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Preface

This book is based on lectures regularly taught in the fifth year of engineering diplomas or basic graduate courses at the University of Liège and at the Universitat Politècnica de Catalunya. The goal of the book is not to provide an exhaustive account of finite elements in fluids, which is an extremely active area of research. The objective is to present the fundamentals of stabilized finite element methods for the analysis of steady and time-dependent convection–diffusion and fluid dynamics problems with an engineering rather than a mathematical bias. Organized into six chapters, it combines theoretical aspects and practical applications and covers some of the recent research in several areas of computational fluid dynamics.

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J. DONEA AND A. HUERTA

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Introduction and preliminaries

1.1 FINITE ELEMENTS IN FLUID DYNAMICS

Introduced in the late 1950s in the aircraft industry, see, for instance, the historical outline by Felippa (2001), the finite element method (FEM) has emerged as one of the most powerful numerical methods so far devised. Among the basic attributes of the method which have led to its widespread acceptance and use are the ease in modeling complex geometries, the consistent treatment of differential-type boundary conditions and the possibility to be programmed in a flexible and general purpose format.

Standard finite element approximations are based upon the Galerkin formulation of the method of weighted residuals. This formulation has proven eminently successful in application to problems in solid/structural mechanics and in other situations, such as heat conduction, governed by diffusion-type equations. The reason for this success is that, when applied to problems governed by self-adjoint elliptic or parabolic partial differential equations, the Galerkin finite element method leads to symmetric stiffness matrices. In this case the difference between the finite element solution and the exact solution is minimized with respect to the energy norm, see, for instance, Strang and Fix (1973). In practice, the Galerkin formulation is optimal in problems governed by self-adjoint equations. In such cases, there exists a quadratic functional the minimum of which corresponds to satisfying the partial differential equation governing the problem at hand. For instance, in linear elasticity the equilibrium position of a structure corresponds to the minimum of the quadratic functional expressing the total potential energy of the system. Similarly, in steady heat conduction problems the thermal equilibrium resulting from satisfying the Laplace or Poisson equation

corresponds to the minimum of a quadratic functional expressed in terms of the thermal flux, which physically represents the total thermal energy of the system.

The success of the Galerkin finite element method in structural mechanics, heat conduction and other problems of the potential type provided, in the early 1970s, a strong impetus for the utilization of the method in the simulation of problems in fluid dynamics. It was thought that the significant advantages gained in structural mechanics and diffusion-type problems would again be open to exploitation in the area of fluid flow simulation. Actually, this proved to be an optimistic point of view, especially with regard to modeling convection-dominated transport phenomena.

The main difficulty was due to the presence of convection operators in the formulation of flow problems based on kinematical descriptions other than Lagrangian. Convection operators are, in fact, non-symmetric and thus the best approximation property in the energy norm of the Galerkin method, which is the basis for success in symmetric cases, is lost when convection dominates the transport process.

In practice, solutions to convection-dominated transport problems by the Galerkin method are often corrupted by spurious node-to-node oscillations. These can only be removed by severe mesh (and time-step) refinement which clearly undermine the practical utility of the method. This has motivated the development of alternatives to the standard Galerkin formulation which preclude oscillations without requiring mesh or time-step refinement. Such alternatives are called stabilization techniques and have provided a major breakthrough in finite element modeling of problems in fluid dynamics.

In truly transient situations, another important issue is to ensure the proper coupling between spatial and temporal approximations. In fact, a stable and accurate spatial representation will be quickly spoiled if the algorithm used for transporting the solution in time is not of comparable accuracy. Space–time coupling is indeed particularly crucial when convection dominates the transport process, due to the directional character of propagation of information in hyperbolic problems. Significant progress has also been achieved in recent years in the development of algorithms for accurately tracing the transient solution of highly convective transport problems.

1.2 SUBJECTS COVERED

The purpose of this book is to describe methods of finite element analysis for steady and time-dependent convection–diffusion and fluid dynamics problems. The intent is to provide an introduction to a variety of modern methods, while preserving a pedagogical character through the presentation of simple worked examples.

The present chapter starts with a review of the basic kinematical descriptions used in fluid mechanics and recalls the conservation laws for mass, momentum and energy in differential and integral forms. It then provides an introduction to the basic ingredients of the finite element analysis of flow problems.

Chapter 2 introduces stabilized finite element methods for steady convection-dominated transport problems (elliptic equations). The difficulties of Galerkin finite elements are first recognized. This allows the design of possible cures for the node-

to-node oscillations. The first alternative formulations proposed in the early 1970s to improve the standard Galerkin method tried to reproduce in the finite element context the effect of upwind differencing used in the finite difference context to stabilize the oscillatory results obtained by central difference approximations. These early, but not fully satisfactory, developments were quickly followed by more convincing finite element procedures, such as the Streamline-Upwind Petrov–Galerkin (SUPG) and the Galerkin/Least-squares (GLS) methods. Such methods do enjoy interesting stability and consistency properties and are nowadays widely used by the finite element community for solving convection-dominated transport problems.

After presenting the difficulties and remedies of Galerkin finite elements in steady convection-dominated problems, transient problems are introduced. In fact, Chapter 3 is devoted to pure convection. The scalar linear first-order hyperbolic equation allows discussion of time-dependent situations. In these problems, the objective has been to develop spatially stable and time-accurate finite element methods that take into account the role of the characteristics in the wave-like solution of hyperbolic equations. This has favored the development of solution algorithms in which attention is focused on achieving a proper coupling between the spatial discretization provided by the finite element method and the time discretization. Methods in this class include various characteristic Galerkin techniques, some classical time-stepping schemes and Taylor–Galerkin methods. The concept of accuracy and numerical stability is introduced. Moreover, spatial formulations especially suited for hyperbolic problems (least-squares and discontinuous Galerkin) are also introduced. A simple model problem also motivates a brief introduction to more recent techniques such as space–time formulations.

Engineering practice goes beyond linear scalar equations. Chapter 4 extends the concepts of the previous chapter to systems of nonlinear equations. In fact, it is concerned with a particular problem: the finite element modeling of inviscid compressible flows governed by the Euler equations of gas dynamics. Moreover, this extension allows discussion of the specificities of numerical methods to capture shocks. First, a brief review of the basic mathematical properties of nonlinear hyperbolic equations is presented. Second, a simple two-step procedure is introduced for the explicit integration of the governing conservation equations of mass, momentum and energy. It ensures second-order accuracy in the smooth part of the flow and, at the same time, allows easy incorporation of a modulated dissipation to avoid oscillatory results in the vicinity of shocks and other discontinuities in the flow. Then, various high-resolution shock-capturing techniques are described and their implementation in the finite element context is illustrated by several worked examples. The chapter closes with a discussion on the use of the Arbitrary Lagrangian–Eulerian (ALE) description for the finite element simulation of problems involving fluid–structure interaction. Both academic and industrial examples are proposed to illustrate the flexibility of the ALE technique in the modeling of coupled transient dynamic problems.

Once the basis of time integration (hyperbolic equations), Chapters 3 and 4, and spatial stabilization in steady problems (elliptic equations), Chapter 2, have been discussed, both methodologies converge in transient convection–diffusion problems (parabolic equations). Chapter 5 is still concerned with accurate time integration but

has to deal with the second-order spatial operator introduced by the diffusion. This allows the incorporation in the transient schemes of the spatial stabilization introduced in Chapter 2, and, moreover, discussion of specific time integration techniques for convection–diffusion problems in order to obtain high-order accurate schemes.

The generalization to nonlinear systems of equations is done in Chapter 6. It provides an introduction to the finite element modeling of incompressible viscous flows governed by the Navier–Stokes equations. And consequently, apart from the numerical difficulties due to the presence of the nonlinear convective term, the incompressibility condition is also a major issue in this chapter. The problem is formulated in the primitive variables, namely velocity and pressure. Mixed and penalty methods in the framework of Stokes and Navier–Stokes equations are introduced. And a brief account is given of stabilization procedures capable of rendering convergent mixed finite element formulations which are unstable in the traditional Galerkin approach. To treat unsteady incompressible flows, a basic fractional-step projection method is introduced and some variants of it are discussed. Emphasis is placed on the treatment of the boundary conditions in each step of the time integration procedure and on the stable treatment of the pressure/incompressibility phase. The chapter closes with applications of the fractional-step method to forced and natural convection problems.

1.3 KINEMATICAL DESCRIPTIONS OF THE FLOW FIELD

In this section and the next one we summarize the continuum mechanics concepts that are needed for the mathematical description of flow problems. Classical references for the basic theory of fluid mechanics are Batchelor (1999), Landau and Lifshitz (1959) and Lamb (1993).

An important consideration when simulating fluid flow problems by any numerical method is the choice of an appropriate *kinematical description* of the flow field. The algorithms of continuum mechanics make use of three distinct types of description of motion: the Lagrangian description, the Eulerian description and the ALE description.

Lagrangian algorithms, in which each individual node of the computational mesh follows the associated material particle during motion, are mainly used in structural mechanics. Classical applications of the Lagrangian description in large deformation problems are the simulation of vehicle crash tests and the modeling of metal forming operations. In these applications, the Lagrangian algorithms are used in combination with both solid and structural (beam, plate, shell) elements. Numerical solutions are often characterized by large displacements and deformations and history-dependent constitutive relations are employed to describe elasto-plastic and visco-plastic material behavior. The Lagrangian description allows easy tracking of free surfaces and interfaces between different materials. Its weakness is its inability to follow large distortions of the computational domain without recourse to frequent remeshing operations.

Eulerian algorithms are widely used in fluid mechanics. Here, the computational mesh is fixed and the fluid moves with respect to the grid. The Eulerian formulation facilitates the treatment of large distortions in the fluid motion and is indispensable for

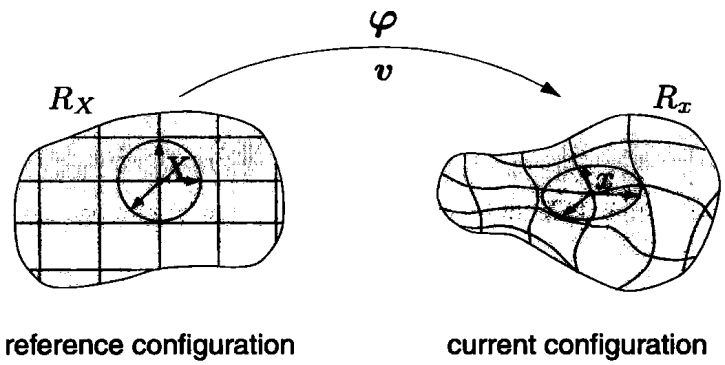

Fig. 1.1 Lagrangian description of motion.

the simulation of turbulent flows. Its handicap is the difficulty to follow free surfaces and interfaces between different materials or different media (e.g., fluid–fluid and fluid–solid interfaces).

ALE algorithms are particularly useful in flow problems involving large distortions in the presence of mobile and deforming boundaries. Typical examples are problems describing the interaction between a fluid and a flexible structure and the simulation of metal forming processes. The key idea in the ALE formulation is the introduction of a computational mesh which can move with a velocity independent of the velocity of the material particles. With this additional freedom with respect to the Eulerian and Lagrangian descriptions, the ALE method succeeds to a certain extend in minimizing the problems encountered in the classical kinematical descriptions, while combining at best their respective advantages.

1.3.1 Lagrangian and Eulerian descriptions

Two domains are commonly used in continuum mechanics: the material domain $R_{\boldsymbol{X}} \subset \mathbb{R}^{n_{sd}}$, with n_{sd} spatial dimensions, made up of material particles $\boldsymbol{X}$, and the spatial domain $R_{\boldsymbol{x}}$, consisting of spatial points $\boldsymbol{x}$.

The Lagrangian viewpoint consists of following the material particles of the continuum in their motion. To this end, one introduces, as suggested in Figure 1.1, a computational grid which follows the continuum in its motion, the grid nodes being permanently connected to the same material points. The material coordinates, $\boldsymbol{X}$, allow us to identify the reference configuration, $R_{\boldsymbol{X}}$.

In the *total Lagrangian formulation*, $R_{\boldsymbol{X}}$ is considered fixed and it corresponds usually to the configuration of the continuum at the initial time. In the *updated Lagrangian formulation*, the reference configuration changes during the calculation and generally corresponds to the configuration relative to the previous time (or load) step.

The motion of the material points relates the material coordinates, $\boldsymbol{X}$, to the spatial ones, $\boldsymbol{x}$. It is defined by an application $\boldsymbol{\varphi}$ such that

$$\begin{aligned} \boldsymbol{\varphi} \; : \; R_{\boldsymbol{X}} \times [t_0, t_{\text{final}}[\; &\longrightarrow \; R_{\boldsymbol{x}} \times [t_0, t_{\text{final}}[\\ (\boldsymbol{X}, t) \; &\longmapsto \; \boldsymbol{\varphi}(\boldsymbol{X}, t) = (\boldsymbol{x}, t), \end{aligned} \tag{1.1}$$

which allows us to link $\boldsymbol{X}$ and $\boldsymbol{x}$ during time by the law of motion, namely

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t), \qquad t = t,$$

which explicitly states the particular nature of $\boldsymbol{\varphi}$: first, the spatial coordinates $\boldsymbol{x}$ depend on both the material particle, $\boldsymbol{X}$, and time t, and, second, physical time is measured by the same variable t in both material and spatial domains. For every fixed instant t, the mapping $\boldsymbol{\varphi}$ defines a configuration in the spatial domain. It is convenient to employ a matrix representation for the gradient of $\boldsymbol{\varphi}$,

$$\frac{\partial \boldsymbol{\varphi}}{\partial (\boldsymbol{X}, t)} = \begin{pmatrix} \dfrac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} & \boldsymbol{v} \\ \mathbf{0}^T & 1 \end{pmatrix},$$

where $\mathbf{0}^T$ is a null row vector and the material velocity $\boldsymbol{v}$ is

$$\boldsymbol{v}(\boldsymbol{X}, t) = \left. \frac{\partial \boldsymbol{x}}{\partial t} \right|_{\boldsymbol{X}}, \tag{1.2}$$

with $\Big|_{\boldsymbol{X}}$ meaning "holding $\boldsymbol{X}$ fixed".

Obviously, the one-to-one mapping $\boldsymbol{\varphi}$ must verify $\det(\partial \boldsymbol{x} / \partial \boldsymbol{X}) > 0$ (non-zero to impose a one-to-one correspondence and positive to avoid change of orientation in the reference axes) at each point $\boldsymbol{X}$ and instant $t > t_0$. This allows us to keep track of the history of motion and, by the inverse transformation $(\boldsymbol{X}, t) = \boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t)$, to identify at any instant the initial position of the material particle occupying position $\boldsymbol{x}$ at time t.

Since the material points coincide with the same grid points during the whole motion, there are no convective effects in Lagrangian calculations: the material derivative reduces to a simple time derivative. The fact that each finite element of a Lagrangian mesh always contains the same material particles represents a significant advantage from the computational viewpoint, especially in problems involving materials with history-dependent behavior. These concepts are discussed in detail by Bonet and Wood (1997) in their excellent textbook on nonlinear continuum mechanics for finite element analysis. However, when large material deformations do occur, for instance vortices in fluids, Lagrangian algorithms undergo a loss in accuracy, and may even be unable to conclude a calculation, due to excessive distortions of the computational mesh linked to the material.

The difficulties caused by an excessive distortion of the finite element grid are overcome in the Eulerian formulation. The basic idea in the Eulerian formulation, which is very popular in fluid mechanics, consists in examining as time evolves