the usual heat-work factor will be included. The units of any constant are, after all, defined by the way in which it enters an equation. There is considerable disagreement among the authorities over the correct value of γ . It is probably most important to use one value consistently; in the following work, we shall use $\gamma = 1.400$, a value which appears to be reasonably accurate near zero degrees Fahrenheit, and which has the advantage of computational simplicity. The corresponding specific heat will be $C_p = 5997 \text{ ft}^2/\text{sec}^2 \text{ }^\circ\text{F}$.

The internal energy of a molecule may be divided among its several degrees of freedom in translation and rotation.

It is a basic principle in classical kinetic theory that the internal energy of each degree of freedom of the individual molecules of a gas equals $(1/2)k_r T$, where k_r is the Boltzmann constant. If there are n molecules of gas per unit mass, then $nk_r = R$. The internal energy per degree of freedom per unit mass is then $RT/2$. If there are N degrees of freedom,

$$E = \frac{NRT}{2} = C_v T$$

or

$$C_v = \frac{NR}{2}$$

Then

$$C_p = \frac{NR}{2} + R = \left(\frac{N}{2} + 1\right) R$$

and

$$\gamma = \frac{N + 2}{N} \quad (1.16)$$

A monatomic gas such as helium has only three degrees of freedom in translation, so

$$\gamma = 1.67$$

Air is a mixture of gases, in approximately the following proportions.

Nitrogen	78.06 per cent
Oxygen	21.0 per cent
Argon	0.94 per cent

Since the two principal components, oxygen and nitrogen, are diatomic gases the properties of air follow very closely that type of molecule. A diatomic molecule has five degrees of freedom—three in translation and two in rotation; rotation about the axis connecting the two atoms can be ignored as the moment of inertia of the molecule about this axis is very small. In this case the kinetic theory predicts

$$\gamma = 1.40$$

which is very nearly correct. The agreement is not so good for gas molecules with more than two atoms.

1.5 Enthalpy

Another function of the state of the gas can be defined by letting

$$h = pv + E \quad (1.17)$$

where h is called the enthalpy or total heat. Then

$$\begin{aligned} dh &= p dv + v dp + dE \\ &= dQ + v dp \end{aligned}$$

Thus, if heat is added to the gas at constant pressure, then $dh = dQ$; that is, all the heat added goes into increasing the enthalpy.

A “throttling” process in a pipe can be shown to result in constant enthalpy. In Fig. 1.5 gas at pressure p_1 flows to the right through the porous resistance plug P and emerges at pressure p_2 . The throttling consists of the reduction in pressure as the gas flows through the porous plug. It is an

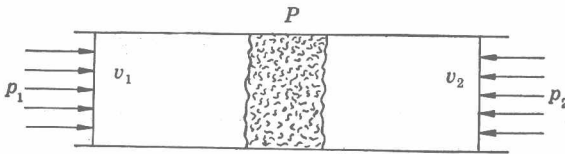


FIG. 1.5. Throttling process.

irreversible process in which mechanical energy is dissipated or used up in friction in the plug. The actual rate of flow of the gas through the passages in the plug must be small; hence the kinetic energy due to the mean velocity of the gas is negligible. As a unit mass of gas flows through the plug, work $p_1 v_1$ is done on this unit mass, and it accomplishes work $p_2 v_2$, making the net work done equal to $p_2 v_2 - p_1 v_1$. Since no heat is introduced from outside,

$$0 = E_2 - E_1 + p_2 v_2 - p_1 v_1$$

or

$$E_1 + p_1 v_1 = E_2 + p_2 v_2$$

so that $h_1 = h_2$. If previous results for a perfect gas are used, it is now possible to write

$$h = RT + C_v T + E_0$$

or, if the enthalpy is measured from its value at $T = 0$,

$$h = C_p T \quad (1.18)$$

1.6 Entropy

If some quantity F is a function only of the state of a gas, and if the gas undergoes a process changing its state, the change in F must be a function only of the initial and final states, and not of the type of process. This is obviously true, for instance, for temperature and pressure. Mathematically speaking, this means that a small increment in F must be a perfect differential function of two variables.

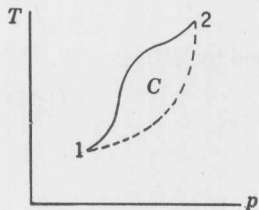


FIG. 1-6. Change of state paths.

$$dF = \left(\frac{\partial F}{\partial p} \right)_T dp + \left(\frac{\partial F}{\partial T} \right)_p dT$$

Suppose the gas is in a state represented by point 1 in Fig. 1-6. Then if F is a function only of the state of the gas, $F_2 - F_1 = \int_1^2 dF$, independent of the path of integration. If the differential of the function F is given by $dF = M dp + N dT$, this is equivalent to saying

$$\oint_C dF = 0 = \oint_C (M dp + N dT)$$

where C is any closed contour in the plane. Green's theorem shows that a necessary and sufficient condition for this to be true is that

$$\frac{\partial M}{\partial T} = \frac{\partial N}{\partial p}$$

if the derivatives are continuous and single valued. This condition is satisfied by

$$M = \frac{\partial F}{\partial p} \quad N = \frac{\partial F}{\partial T}$$

since

$$\frac{\partial}{\partial T} \left(\frac{\partial F}{\partial p} \right) = \frac{\partial}{\partial p} \left(\frac{\partial F}{\partial T} \right)$$

This is known to be true whenever F is an explicitly given function of p and T . However, it is conceivable that the derivatives M and N are given such that the condition is not satisfied. For instance,

$$dQ = C_p dT - v dp$$

Now $(\partial C_p / \partial p)_T = 0$ and $(\partial v / \partial T)_p \neq 0$, so Q cannot be a function of T and p only. This is physically obvious, since the heat added during a process certainly depends on the nature of the process.

However, dQ can be made a perfect differential by multiplying by an "integrating factor." In the case of a perfect gas, for which $pv = RT$, it may be seen that $1/T$ is an integrating factor, for

$$\frac{dQ}{T} = C_p \frac{dT}{T} - R \frac{dp}{p} \quad (1.19)$$

which can obviously be integrated immediately. So, apparently, dQ/T is the perfect differential of some quantity which is a function only of the state of the gas. This quantity is called entropy, defined by

$$dS = \frac{dQ}{T}$$

It can be given as a function of p and T by integrating equation 1.19.

$$S = C_p \log T - R \log p + S_0 \quad (1.20)$$

The constant of integration S_0 is not needed here because it simply establishes the datum from which the entropy is measured; but changes of entropy are of prime interest. Changes in entropy may be written in the form

$$S - S_0 = C_p \log \frac{T}{T_0} - R \log \frac{p}{p_0}$$

The dQ defining the entropy differential must be that used in the second form of the first law, equation 1.8. It is therefore the sum of dQ_0 , the heat introduced from outside, plus any heat arising from dissipation of kinetic energy into heat inside the gas.

If dQ_0 is zero, i.e., the process is conducted with no addition of heat from the outside, it is called adiabatic. This does not necessarily mean that $dQ = 0$ however.

If $dQ = dQ_0$, the process is reversible, because there is no dissipation. If the process is both reversible and adiabatic, $dQ = dQ_0 = 0$. In this case, $dS = 0$, so the reversible adiabatic process is called an isentropic process.

Most gas processes occurring in hydrodynamics are adiabatic, but not all are isentropic. The latter requires friction-free flow, so that the flow in a boundary layer is not isentropic.