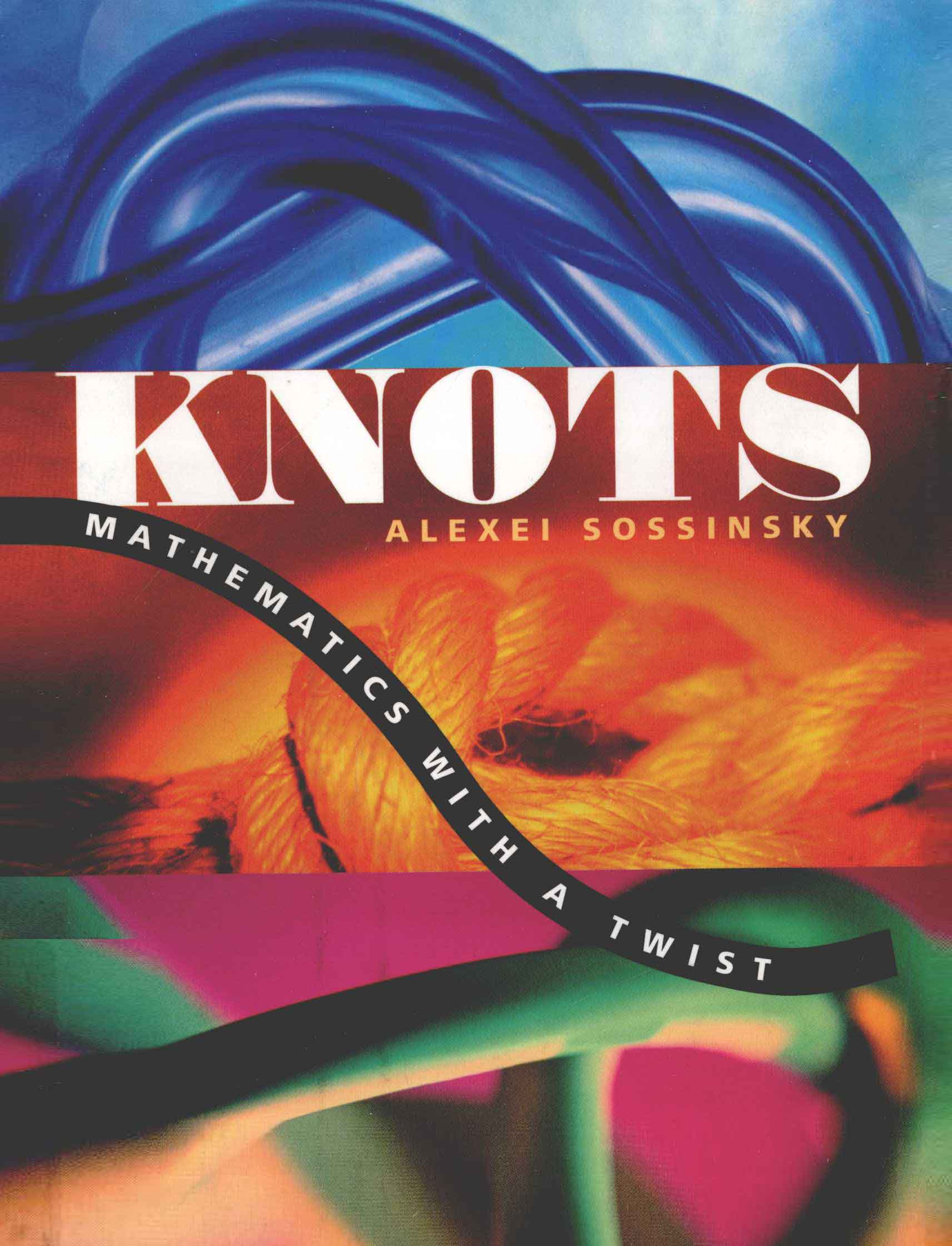
KNOTS
ALEXEI SOSSINSKY
MATHEMATICS WITH A TWIST

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TRANSLATED BY GISELLE WEISS

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KNOTS

PREFACE

Butterfly knot, clover hitch knot, Gordian knot, hangman's knot, vipers' tangle—knots are familiar objects, symbols of complexity, occasionally metaphors for evil. For reasons I do not entirely understand, they were long ignored by mathematicians. A tentative effort by Alexandre-Théophile Vandermonde at the end of the eighteenth century was short-lived,[1] and a preliminary study by the young Karl Friedrich Gauss was no more successful. Only in the twentieth century did mathematicians apply themselves seriously to the study of knots. But until the mid-1980s, knot theory was regarded as just one of the branches of topology: important, of course, but not very interesting to anyone outside a small circle of specialists (particularly Germans and Americans).

Today, all that has changed. Knots—or more accurately, mathematical theories of knots—concern biologists, chemists, and physicists. Knots are trendy. The French "nouveaux philosophes" (not so new anymore) and postmodernists even talk about knots on television, with their typical nerve and incompetence. The expressions "quantum group" and "knot polynomial" are used indiscriminately by people with little scientific expertise. Why the interest? Is it a passing fancy or the provocative beginning of a theory as important as relativity or quantum physics?

This book addresses this question, at least to some extent, but its aim is certainly not to provide peremptory answers to global inquiries. Rather, it presents specific information about a subject that is difficult to grasp and that, moreover, crops up in many guises, often imbued with mystery and sometimes with striking and unexpected beauty.

This book is intended for three groups of readers: those with a solid scientific background, young people who like mathematics, and others, more numerous, who feel they have no aptitude for math as a result of their experience in school but whose natural curiosity remains intact. This last group of readers suffers from memories of daunting and useless "algebraic expressions," tautological arguments concerning abstractions of dubious interest, and lifeless definitions of geometric entities. But mathematics was a vibrant field of inquiry before lackluster teaching reduced it to pseudoscientific namby-pamby. And the story of its development, with its sudden brainstorms, dazzling advances, and dramatic failures, is as emotionally rich as the history of painting or poetry.

The hitch is that understanding this history, when it is not reduced to simple anecdotes, usually calls for mathematical sophistication. But it so happens that the mathematical theory of knots—the subject of this book—is an exception to the rule. It doesn't necessarily take a graduate of an elite math department to understand it. More specifically, the reader will see that the only mathematics in this book are simple calculations with polynomials and transformations of little diagrams like these:

Readers will also have to draw on their intuition of space or, failing that, fiddle with strings and make actual knots.

My desire to avoid overly abstract and technically difficult mathematics led me to leave out completely the most classical tool of the theory of knots (and the most efficient at the early stages), the so-called fundamental group. The first successes with the theory—those of the mathematicians of the German school (N. G. Van Kampen, H. Seifert, M. Dehn), the Dane J. Nielsen, and the American J. W. H. Alexander—were based on the judicious use of this tool. Their work will barely be mentioned here.

Given the diversity of the topics tackled in this book, I have not tried to provide a systematic and unified exposition of the theory of knots; on the contrary, various topics are scattered throughout the chapters, which are almost entirely independent of each other. For each topic, the starting point will be an original idea, as a rule simple, profound, and unexpected, the work of a particular researcher. We will then follow the path of his thinking and that of his followers, in an attempt to understand the major implications of the topic for contemporary science, without going into technical details. Accordingly, the chapters are ordered more or less chronologically. But I have striven to minimize cross-references (even if it means repeating certain passages), so that the chapters can be read in whatever order the reader chooses.

Before I review the topics taken up in each chapter, it is worth mentioning that prior to becoming the object of a theory, knots were associated with a variety of useful activities. Of course, those activities are not the subject of this book, but talking a little about their practical charms will make it easier to glimpse the beauty of the theory.

Since Antiquity, the development of knot making was motivated by practical needs, especially those of sailors and builders. For each specific task, sailors invented an appropriate knot, and the best knots survived, passing from generation to generation (see Adams, 1994). To tie a rope to a rigid pole (a mooring or a mast), one uses the clover hitch knot (see Figure P.1a), the rolling hitch knot (b), or the camel knot (c); to tie two ropes together, the square knot (d) or the fisherman's knot (f) (when they are the same size) or the sheet bend (e) (when one rope is thicker than the other). And there are many other knots adapted to these special tasks (see Figure P.2). Sailors use knots not only to moor boats, rig sails, and hoist loads, but also to make objects as varied as the regrettably famous "cat o' nine tails" and straw mats woven in Turk's head knots (Figure P.2b).

In the Age of Enlightenment (in England even earlier), oral transmission of maritime knot making was supplanted by specialized books about knots. One of the first authors in this genre was the Englishman John Smith, much better known for his romantic adventures with the beautiful Indian princess Pocahontas. At the same time, the terminology associated with knots became codified; it was even the subject of a detailed article in Diderot's and d'Alembert's *Enclopédie.*

Sailors were not the only inventors of knots. The fisherman's hook knot (Figure P.2f), the alpinist's chair knot (d), the engineer's constrictor knot (c), and the knitter's rice stitch (e) are only a few examples among many. The classic reference for knot making is Ashley's famous *Book of Knots* (1944). A few knots in particular derive from one of the greatest technological inventions of the Middle Ages: the pulley (Figure P.3a), together with the compound pulley (b and c). This work-saving device, a sort of Archimedes' lever with ropes, unites two major

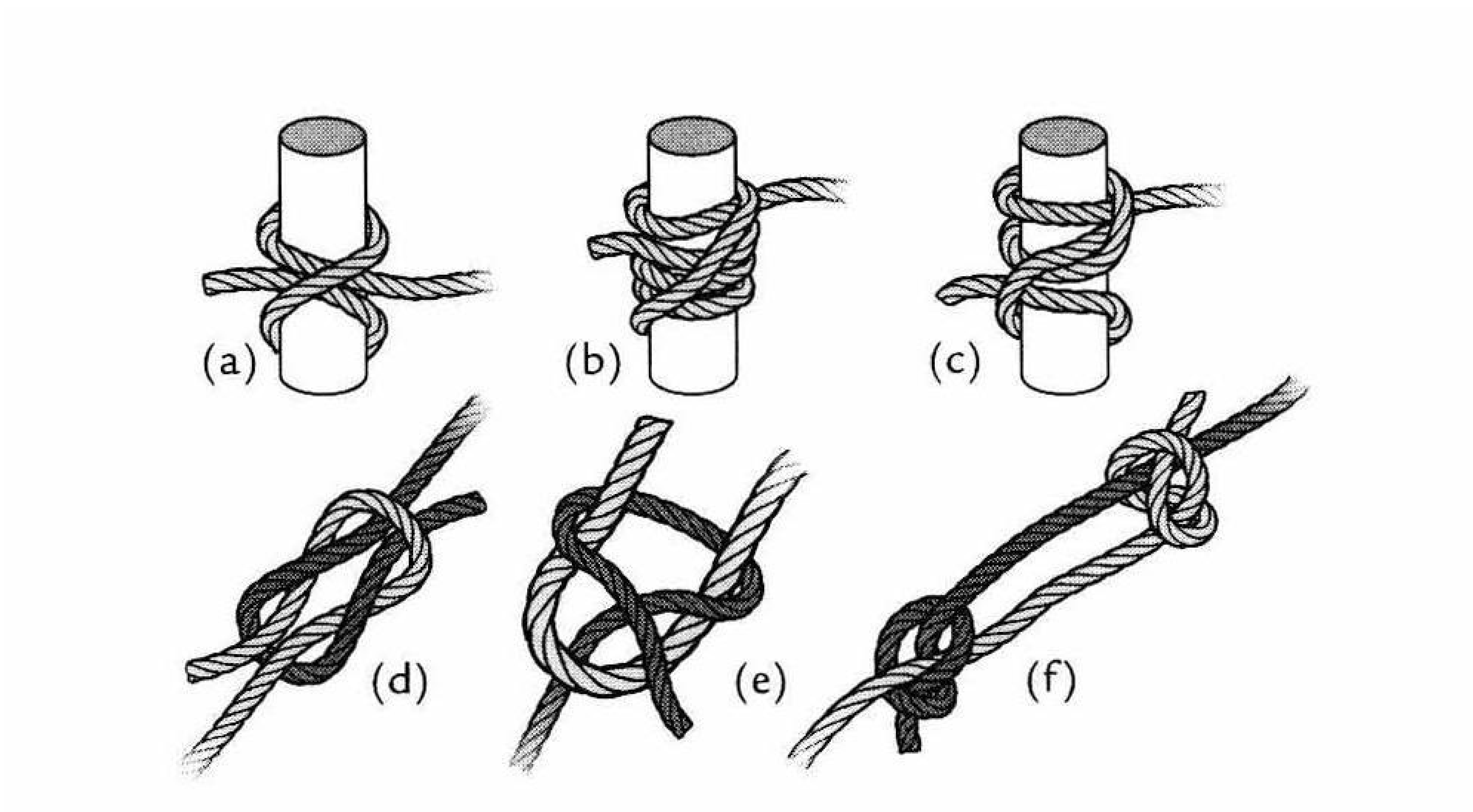

Figure P.1. Some sailors' knots.

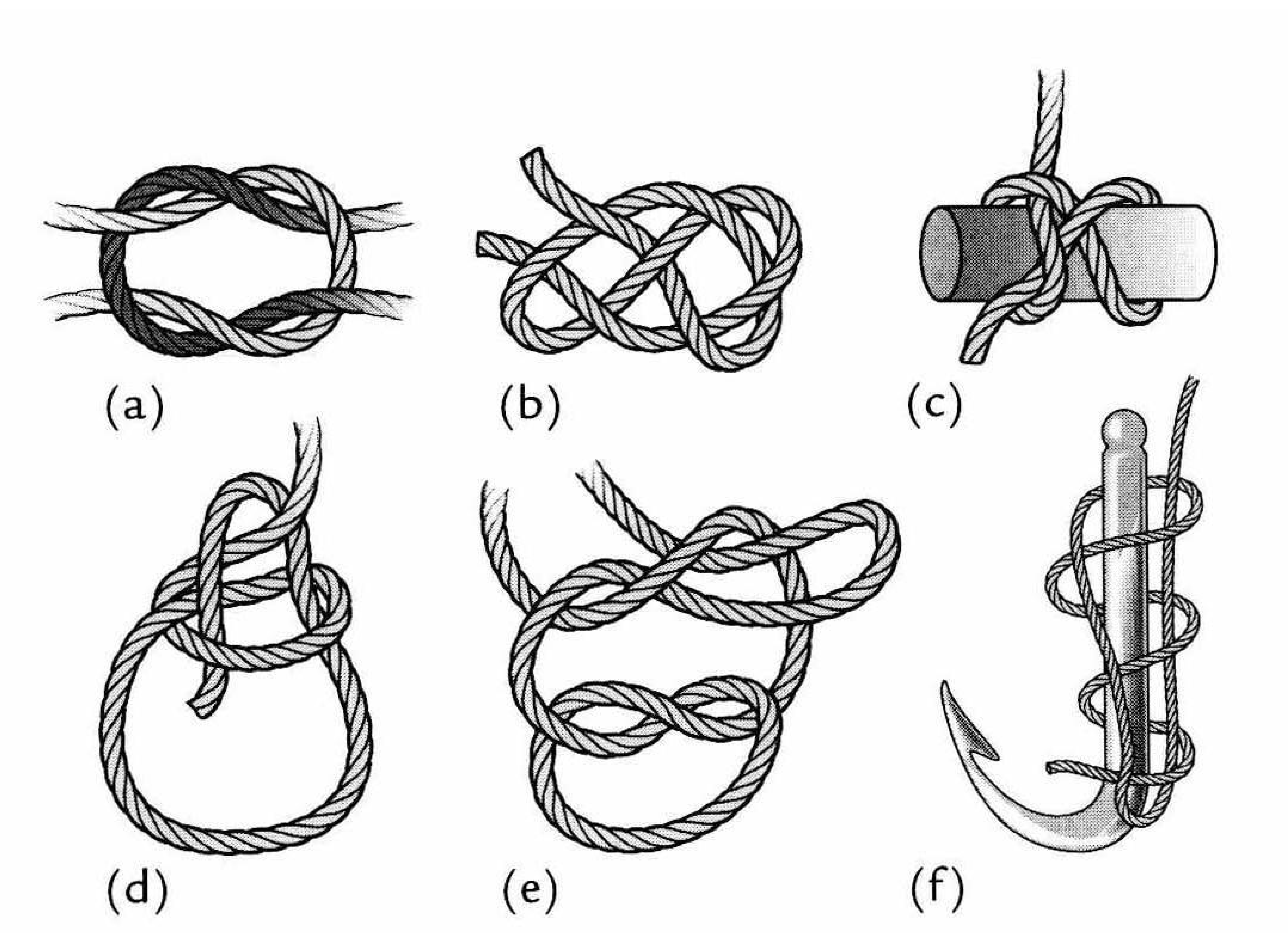

Figure P.2. Other knots.

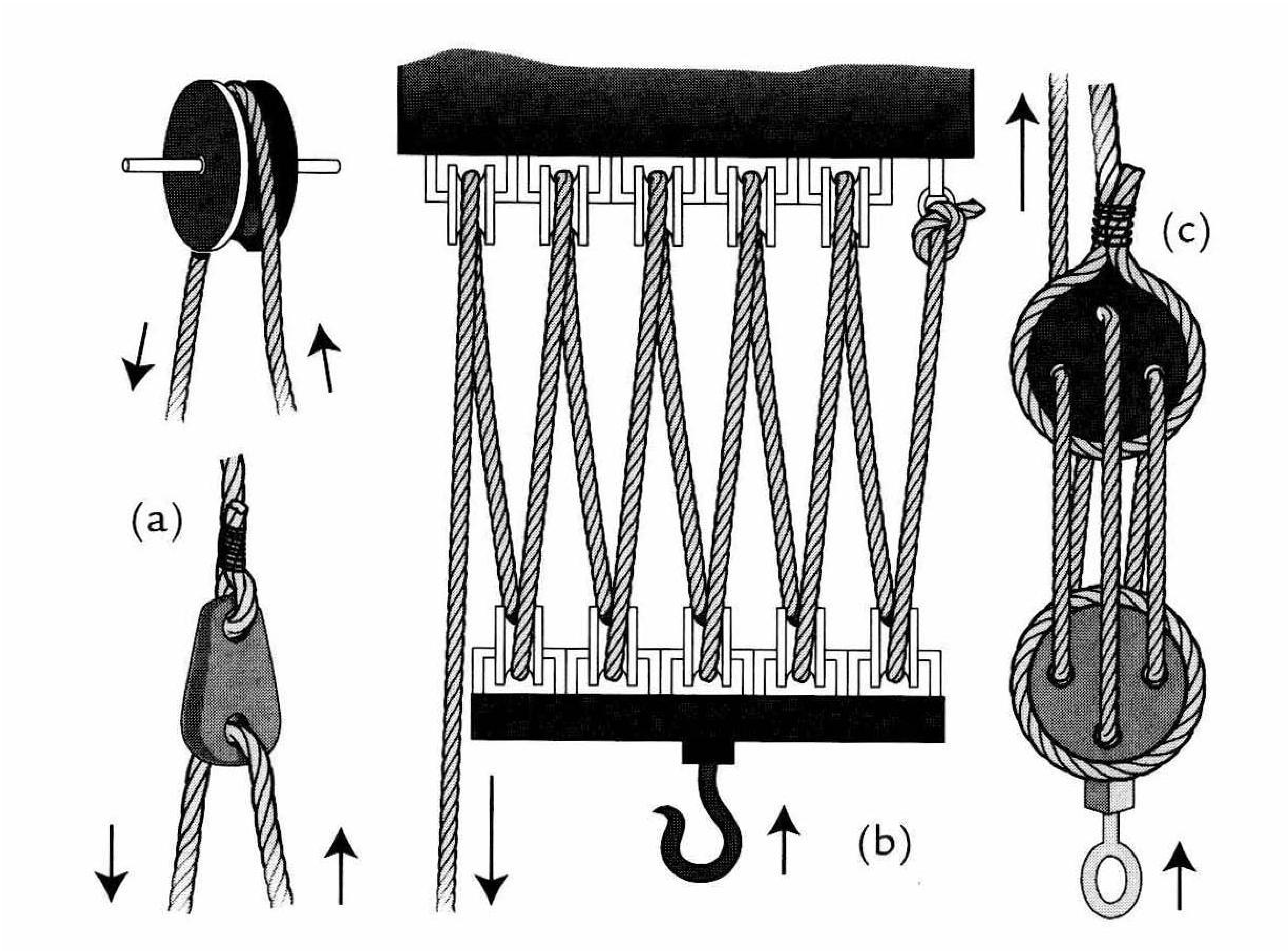

Figure P.3. Pullies and a hoist.

inventions of Antiquity: the wheel and the rope. It is used to pull or to lift all kinds of loads, usually also attached with the help of suitable knots. Thanks to knots, the rope became the universal technological tool of the age.

The technology for producing ropes (and cables) themselves—braiding—became very important. Fibers (once made of plants such as hemp, but synthetic in our times) had to be twisted into threads that were then braided into thicker strands, called lines, which in turn were braided in a specific way (generally involving three lines) to make a rope (see Figure P.4). The procedure for making cables is more com-

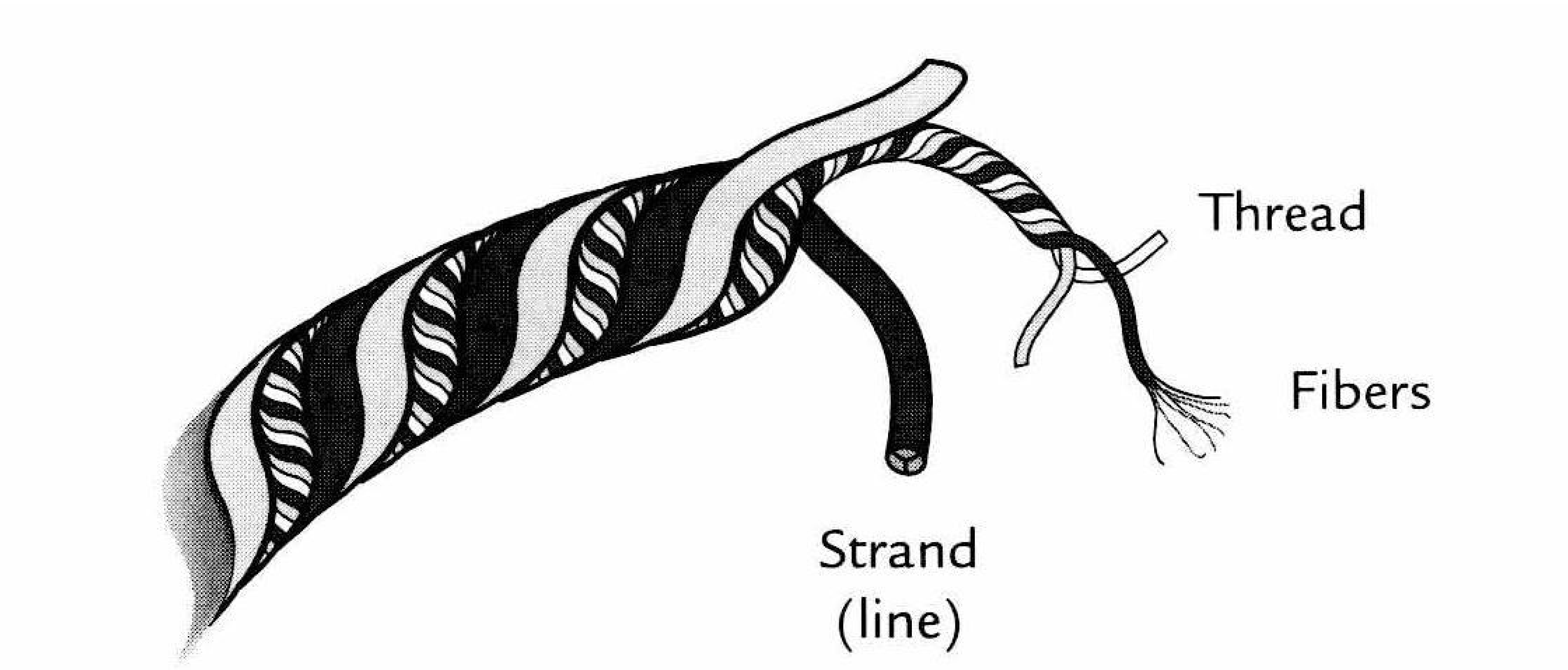

Figure P.4. Anatomy of a rope.

plex and involves four (or more) levels of cords, lines, and braided ropes. For the mathematician, the technology of braiding is the model for a basic idea in topology (as well as in mechanics)—the *braid*—which we will discuss in detail in Chapter 2.

Utilitarian and technological considerations aside, knots also have an aesthetic, mysterious, and magical aspect. As far as I know, it is precisely this feature of knots that is responsible for their first traces in our civilization. I have in mind the remarkable representations of knots on the megaliths and burial stones engraved by Neolithic peoples, in particular the Celts, during the fourth century B.C. Actually, these are chains of knots connected to one another (mathematicians call them *links*), as shown schematically in Figure P.5. We do not know the mystical and religious meaning of the links represented on menhirs (upright monuments also known as standing stones), but the geometric technology (based on regular figures) used to create these bewitching designs has been decoded by mathematicians (see Mercat, 1996).

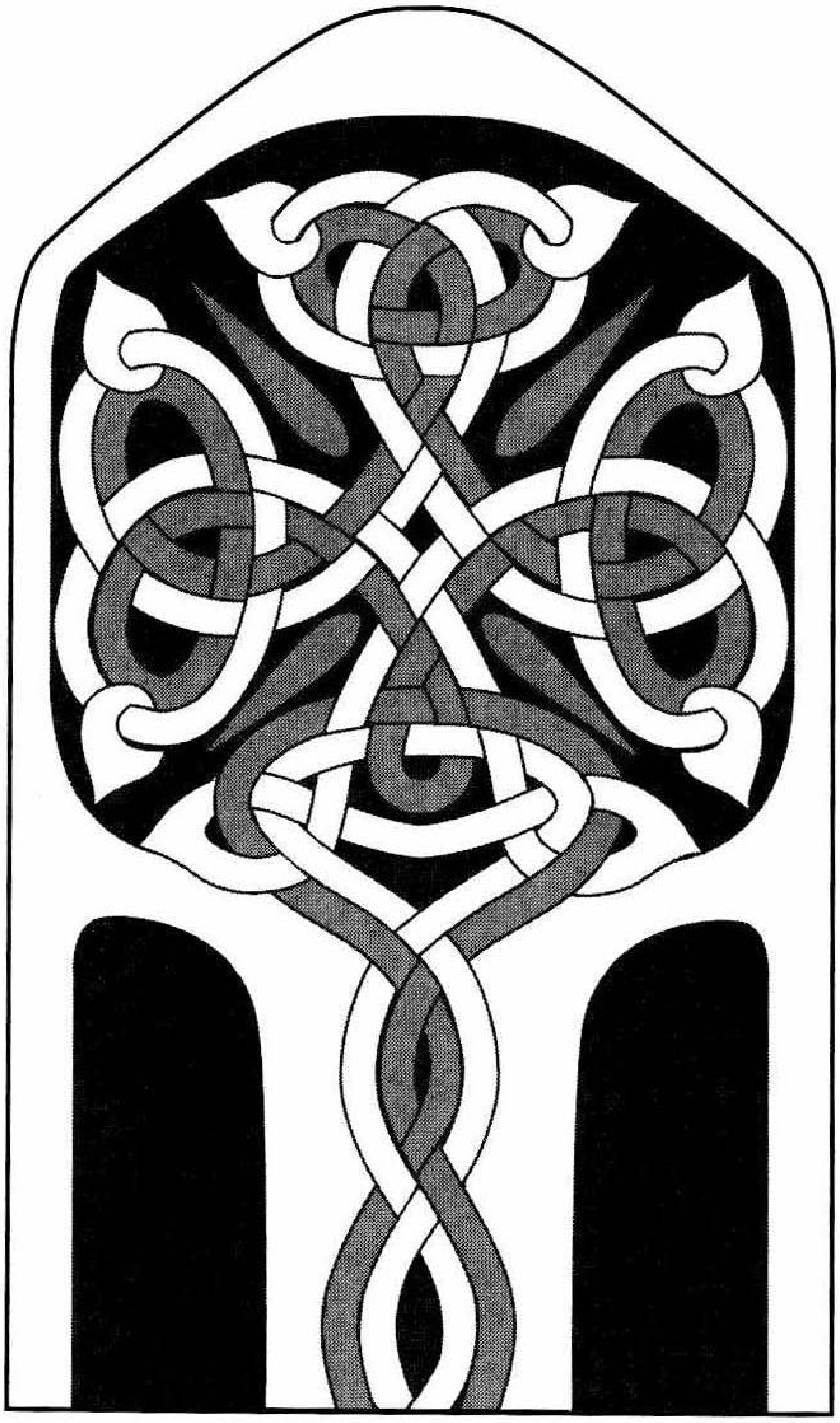

Figure P.5. Links on a megalith.

Neolithic peoples were not alone in using links to decorate their objects of worship. Links are also found in the Middle Ages, in illuminated manuscripts, in the architecture of certain Eastern civilizations (friezes and other ornaments of the famous Alhambra palace in Spain are examples), and in the decorative elements framing icons in orthodox churches in northern Russia.

To end this overview of knots on a lighter note, think of the essential role they play in the magician's arsenal: knots that aren't, ropes that come undone instead of strangling the sexy magician's assistant, and so on. Some of these tricks (which amateur magicians can do) are described, from a mathematical vantage point, elsewhere (Prasolov and Sossinsky, 1997; Walker, 1985).

Let us move on to a summary of this book, to give a brief idea of what is to come and to allow those who don't intend to read the book from beginning to end to choose which chapters they wish to take in.[2] (Remember that the chapters are relatively independent.)

The first chapter has to do with the beginnings of the mathematical theory of knots, which was not the work of mathematicians—what a shame for them!—but that of physicists, more precisely, William Thomson (alias Lord Kelvin). The starting point (dating from around 1860) was Thomson's idea of using knots as models for the atom, models he dubbed "vortex atoms." To study the theory of matter from this point of view, he had to begin with knots. Fortunately for the self-esteem of mathematicians, Kelvin's theory ran aground and was soon forgotten, but not without leaving to posterity a series of problems (the *Tait conjectures*), which physicists were unable to solve at the time but mathematicians took care of a century later. The chapter not only deals with this spectacular failure of a beautiful physical theory, it also reviews various aspects of knot theory: Tait's tables of alternating knots, the superb *wild knots,* and *Antoine's necklace.* This last object provides us with an opportunity to talk about . . . blind geometers. The chapter ends with a brief discussion of the reasons for the failure of Thomson's theory.

The second chapter deals with the fundamental connection between knots and braids discovered by the American J. W. H. Alexander a half-century after Kelvin's abortive start. The mathematical theory of braids, which was formulated about the same time by the young German researcher Emil Artin, is more algebraic (and consequently simpler and more efficient) than knot theory. The connection in question (a geometric construction of childlike simplicity: the so-called closure of braids) enables one to obtain all knots from braids—Alexander's result. And because Artin rapidly established the classification of braids, it was natural to try to deduce the classification of knots from it. Efforts in this direction were unsuccessful, but they gave nice results, among which are the algorithms and software recently devised by French researchers.

In Chapter 3, I present a clever but simple geometric construction by the German mathematician Kurt Reidemeister, which reduces the study of knots in space to their planar projections (called *knot diagrams*). This gives us a chance to talk a little about catastrophe theory, encoding of knots, and working with knots on the computer. We will see that an algorithm invented by Reidemeister's compatriot Wolfgang Haken to determine whether a given knot can be untied does indeed exist, though it is very complex. That is because untying a knot often means first making it more complicated (alas, also true in real life). Finally, the functioning of an unknotting algorithm (which is fairly simple but has the disadvantage of futility when it comes to trying to unknot non-unknottable knots) will be explained: there, too, the modern computer does a better job of unknotting than we poor *Homo sapiens.*

Chapter 4 reviews the arithmetic of knots, whose principal theorem (the existence and uniqueness of prime knots) was demonstrated in

1949 by the German Horst Schubert. The curious resemblance between knots equipped with the composition operation (placing knots end to end) and positive integers (with the ordinary product operation) excited all sorts of hopes: Could knots turn out to be no more than a geometric coding of numbers? Could the classification of knots be just a plain enumeration? In Chapter 4 I explain why such hopes were unfulfilled.

Chapter 5 brings us to an invention that seems trite at first. It is due to the Anglo-American John Conway, one of the most original mathematicians of the twentieth century. As in Chapter 3, we will be dealing with small geometric operations carried out on knot diagrams. Contrary to Reidemeister moves, Conway operations can change not only the appearance but also the type of the knot; they can even transform knots into links. They make it possible to define and to calculate, in an elementary way, the so-called *Alexander-Conway polynomial*[3] of a knot (or link). These calculations provide a very easy and fairly efficient way to show that two knots are not of the same type, and in particular that some knots cannot be unknotted. But this method is probably not what the reader of this chapter will find most interesting: a biological digression explains how *topoisomerases* (recently discovered specialized enzymes) actually carry out Conway operations at the molecular level.

Chapter 6 presents the most famous of the knot invariants, the Jones polynomial, which gave new life to the theory fifteen years ago. In particular, it allowed several researchers to establish the first serious connections between this theory and physics. Oddly, it is the physical interpretation[4] of the Jones polynomial that gives a very simple description of the Jones invariants, whose original definition was far from elementary. This description is based on a tool—the Kauffman bracket—that is very simple but that plays no less fundamental a role in modern theoretical physics. This chapter contains several digressions. In one of them, readers will learn that the main ingredient in the Kauffman bracket was already known in the Neolithic age by the Celtic artists mentioned earlier.

Chapter 7 is devoted to the last great invention of knot theory, Vassiliev invariants. Here, too, the original definition, which drew on catastrophe theory and spectral sequences,[5] was very sophisticated, but an elementary description is proposed. Instead of complicated mathematical formulas, readers will find abbreviated calculations involving little diagrams, along with a digression on the sociological approach to mathematics.

The eighth and final chapter discusses connections between knot theory and physics. Contrary to what I tried to do in the other chapters, here I could only sketch out the most rudimentary explanations of what is going on in this area. I had to use some new technical terms from mathematical physics without being able to explain them properly. But I am convinced that even readers closer to the humanities than to the sciences will succeed in getting through this chapter. Even if they cannot grasp the precise meaning of the terms and equations, they can focus on the gist of the discussion, on the role of coincidences, and on the dramatic and emotional side of contemporary research.

The brilliant beginnings of knot theory, over 130 years ago, were marked by a ringing failure—as a physical theory of matter—but the concepts were revived thanks to the repeated efforts of mathematicians, whose only motivation was intellectual curiosity. Progress required new, concrete ideas. And the ideas came, springing from the imagination of the best researchers, often sparking exaggerated hopes. But every failure made it easier to grasp the remaining problems, making the final goal ever more attractive. Today we are in a situation similar to that of 1860: some researchers think, as William Thomson did, that knots play a key role in the basic theory of the structure of matter. But that is not to say that we are back at the beginning: the spiral of knowledge has made a full loop, and we find ourselves at a higher level.

The theory of knots remains just as mysterious and vibrant as ever. Its major problems are still unsolved: knots continue to elude efforts to classify them effectively, and still no one knows whether they possess a complete system of invariants that would be easy to calculate. Finally, the basic role knots are supposed to play in physics has not yet been specified in a convincing way.