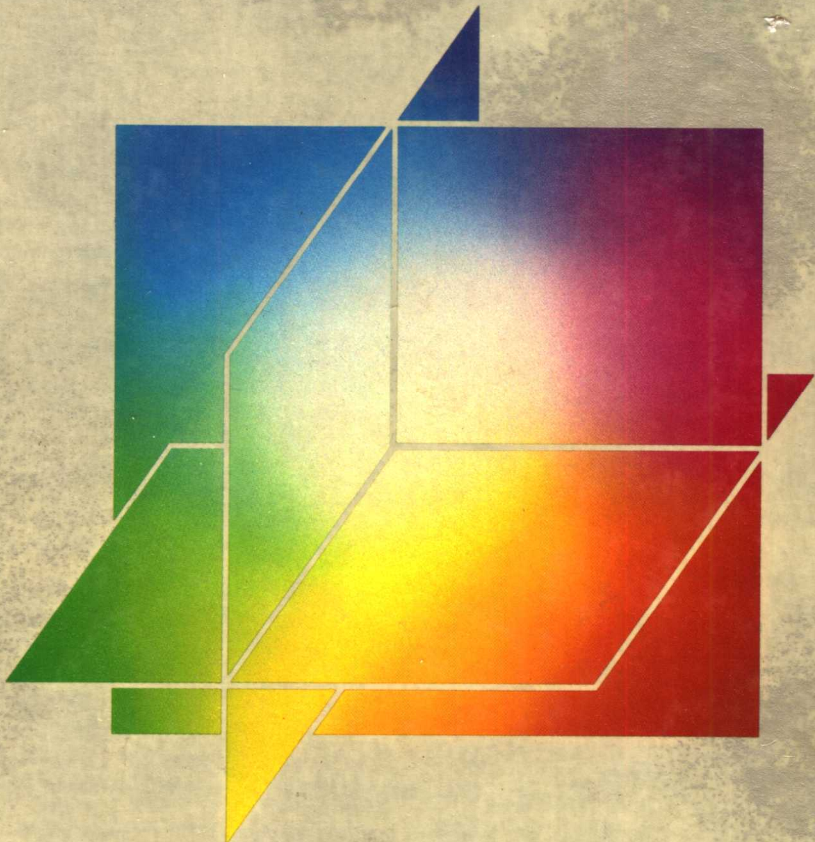

CALCULUS

WITH ANALYTIC GEOMETRY

THIRD EDITION

ROBERT ELLIS DENNY GULICK

CALCULUS

WITH ANALYTIC GEOMETRY

THIRD EDITION

ROBERT ELLIS
University of Maryland
at College Park

DENNY GULICK
University of Maryland
at College Park

HBJ

HARCOURT BRACE JOVANOVICH, PUBLISHERS
San Diego New York Chicago Atlanta Washington, D.C.
London Sydney Toronto

To Rosemarie and Mark

To Frances, David, Barbara, and Sharon

ISBN: 0-15-505737-5

Library of Congress Catalog Card Number: 85-60808

Printed in the United States of America

Technical Art by Vantage Art, Inc.

Photo Credits

p. 19: Courtesy of Educational Development Center, Newton, Ma.
32: © Tiers, Monkmeyer Press Photo Service
205: Courtesy of French Cultural Services
221: NASA
332: UPI/Bettmann News Photos
339: The Cleveland Museum of Natural History
342: NASA
354: The Convention and Tourist Board of Greater St. Louis
430: The Oriental Institute, University of Chicago
431: NASA
469: National Park Service
499: From *Sounds of Music* by Charles Taylor, BBC London
577: NASA
578: The Port Authority of New York and New Jersey
587: National Park Service
659: Reproduced by permission of the publisher of *Physics, 2nd ed.* by the Physical Science Study Committee, © 1965, p. 100, D.C. Heath and Company.
677: UPI/Bettmann News Photos
697: Stephen J. Potter/STOCK, Boston
706: UPI/Bettmann News Photos
763: Swiss National Tourist Office
952: UPI/Bettmann News Photos

PREFACE

Like its predecessor, the Third Edition of *Calculus with Analytic Geometry* contains all the topics that normally constitute a course in calculus of one and several variables. It is suitable for sequences taught in three semesters or in four or five quarters. In the three-semester case, the first semester will usually include the introductory chapter (Chapter 1), the three chapters on limits and derivatives (Chapters 2–4), and the initial chapter on integrals (Chapter 5). The second semester would then include the rest of the discussion of integration (Chapters 6–8) and some combination of the chapters on series (Chapter 9), conic sections (Chapter 10), and the introduction to vectors and vector-valued functions (Chapters 11 and 12). The third semester would include the remainder of those chapters, along with the material on calculus of several variables (Chapters 13 and 14) and Chapter 15, which includes the theorems of Green and Stokes as well as the Divergence Theorem. Finally, an optional chapter on differential equations can be covered in whole or in part during the second or third semester.

The revisions in the Third Edition have been made with the benefit of comments from many users of the previous editions, as well as from our experience in teaching from those editions. The most noteworthy changes are the following:

- Cumulative Review Exercises now conclude Chapters 3 through 15, thereby providing continuous reinforcement of major concepts from the preceding chapters.
- Riemann sums, fundamental to applications of the integral, are now presented along with the initial presentation of the integral, in Chapter 5.
- Derivations of formulas for applications of the integral have been altered so as to be more intuitive.
- The complete single variable discussion of area is now placed in Chapter 5, in order to give added emphasis to the concept.
- We have used an algorithmic approach for the bisection and Newton-Raphson methods, because of the potential use of calculators and computers in calculus. In addition, we have increased the number and variety of exercises for which the Newton-Raphson method is appropriate.
- We have streamlined the introductory discussion of Taylor's Theorem at the outset of Chapter 9, thereby facilitating progress toward the topics of sequences and series.
- Finally, a selection of computer-drawn figures appears in Chapter 13 to help students visualize surfaces that are not easily drawn by hand.

Although we develop the material in the order that we have found pedagogically most effective, instructors will have considerable flexibility in choosing topics. Chapter 1 (which includes a section on trigonometry, so that trigonometric functions can serve as examples throughout the book) is preliminary and can be covered quickly if the student's preparation is sufficient. With a little care, applications of the integral (Chapter 8) can be discussed before techniques of integration (Chapter 7). Sequences and series (Chapter 9) can be studied any time after Chapter 8, conic sections (Chapter 10) any time after Chapter 4, and differential equations (Chapter 16) any time after Chapter 7.

Whenever possible, we use geometric and intuitive motivation to introduce concepts and results, so that students may readily absorb the carefully worded definitions and theorems that follow. The topical development, in which we employ numerous worked examples and almost 900 illustrations, aims for clarity and precision without overburdening the reader with formalism. In keeping with this goal, we have proved most theorems of first-year calculus in the main body of the text but have placed the more difficult proofs in the Appendix. In the chapters on calculus of several variables we have proved selected theorems that aid comprehension of the material.

Exercises appear both at the ends of sections and, for review, at the end of each chapter. Each set begins with a full complement of routine exercises to provide practice in using the ideas and methods presented in the text. These are followed by applied problems and by other exercises of a more challenging nature (identified with an asterisk). To supplement the usual problems from physics and engineering, we have included many from business, economics, biology, chemistry, and other disciplines, as well as a smaller number of exercises suitable for solution on a calculator (indicated by the symbol ■). In addition, Chapters 3–15 each end with a collection of cumulative review exercises, which are intended to reinforce the main ideas of the previous chapters. In the interest of accuracy every exercise has been completely worked by each of the authors. Answers to odd-numbered exercises (except those requiring longer explanations) appear at the back of the book.

Throughout the book, statements of definitions, theorems, lemmas, and corollaries, as well as important formulas, are highlighted with tints for easy identification. Numbering is consecutive throughout each chapter for definitions and theorems, and consecutive within each section for examples and formulas. We use the symbol ■ to signal the end of a proof and □ for the end of the solution to an example.

Lists of Key Terms and Expressions, Key Formulas, and Key Theorems appear at the end of each chapter. On the endpapers we have assembled important formulas and results that the student will want to have handy, both for course review and for reference in later studies. Pronunciation of difficult terms and names is shown in footnotes on the pages where they first appear.

We are very grateful to many people who have helped us in a variety of ways as we prepared the various editions of this book. Our thanks go to reviewers Daniel D. Anderson (*University of Iowa*), Raymond J. Cannon, Jr. (*Baylor University*), Douglas Crawford (*College of San Mateo*), Arthur Crummer (*University of Florida*), Robert M. Dieffenbach (*Miami University, Ohio*), J. R. Dorroh (*Louisiana State University*), Daniel Drucker (*Wayne State University*), Bruce Edwards (*University of Florida*), Murray Eisenberg (*University of Massachusetts*), Charles H. Franke (*Seton Hall University*), Robert Gold (*Ohio State University at Columbus*), Jack Goldberg (*University of Michigan*), Stuart Goldenberg (*California Polytechnic University, San Luis Obispo*), Robert B. Hughes (*Boise State University*), Richard Koch (*University of Oregon*), J. D. Konhauser (*Macalester College*), Theodore Laetsch (*University of Arizona*), Peter Lindstrom (*Genessee Community College*), David J. Lutzer (*Miami University, Ohio*), Hugh B. Maynard (*University of Texas at San Antonio*), Peter Nyikos (*University of Southern California*), Jack Robertson (*Washington State University*), M. M. Subramaniam (*Pennsylvania State University, Delaware Campus*), John Thorpe (*State University of New York at Stonybrook*), Mark S. Ubelhor (*Scott Community College*), Abraham Weinstein (*Nassau Community College*), and Paul Zorn (*St. Olaf College*). Many of our colleagues at the University of Maryland have made contributions to the original writing of this book and to the revisions; we wish to express our appreciation to William Adams, Stuart Antman, Douglas Arnold, Joseph Auslander, Kenneth Berg, Ellen Correl, Jerome Dancis, Gertrude Ehrlich, Craig Evans, Seymour Goldberg, Jacob Goldhaber, Paul Green, Frances Gulick, Bert Hubbard, James Hummel, Nelson Markley, James Owings, Jonathan Rosenberg, Karl Stellmacher, C. Robert Warner, Peter Wolfe, James Yorke, and Mishael Zedek. In addition, we are grateful for comments and suggestions from Bruce L. Aborn (*Bentley College*), Steven Agronsky (*California Polytechnic University, San Luis Obispo*), Robert Baer (*Miami University*), David W. Bange (*University of Wisconsin, LaCrosse*), Don Blevins (*Trinity College*), Thomas T. Bowman (*University of Florida*), Art Bukowski (*University of Alaska at Anchorage*), Martin Buntinas (*Loyola University*), Lawrence O. Cannon (*Utah State University*), Ray Cannon (*Stetson University*), Elizabeth B. Chang (*Hood College*), F. Lee Cook (*University of Alabama, Huntsville*), Craig Cordes (*Louisiana State University*), Brad Crain (*Portland State University*), Hall Crannell (*Catholic University of America*), John S. Cross (*University of Northern Iowa*), Randall Dahlberg (*Seton Hall University*), Leroy Damewood (*Eastern Oregon State College*), Lynn K. Davis (*University of Cincinnati*), Loyal Farmer (*Cameron University*), Gerald Farrell (*California Polytechnic University, San Luis Obispo*), Bill Finch (*University of Florida*), Gregory D. Foley (*North Harris County College*), Robert Fontenot (*Whitman College*), Juan A. Gatica (*University of Iowa*), Donald Gray (*Iowa Western Community College*), Harvey C. Greenwald (*California Polytechnic University, San Luis Obispo*), Charles Groetsch (*University of Cincinnati*), Edwin Halfar (*University of Nebraska*), Leona Henry (*Mercy College*), Stephen R. Hilding (*Gustavus Adolphus College*), Tim Hodges (*University of Cincinnati*), Dean W. Hooner (*Alfred University*), Brindell Horelick (*University of Maryland, Baltimore County*), Shirley Huffman (*Southwest Missouri State University*), Ronald Infante (*Seton Hall University*), Cassius T. Ionescu Tulcea (*Northwestern University*) Bernice Kastner (*Montgomery College*), Dan Kemp (*South Dakota State University*), John T. Kemper (*College of St. Thomas*), Frank Kost (*State University of New York at Oneonta*), Charles Lanski (*University of Southern California*), David Lehmann (*Southwest Missouri State*), Verlyn Lindell (*Augustana College*), Lowell Lynde (*University of Arkansas*), Danny W. McCarthy (*Tulane University*), Jim McKinney (*California State Polytechnic University, Pomona*), Jerome H. Manheim (*California State University, Long Beach*), Bill Marion (*Valparaiso University*), Frank Mathis (*Baylor University*), John Moriarty (*University of Cincinnati*), Roger H. Moritz (*Alfred University*), Kent Morrison (*California Polytechnic State University, San Luis Obispo*), James M. Nare (*University of Tennessee, Chattanooga*), Michael J. Nowak (*United States International

University, San Diego), Jim Osterburg (*University of Cincinnati*), Nancy Jim Poxon (*California State University, Sacramento*), Lolan Redden (*Cumberland College*), Robert Reisel (*Loyola University*), Wayne Rich (*Utah State University*), Joyce Riseberg (*Montgomery College*), Norlin Rober (*Marshalltown Community College*), Larry P. Runyan (*Shoreline Community College*), John Saccoman (*Seton Hall University*), Helen Salzberg, (*Rhode Island College*), Rollin T. Sandberg (*California State University, Fullerton*), Franklin E. Schroeck (*Florida Atlantic University*), Karen J. Schroeder (*Bentley College*), John Shupert (*Shawnee State Community College*), Kermit Sigmon (*University of Florida*), Madelyn Smith (*Gloucester County College*), James M. Sobota (*University of Wisconsin, La Crosse*), Ken Solem (*Gloucester County College*), Lavinia Spilman (*University of Alabama*), Robert Steward (*Rhode Island College*), Alexander P. Stone (*University of New Mexico*), Keith Stroyan (*University of Iowa*), Virginia Taylor (*University of Lowell*), Terry R. Tiballi (*North Harris County College*), George Van Zwalenberg (*Calvin College*), David S. Watkins (*Utah State University*), Philip M. Whitman (*Rhode Island College*), Robert C. Williams (*Alfred University*), Stephen J. Willson (*Iowa State University*), and Elmar Zemgalis (*Highline Community College*).

Finally, we are grateful to the staff of Harcourt Brace Jovanovich, Inc., for its assistance in the preparation of all three editions of this text: To Marilyn Davis and Judy Burke for their invaluable help in the first two editions, to Richard Wallis for his patience and encouragement as we prepared the Third Edition, to Audrey Thompson and Kim Miller for their careful editing, to Merilyn Britt for her attractive cover design, and to Sharon Weldy for her efforts in maintaining the production schedule for the Third Edition.

Robert Ellis • Denny Gulick

TO THE READER

When you begin to study calculus, you will find that you have encountered many of its concepts and techniques before. Calculus makes extensive use of plane geometry and algebra, two branches of mathematics with which you are already familiar. However, added to these is a third ingredient, which may be new to you: the notion of limit and of limiting processes. From the idea of limit arise the two principal concepts that form the nucleus of calculus; these are the derivative and the integral.

The derivative can be thought of as a rate of change, and this interpretation has many applications. For example, we may use the derivative to find the velocity of an object, such as a rocket, or to determine the maximum and minimum values of a function. In fact, the derivative provides so much information about the behavior of functions that it greatly simplifies graphing them. Because of its broad applicability, the derivative is as important in such disciplines as physics, engineering, economics, and biology as it is in pure mathematics.

The definition of the integral is motivated by the familiar notion of area. Although the methods of plane geometry enable us to calculate the areas of polygons, they do not provide ways of finding the areas of plane regions whose boundaries are curves other than circles. By means of the integral we can find the areas of many such regions. We will also use it to calculate volumes, centers of gravity, lengths of curves, work, and hydrostatic force.

The derivative and the integral have found many diverse uses. The following list, taken from the examples and exercises in this book, illustrates the variety of fields in which these powerful concepts are employed.

	Section
Windpipe pressure during a cough	4.5
Cost of insulating an attic floor	4.5
Blood resistance in vascular branching	4.5
Surface area of a cell in a beehive	4.5
Buffon's needle problem	5.4
Growth of a paramecium population	6.3
Magnitude of an earthquake	6.4
Dating of a lunar rock sample	6.5
Amount of an anesthetic needed during an operation	6.5
Mass of binary stars	7.1
Pareto's Law of distribution of income	7.6
Volume of the great pyramid of Cheops	8.1
Force of water on an earth-filled dam	8.8
Harmonics of a stringed instrument	9.2
Multiplier effect in economics	9.4
Location of the source of a sound	10.3
Rated speed of a banked curve	12.5
Kepler's Laws of planetary motion	12.7
Escape velocity from the earth's gravitational field	12.7
Analysis of a rainbow	13.3
Effect of taxation on production of a commodity	13.3
Law of conservation of energy	15.3
Electric field produced by a charged telephone wire	15.6
Terminal speed of a falling object	16.4
Motion of a spring	16.7

The concepts basic to calculus can be traced, in uncrystallized form, to the time of the ancient Greeks. However, it was only in the sixteenth and early seventeenth centuries that mathematicians developed refined techniques for determining tangents to curves and areas of plane regions. These mathematicians and their ingenious techniques set the stage for Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), who are usually credited with the "invention" of calculus because they codified the techniques of calculus and put them into a general setting; moreover, they recognized the importance of the fact that finding derivatives and finding integrals are inverse processes.

During the next 150 years calculus matured bit by bit, and by the middle of the nineteenth century it had become, mathematically, much as we know it today. Thus the definitions and theorems presented in this book were all known a century ago. What is newer is the great diversity of applications, with which we will try to acquaint you throughout the book.

Robert Ellis • Denny Gulick

CONTENTS

7 TECHNIQUES OF INTEGRATION 371

8 APPLICATIONS OF THE INTEGRAL 426

9 SEQUENCES AND SERIES 489

10 CONIC SECTIONS 572

11 VECTORS, LINES, AND PLANES 609

12 VECTOR-VALUED FUNCTIONS 654

13 PARTIAL DERIVATIVES 714

1

FUNCTIONS

In this chapter we will review the basic properties of real numbers, introduce the concept of function, and discuss different types of functions. If you are already familiar with most of the definitions and concepts given, we suggest that you read Chapter 1 quickly and proceed to Chapter 2.

1.1 THE REAL NUMBERS

Real numbers, their properties, and their relationships are basic to calculus. Therefore we begin with a description of some important properties of real numbers.

Types of Real Numbers and the Real Number Line

The best known real numbers are the **integers**:

$$0, \pm 1, \pm 2, \pm 3, \ldots$$

From the integers we derive the **rational numbers**. These are the real numbers that can be written in the form p/q, where p and q are integers and $q \neq 0$. Thus $\frac{48}{37}$, -17, and -1.41 (which is equal to $-\frac{141}{100}$) are rational numbers. Any real number that is not rational is called an **irrational number**. Examples of irrational numbers are π and $\sqrt{2}$. (See Exercise 84 at the end of this section for a proof that $\sqrt{2}$ is irrational.)

There is an order $<$ on the real numbers. If $a \neq b$, then either $a < b$ or $a > b$. For example, $5 < 7$ and $-1 > -2$. If a is less than or equal to b, we write $a \leq b$. If

a is greater than or equal to b, we write $a \geq b$. For example, $x^2 \geq 0$ for any real number x. We say that a is **positive** if $a > 0$ and **negative** if $a < 0$. If $a \geq 0$, we say that a is **nonnegative**.

The real numbers can be represented as points on a horizontal line in such a way that if $a < b$, then the point on the line corresponding to the number a lies to the left of the point on the line corresponding to the number b (Figure 1.1).

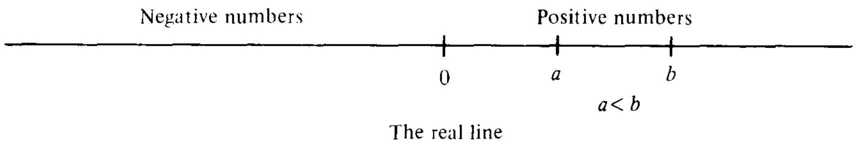

FIGURE 1.1

Such a line is called the **real number line**, or **real line**. We think of the real numbers as points on the real line, and *vice versa*. Thus we say that the negative numbers lie to the left of 0 and the positive numbers lie to the right of 0.

Intervals

Certain sets of real numbers, called **intervals**, appear with great frequency in calculus. They can be grouped into nine categories:

Name	Notation	Description
Open interval	(a, b)	all x such that $a < x < b$
Closed interval	$[a, b]$	all x such that $a \leq x \leq b$
Half-open interval	$(a, b]$	all x such that $a < x \leq b$
Half-open interval	$[a, b)$	all x such that $a \leq x < b$
Open interval	(a, ∞)	all x such that $a < x$
Open interval	$(-\infty, a)$	all x such that $x < a$
Closed interval	$[a, \infty)$	all x such that $a \leq x$
Closed interval	$(-\infty, a]$	all x such that $x \leq a$
The real line	$(-\infty, \infty)$	all real numbers

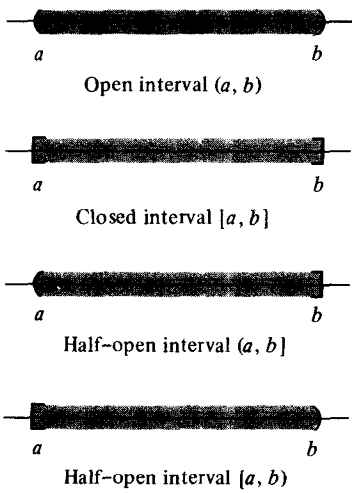

FIGURE 1.2

Intervals of the form (a, b), $[a, b]$, $(a, b]$, and $[a, b)$ are **bounded intervals**, and a and b are the **endpoints** of each of these intervals. Figure 1.2 shows the four types of bounded intervals. Intervals of the form (a, ∞), $(-\infty, a)$, $[a, \infty)$, $(-\infty, a]$, and $(-\infty, \infty)$ are **unbounded intervals**, and a is the **endpoint** of each of the first four of these intervals. A number that is in an interval but is not an endpoint of the interval is called an **interior point** of the interval.

Caution: The symbols ∞ and $-\infty$ used above are called "infinity" and "minus infinity," respectively. They do not represent numbers.

Notice that (b, b), $(b, b]$, and $[b, b)$ contain no numbers. More generally, (a, b), $(a, b]$, and $[a, b)$ contain no numbers if $b \leq a$. Whenever we write (a, b), $(a, b]$, or

$[a, b)$, we make the implicit assumption that $a < b$. Likewise, we write $[a, b]$ only when $a \leq b$.

Inequalities and Their Properties

Statements such as $a < b$, $a \leq b$, $a > b$, and $a \geq b$ are called **inequalities**. We list several basic laws for inequalities. In what follows a, b, c, and d are assumed to be real numbers.

Trichotomy: Either $a < b$, or $a > b$, or $a = b$, and only one of these holds for any given a and b. (1)

Transitivity: If $a < b$ and $b < c$, then $a < c$. (2)

Additivity: If $a < b$ and $c < d$, then $a + c < b + d$. (3)

Positive multiplicativity: If $a < b$ and $c > 0$, then $ac < bc$. (4)

Negative multiplicativity: If $a < b$ and $c < 0$, then $ac > bc$. (5)

Replacing $<$ by $\leq$ and $>$ by $\geq$ in laws (2)–(5) yields four new laws for inequalities, which we will also find useful.

The word "trichotomy" in (1) means a threefold division. The trichotomy law states that any two numbers a and b are related in exactly one of the three ways listed in (1). For example, given the two numbers 3.1416 and π, we have either $3.1416 < \pi$, $3.1416 = \pi$, or $3.1416 > \pi$. (The last is actually correct.)

For simplicity of notation, two inequalities are sometimes combined. For example, if $a \leq b$ and $b \leq c$, then we can write $a \leq b \leq c$.

Caution: The multiplication laws, (4) and (5), must be carefully observed. To illustrate their use, we will present several examples. In each, the problem is to "solve an inequality," which means to find all real numbers that satisfy the inequality.

Example 1 Solve the inequality $1/x < 3$.

Solution First we observe that 0 cannot be a solution because division by 0 is impossible. Next we multiply through by x to eliminate x from the denominator. For positive x, (4) yields $1 < 3x$ or $\frac{1}{3} < x$. Thus the numbers in $(\frac{1}{3}, \infty)$ constitute one part of the solution of the given inequality. For negative x, (5) yields $1 > 3x$, or $\frac{1}{3} > x$. Since the last inequality is satisfied by all $x < 0$, a second part of the solution consists of all x in $(-\infty, 0)$. Therefore the complete solution consists of all numbers in the interval $(-\infty, 0)$ and all numbers in the interval $(\frac{1}{3}, \infty)$. □

When the solution of an inequality forms only one interval, we will write only that interval as the solution. Moreover, if the solution of an inequality consists of more than one interval, we will refer to the solution as the **union** of these intervals. Thus the solution of the inequality $1/x < 3$ in Example 1 consists of the union of the intervals $(-\infty, 0)$ and $(\frac{1}{3}, \infty)$.

Example 2 Solve the inequality $-1 < -2x + 3 \leq 2$.

Solution The given inequality is equivalent to the following pair of inequalities:

$$-1 < -2x + 3 \quad \text{and} \quad -2x + 3 \leq 2$$

First subtracting 3 throughout, we find that

$$-4 < -2x \quad \text{and} \quad -2x \leq -1$$

Then dividing throughout by -2 and reversing the inequality signs, we obtain

$$2 > x \quad \text{and} \quad x \geq \frac{1}{2}$$

Thus the solution consists of all numbers x satisfying $\frac{1}{2} \leq x < 2$, that is, the interval $[\frac{1}{2}, 2)$. □

Since we performed the same algebraic manipulation on both inequalities $-1 < -2x + 3$ and $-2x + 3 \leq 2$, we could have solved the original inequality without splitting it up:

$$\begin{aligned} -1 < -2x + 3 &\leq 2 \\ -4 < -2x &\leq -1 \\ 2 > x &\geq \tfrac{1}{2} \end{aligned}$$

In solving most inequalities, we will need to find values of x for which a certain expression in x is positive (or negative). We will need to be careful to observe the negative multiplicativity rule (5) when we multiply negative numbers.

Example 3 Solve the inequality

$$\frac{(x-1)(x-3)}{x+2} > 0$$

Solution First we draw a diagram that shows the signs of the factors $x - 1$ and $x - 3$ of the numerator and $x + 2$ of the denominator.

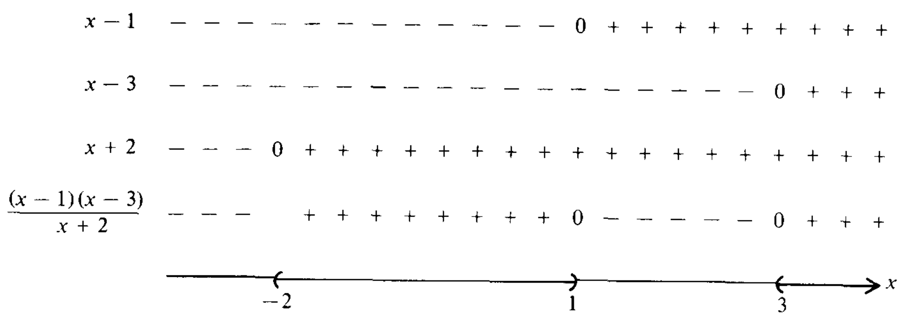

FIGURE 1.3

Then we deduce the sign of $(x-1)(x-3)/(x+2)$ for various values of x, and determine where it is positive. From Figure 1.3 we see that the solution of the given inequality is the union of the intervals $(-2, 1)$ and $(3, \infty)$. □

If we had wished to solve the inequality

$$\frac{(x-1)(x-3)}{x+2} \leq 0$$

we would have used the same diagram, but at the end we would have selected the union of those intervals on which $(x-1)(x-3)/(x+2)$ is nonpositive, namely $(-\infty, -2)$ and $[1, 3]$.

In Section 2.6 we will discuss a second method of solving inequalities that uses results from calculus.

Absolute Value

The **distance** between a and b on the real line is either $a-b$ or $b-a$, whichever is nonnegative (Figure 1.4). Likewise, the distance between 0 and b is either $b-0=b$ or $0-b=-b$, whichever is nonnegative. The distance between a point on the real line and 0 is the basis for the definition of the absolute value of a number.

FIGURE 1.4

DEFINITION 1.1

The **absolute value** of any real number b is b if $b \geq 0$ and is $-b$ if $b < 0$. The absolute value of b is denoted $|b|$.

For example, $|6| = 6$, $|0| = 0$, $|-5| = -(-5) = 5$, and $|8-17| = |-9| = -(-9) = 9$. Notice that $|b|$ is the larger of b and $-b$, whichever is nonnegative. Geometrically, $|b|$ is the distance between 0 and b. More generally, $|a-b|$ is the distance between the numbers a and b.

We will use the following properties of absolute value in the remainder of this book:

$$|-a| = |a|, \quad \text{and} \quad |a-b| = |b-a| \tag{6}$$

$$|ab| = |a||b|, \quad \text{and} \quad |b^2| = |b|^2 \tag{7}$$

$$-|b| \leq b \leq |b| \tag{8}$$

$$|a+b| \leq |a| + |b| \tag{9}$$

$$|a-b| \geq ||a| - |b|| \tag{10}$$

Except for (9) and (10), these properties follow directly from Definition 1.1. We will verify (9) and leave (10) as an exercise. To verify (9), we first use (8):

$$-a \leq |a| \quad \text{and} \quad a \leq |a|$$

$$-b \leq |b| \quad \text{and} \quad b \leq |b|$$

Adding these inequalities vertically yields

$$-(a + b) = -a - b \leq |a| + |b| \quad \text{and} \quad a + b \leq |a| + |b|$$

Since $|a + b|$ is the larger of $a + b$ and $-(a + b)$, it follows that

$$|a + b| \leq |a| + |b|$$

Next we show that if $b > 0$, then

$$|x| < b \quad \text{if and only if} \quad -b < x < b \tag{11}$$

To verify (11) we notice that $|x| < b$ means that

$$\text{if } x \geq 0, \quad \text{then} \quad x < b$$

and

$$\text{if } x < 0, \quad \text{then} \quad -x < b, \text{ or equivalently, } -b < x$$

From (11) we see that the solution of the inequality $|x| < b$ is the open interval $(-b, b)$. Statements analogous to (11) pertain to inequalities of the form $|x| \leq b$, $|x| > b$, and $|x| \geq b$.

Example 4 Solve the inequality $|x - 1| < 3$.

Solution By (11) the given inequality is equivalent to

$$-3 < x - 1 < 3$$

or equivalently, $-2 < x < 4$. Thus the solution is $(-2, 4)$. □

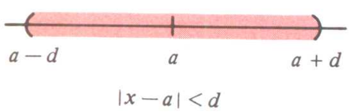

FIGURE 1.5

Geometrically, $|x - 1| < 3$ means that the distance between x and 1 is less than 3. More generally, $|x - a| < d$ means the distance between x and a is less than d. Thus $|x - a| < d$ if and only if x lies in the interval $(a - d, a + d)$ (Figure 1.5).

Example 5 Find all values of x such that $|x - 7| \geq 9$.

Solution Here we wish to find the values of x whose distance from 7 is no less than 9. Thus the required values of x are those in the union of the intervals $(-\infty, -2]$ and $[16, \infty)$. □

Example 6 Find all values of x such that $0 < |x - a| < d$, where a is any number and d is any positive number.

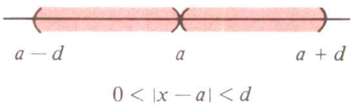

FIGURE 1.6

Solution The double inequality $0 < |x-a| < d$ means that

$$0 < |x-a| \quad \text{and} \quad |x-a| < d$$

From $0 < |x-a|$ we know that $x \neq a$. By our comments above, the values of x satisfying $|x-a| < d$ lie in the interval $(a-d, a+d)$. These two observations give the complete solution, which is the union of the intervals $(a-d, a)$ and $(a, a+d)$ (Figure 1.6). □

This concludes our discussion of real numbers, inequalities, and absolute values. The concepts and rules we have given will play an important part in our study of calculus.

EXERCISES 1.1

In Exercises 1–4 determine whether $a < b$ or $a > b$.

1. $a = \frac{4}{9}, b = \frac{7}{16}$
2. $a = -\frac{1}{7}, b = -0.142857$
3. $a = \pi^2, b = 9.8$
4. $a = (3.2)^2, b = 10$
5. Use the fact that $(\sqrt{2})^2 = 2$ to determine whether $\sqrt{2} < 1.41$, $\sqrt{2} = 1.41$, or $\sqrt{2} > 1.41$.
6. Use the fact that $(\sqrt{11})^2 = 11$ to determine whether $\sqrt{11} < 3.3$, $\sqrt{11} = 3.3$, or $\sqrt{11} > 3.3$.

In Exercises 7–14 state whether the interval is open, half-open, or closed and whether it is bounded or unbounded. Then sketch the interval on the real line.

7. $[-4, 5]$
8. $(-2, -1)$
9. $(-\infty, 3)$
10. $[\frac{3}{2}, \frac{5}{2})$
11. $[0, \infty)$
12. $(5, 7)$
13. $(-\infty, -1]$
14. $[-\frac{1}{2}, \frac{1}{2}]$

In Exercises 15–18 write the union of the two intervals as a single interval.

15. $(-3, 2)$ and $[1, 4)$
16. $(-\infty, 0]$ and $[0, 3)$
17. $(1, 3)$ and $(2, \infty)$
18. $(-\infty, \frac{1}{2}]$ and $(0, \infty)$

In Exercises 19–38 solve the inequality.

19. $-6x - 2 > 5$
20. $4 - 3x \geq 7$
21. $-1 \leq 2x - 3 < 4$
22. $-0.1 < 3x + 4 < 0.1$
23. $(x-1)(x+\frac{1}{2}) \geq 0$
24. $(x-1)(x-2)(x-3) \leq 0$
25. $x(x-\frac{2}{3})(x+\frac{1}{3}) < 0$
26. $\dfrac{x}{(x-1)(x+2)} > 0$
27. $\dfrac{(2x-1)^2}{(x+1)(x+3)} \geq 0$
28. $\dfrac{(2x-3)(4x+1)}{x-2} \leq 0$
29. $4x^3 - 6x^2 \leq 0$
30. $3x^2 - 2x - 1 \geq 0$
31. $8x - \dfrac{1}{x^2} > 0$
32. $8x + \dfrac{1}{x^2} < 0$
33. $\dfrac{4x(x^2-6)}{x^2-4} < 0$
34. $\dfrac{2x(x^2-3)}{(x^2+1)^3} \geq 0$
35. $\dfrac{t^2+t-2}{(t^2-1)^3} \geq 0$
36. $\dfrac{t^2-2t-3}{t^2-8t+15} > 0$
37. $\dfrac{2-x}{\sqrt{9-6x}} > 0$
38. $\dfrac{2x^2-1}{(1-x^2)^{1/2}} < 0$

In Exercises 39–42 solve the inequality.

39. $\dfrac{1}{x+1} > \dfrac{3}{2}$ (*Hint:* Write the inequality as $1/(x+1) - 3/2 > 0$, and then write the left side as a single fraction.)
40. $\dfrac{1}{3-x} < -2$
41. $\dfrac{x+1}{x-1} \leq \dfrac{1}{2}$
42. $\dfrac{2-5x}{3-4x} \geq -2$

In Exercises 43–46 evaluate the expression.

43. $-|-3|$
44. $|-\sqrt{2}|^2$
45. $|-5| + |5|$
46. $|-5| - |5|$

In Exercises 47–58 solve the equation.

47. $|x| = 1$
48. $|x| = \pi$
49. $|x-1| = 2$
50. $|2x - \frac{1}{2}| = \frac{1}{2}$
51. $|6x+5| = 0$
52. $|3-4x| = 2$
53. $|x| = |x|^2$
54. $|x| = |1-x|$
55. $|x+1|^2 + 3|x+1| - 4 = 0$
56. $|x-2|^2 - |x-2| = 6$
57. $|x+4| = |x-4|$
58. $|x-1| = |2x+1|$

In Exercises 59–70 solve the inequality.

59. $|x-2| < 1$
60. $|x-4| < 0.1$
61. $|x+1| < 0.01$
62. $|x+\frac{1}{2}| \leq 2$