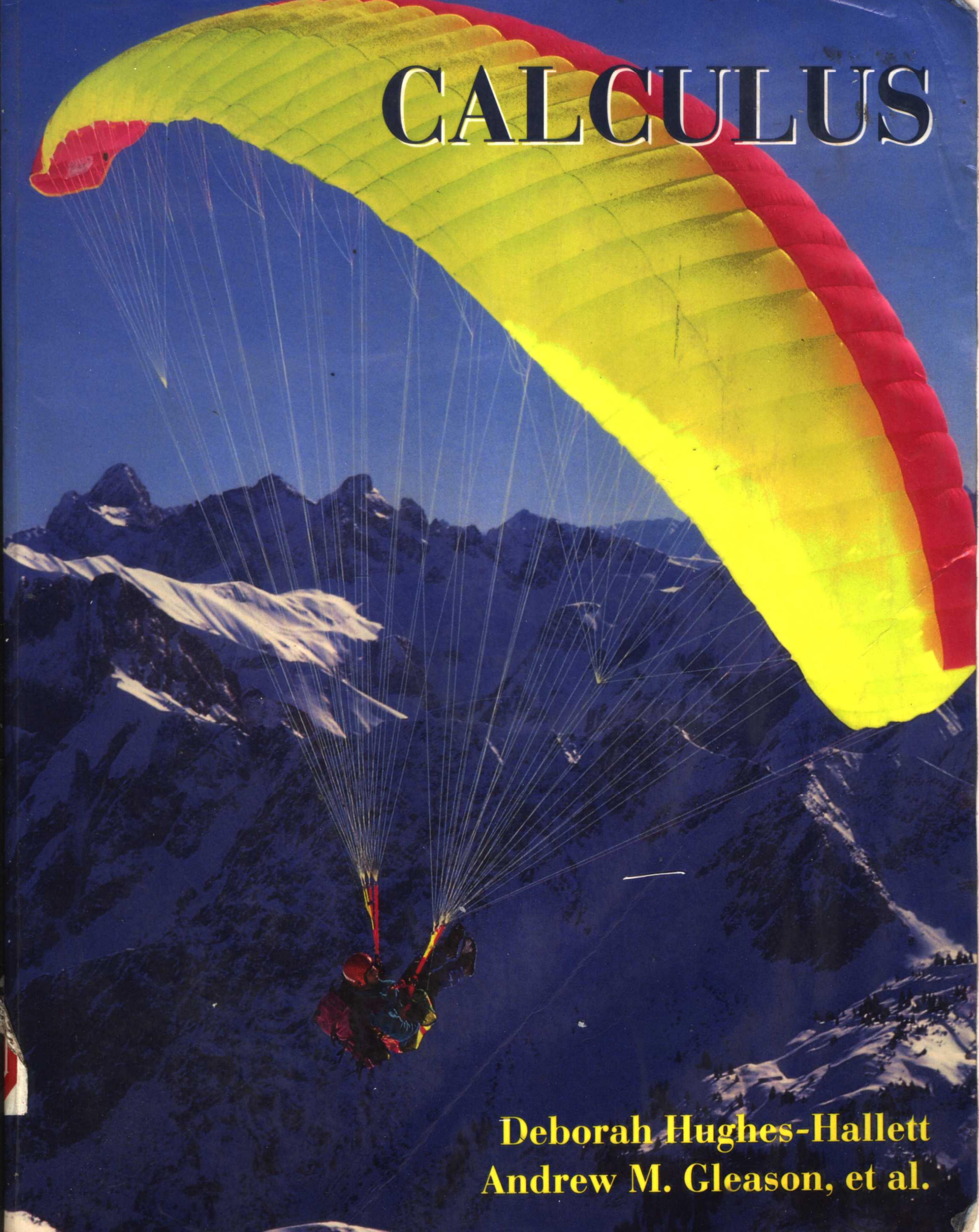
CALCULUS



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Dedicated to Alex, Alice, Eric, and Alex for their resourcefulness, creativity, and endless good humor.

PREFACE

Calculus is one of the greatest achievements of the human intellect. Inspired by problems in astronomy, Newton and Leibniz developed the ideas of calculus 300 years ago. Since then, each century has demonstrated the power of calculus to illuminate questions in mathematics, the physical sciences, engineering, and the social and biological sciences.

Calculus has been so successful because of its extraordinary power to reduce complicated problems to simple rules and procedures. Therein lies the danger in teaching calculus: it is possible to teach the subject as nothing but the rules and procedures – thereby losing sight of both the mathematics and of its practical value. With the generous support of the National Science Foundation, our consortium set out to create a new calculus curriculum that would restore that insight. This book is part of that endeavor.

Basic Principles

Two principles guided our efforts. The first is our prescription for restoring the mathematical content to calculus:

> **The Rule of Three:** *Every topic should be presented geometrically, numerically, and algebraically.*

We continually encourage students to think about the geometrical and numerical meaning of what they are doing. It is not our intention to undermine the purely algebraic aspect of calculus, but rather to reinforce it by giving meaning to the symbols. In the homework problems dealing with applications, we continually ask students to explain verbally what their answers mean in practical terms.

The second principle, inspired by Archimedes, is our prescription for restoring practical understanding:

> **The Way of Archimedes:** *Formal definitions and procedures evolve from the investigation of practical problems.*

Archimedes believed that insight into mathematical problems is gained by first considering them from a mechanical or physical point of view.[1] For the same reason, our text is problem driven. Whenever possible, we start with a practical problem and derive the general results from it. By practical problems we usually, but not always, mean real world applications. These two principles have led to a dramatically new curriculum – more so than a cursory glance at the table of contents might indicate.

[1]. . . I thought fit to write out for you and explain in detail . . . the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. From *The Method*, in *The Works of Archimedes* edited and translated by Sir Thomas L. Heath (Dover, NY)

Technology

We take advantage of computers and graphing calculators to help students learn to think mathematically. For example, using a graphing calculator to zoom in on functions is one of the best ways of seeing local linearity. Furthermore, the ability to use technology effectively as a tool is itself of the greatest importance. Students are expected to use their own judgement to determine where technology is useful.

However, the book does not require any specific software or technology. Test sites have used the materials with graphing calculators, graphing software, and computer algebra systems. Any technology with the ability to graph functions and perform numerical integration will suffice.

What Student Background is Expected?

We have found this curriculum to be thought-provoking for well-prepared students while still accessible to students with weak algebra backgrounds. Providing numerical and graphical approaches as well as the algebraic gives students several ways of mastering the material. This approach encourages students to persist, thereby lowering failure rates.

Content

When we designed this curriculum we started with a clean slate. We included some new topics, such as differential equations, and omitted some traditional topics whose inclusion we could not justify after discussions with mathematicians, engineers, physicists, chemists, biologists, and economists. In the process, we also changed the focus of certain topics. In order to meet individual needs or course requirements, topics can easily be added or deleted, or the order changed.

Chapter 1: A Library of Functions

Chapter 1 introduces all the elementary functions to be used in the book. Although the functions are probably familiar, the graphical, numerical, and modeling approach to them is fresh. Our purpose is to acquaint the student with each function's individuality: the shape of its graph, characteristic properties, comparative growth rates, and general uses. We expect to give the student the skill to read graphs and think graphically, to read tables and think numerically, and to apply these skills, along with their algebraic skills, to modeling the real world. We introduce exponential functions at the earliest possible stage, since they are fundamental to the understanding of real-world processes. Further attention is given to constructing new functions from old ones—how to shift, flip and stretch the graph of any basic function into a new, related function.

We encourage you to cover this chapter thoroughly, as the time spent on it will pay off when you get to the calculus.

Chapter 2: The Derivative

Chapter 2 presents the key concept of the derivative according to the Rule of Three. The purpose of this chapter is to give the student a practical understanding of the limit definition of the derivative and its interpretation as an instantaneous rate of change without complicating the discussion with differentiation rules. After finishing this chapter, a student will be able to find derivatives numerically (by taking arbitrarily fine difference quotients), visualize derivatives graphically as the slope of the graph, and interpret the meaning of first and second derivatives in various applications. The student will also understand local linearity and recognize the derivative as a function in its own right.

Chapter 3: The Definite Integral

Chapter 3 presents the key concept of the definite integral, along the same lines as Chapter 2. Some instructors using preliminary versions of the book have delayed covering Chapter 3 until after Chapter 5 without any difficulty.

The purpose of this chapter is to give the student a practical understanding of the definite integral as a limit of Riemann sums, and to bring out the connection between the derivative and the definite integral in the Fundamental Theorem of Calculus. We use the same method as in Chapter 2, introducing the fundamental concept in depth without going into technique. The motivating problem is computing the total distance traveled from the velocity function. The student will finish the chapter with a good grasp of the definite integral as a limit of Riemann sums, with the ability to compute it numerically, and with an understanding of how to interpret the definite integral in various contexts.

Chapter 4: Short-Cuts to Differentiation

Chapter 4 presents the symbolic approach to differentiation. The title is intended to remind the student that the basic methods of differentiation are not to be regarded as the definition of the derivative. The derivatives of all the functions in Chapter 1 are introduced as well as the rules for differentiating the combinations discussed in Chapter 1. Implicit differentiation is introduced and used to find derivatives of several basic functions. We give informal but mathematically sound justifications, introducing graphical and numerical reasoning where appropriate. The student will finish this chapter with basic proficiency in differentiation and an understanding of why the various rules are true.

Chapter 5: Using the Derivative

Chapter 5 presents applications of the derivative. It includes an investigation of parametrized families of functions according to the Way of Archimedes, using the graphing technology to observe basic properties and calculus to confirm them.

Our aim in this chapter is to enable the student to use the derivative in solving problems, rather than to learn a catalogue of application templates. It is not meant to be comprehensive, and you do not need to cover all the sections. The student should finish this chapter with the experience of having successfully tackled a few problems that required sustained thought over more than one session.

Chapter 6: Reconstructing a Function from Its Derivative

Chapter 6 focuses on "going backward" from a derivative to the original function, first graphically and numerically, then analytically. The chapter starts with the properties of the definite integral and its interpretation as an area (Sections 6.1 and 6.2) and ends with an analysis of motion under the influence of gravity. After finishing this chapter, students will understand how to "go backwards" from the derivative to the original function.

Chapter 7: The Integral

Chapter 7 investigates methods of finding integrals. We do not restrict our attention to functions that have closed-form antiderivatives. Instead, we emphasize the role of numerical integration as a basic tool. This chapter includes several techniques of integration; others are included in the table of integrals. While we do not specifically make use of computer algebra software, we certainly acknowledge that its existence changes the skills that students need to master.

This chapter threads practical skills with theoretical understanding. There are two groups of sections on computing definite integrals: Sections 7.1–7.5 on using the Fundamental Theorem, and Sections 7.6–7.7 on using numerical methods. Sections 7.8–7.9 are on improper integrals. The student will finish this chapter with proficiency in the basic methods of integration.

Chapter 8: Using the Definite Integral

Chapter 8 addresses applications of the definite integral. We emphasize the idea of subdividing a quantity to produce Riemann sums which, in the limit, yield a definite integral, and aim to show ways the integral is used without resorting to templates. The chapter starts with a discussion of how to set up definite integrals that represent given physical quantities, and then gives examples from geometry, physics, economics, and probability. You do not need to cover all the sections. The student will finish this chapter understanding how to form Riemann sums and knowing how they are used.

Chapter 9: Differential Equations

Chapter 9 introduces differential equations without too many technicalities. It is intended to show the power of the methods we have developed, using more realistic and complex applications. Slope fields are used to visualize the behavior of solutions of first-order differential equations. The emphasis is on qualitative solutions, modeling, and interpretation. We include applications to population models (exponential and logistic), the spread of disease, predator-prey equations, and competitive exclusion. There is also some material on second-order differential equations with constant coefficients: the spring equation, both damped and undamped, and solutions using complex numbers. The student will finish this chapter knowing what a differential equation is, how to approximate its solution graphically and numerically, and how to find some analytic solutions, all in the context of substantial applications.

Chapter 10: Approximations

Chapter 10 is an introduction to Taylor Series and Fourier series via the idea of approximating functions with simpler functions; the Taylor series is a local approximation, the Fourier series a global one. The primary focus is on Taylor polynomials and series. Geometric series and their applications are discussed. The notion of a convergent series is permitted to evolve naturally out of the investigation of Taylor polynomials. The graphical and numerical points of view are kept at the forefront throughout. The student will finish this chapter with a good grasp of Taylor approximations and understand how they differ from Fourier approximations.

Appendices

The appendices contain material on roots and accuracy, on continuity and bounds, on polar coordinates, and on complex numbers.

Our Experiences

In the process of developing the ideas incorporated in this book, we have been conscious of the need to test the materials thoroughly in a wide variety of institutions serving many different types of students. Consortium members have used previous versions of the book for several years at large and small liberal arts colleges, at large and small public universities, at a two-year institution, and at a high school. During the 1991-92 and 92-93 academic years, we were assisted by colleagues at over one hundred schools around the country who class-tested the book and reported their experiences and those of their students. This diverse group of schools used the book in semester and quarter systems, in large lecture sections and small classes, in computer labs, small groups, and traditional settings, and with a number of different technologies. We appreciate the valuable suggestions they made, which we have tried to incorporate into this first edition of the text.

Changes from the Preliminary Edition

Chapter 1. The number e is introduced later, at the same time as the natural logarithm. Compound interest has its own section, and the section on Roots, Continuity, and Accuracy has been moved to Appendix A and Appendix B.

Chapter 2. The dy/dx notation for the derivative is delayed until Section 2.4 where it is used to help students interpret the derivative in practical terms. Slightly more emphasis is put on limits in Section 2.1.

Chapter 3. The interpretation of the definite integral of a rate as total change has been moved to Section 3.4, where it is used to help students appreciate the Fundamental Theorem of Calculus.

Chapter 5. The material on antidifferentiation has been moved to the new Chapter 6: Reconstructing a Function from Its Derivative.

Chapter 6. This new chapter pulls together material on "going backward" from derivative to original function that was previously distributed through several chapters. It can be used to finish a study of Chapters 1–5 or as a starting point for Chapters 6–10.

Chapter 7. Introductory material has been moved to the new Chapter 6. A subsection on integration using partial fractions has been added to Section 7.5.

Chapter 8. The material on interpreting the definite integral as an area is now in Section 6.1. The section on probability and distributions has been split into two sections.

Chapter 9. This chapter has been rewritten to include more emphasis on applications. Section 9.7 is based on the growth of the US population since 1790. Section 9.8 contains the original material on two interacting populations as well as new material on the spread of a disease. Section 9.9 is entirely new and shows students how to investigate the phase plane using nullclines. The section on complex numbers and polar coordinates has been moved to Appendix C and Appendix D.

Chapter 10. The section on geometric series has been added. The section on estimating the error in a Taylor approximation has been rewritten, and now includes the Mean Value Theorem.

Appendices. The material in the appendices has been moved here for flexibility in scheduling.

Supplementary Materials

- **Instructor's Manual with Sample Exams** containing teaching tips, calculator programs, some overhead transparency masters and sample exams.
- **Instructor's Solution Manual** with complete solutions to all problems.
- **Answer Manual** with brief answers to all odd-numbered problems.
- **Student's Solution Manual** with complete solutions to half the odd-numbered problems.
- **Calculus Project Book** containing projects for students, and their solutions.

- **Orientation Video**, an orientation to teaching with the materials.
- **Workshop Video** to guide instructors conducting workshops on the materials in the textbook.
- **University of Arizona Software Manual** ties the University of Arizona software to specific problems in the text and includes data sets for working some problems.
- **Discovering Calculus with Derive**, a problems manual with brief instructions on the use of the software as well as additional problems which correspond to the text.

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To Students: How to Learn from this Book

- This book may be different from other math textbooks that you have used, so it may be helpful to know about some of the differences in advance. At every stage, this book emphasizes the *meaning* (in practical, graphical or numerical terms) of the symbols you are using. There is much less emphasis on "plug-and-chug" and using formulas, and much more emphasis on the interpretation of these formulas than you may expect. You will often be asked to explain your ideas in words or to explain an answer using graphs.
- The book contains the main ideas of calculus in plain English. Success in using this book will depend on reading, questioning, and thinking hard about the ideas presented. It will be helpful to read the text in detail, not just the worked examples.
- There are few examples in the text that are exactly like the homework problems, so homework problems can't be done by searching for similar–looking "worked out" examples. Success with the homework will come by grappling with the ideas of calculus.
- Many of the problems in the book are open-ended. This means that there is more than one correct approach and more than one correct solution. Sometimes, solving a problem relies on common sense ideas that are not stated in the problem explicitly but which you know from everyday life.
- This book assumes that you have access to a calculator or computer that can graph functions, find (approximate) roots of equations, and compute integrals numerically. There are many situations where you may not be able to find an exact solution to a problem, but can use a calculator or computer to get a reasonable approximation. An answer obtained this way is usually just as useful as an exact one. However, the problem does not always state that a calculator is required, so use your own judgement.

  If you mistrust technology, listen to this student, who started out the same way:

  > Using computers is strange, but surprisingly beneficial, and in my opinion is what leads to success in this class. I have difficulty visualizing graphs in my head, and this has always led to my downfall in calculus. With the assistance of the computers, that stress was no longer a factor, and I was able to concentrate on the concepts behind the shapes of the graphs, and since these became gradually more clear, I got increasingly better at picturing what the graphs should look like. It's the old story of not being able to get a job without previous experience, but not being able to get experience without a job. Relying on the computer to help me avoid graphing, I was tricked into focusing on what the graphs meant instead of how to make them look right, and what graphs symbolize is the fundamental basis of this class. By being able to see what I was trying to describe and learn from, I could understand a lot more about the concepts, because I could change the conditions and see the results. For the first time, I was able to see how everything works together . . . .

  That was a student at the University of Arizona who took calculus in Fall 1990, the first time we used the text. She was terrified of calculus, got a C on her first test, but finished with an A for the course.
- This book attempts to give equal weight to three methods for describing functions: graphical (a picture), numerical (a table of values) and algebraic (a formula). Sometimes it's easier to translate a problem given in one form into another. For example, you might replace the graph of a parabola with its equation, or plot a table of values to see its behavior. It is important to be flexible about your approach: if one way of looking at a problem doesn't work, try another.

- Students using this book have found discussing these problems in small groups helpful. There are a great many problems which are not cut-and-dried; it can help to attack them with the other perspectives your colleagues can provide. If group work is not feasible, see if your instructor can organize a discussion session in which additional problems can be worked on.
- You are probably wondering what you'll get from the book. The answer is, if you put in a solid effort, you will get a real understanding of one of the most important accomplishments of the millennium – calculus – as well as a real sense of how mathematics is used in the age of technology.

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CHAPTER ONE

A LIBRARY OF FUNCTIONS

Functions are truly fundamental to mathematics. For example, in everyday language we say, "The price of a ticket is a function of where you sit," or "The speed of a rocket is a function of its payload." In each case, the word *function* expresses the idea that knowledge of one fact tells us another. In mathematics, the most important functions are those in which knowledge of one number tells us another number. If we know the length of the side of a square, its area is determined. If the circumference of a circle is known, its radius is determined.

Calculus starts with the study of functions. This chapter will lay the foundation for calculus by surveying the behavior of the most common functions, including powers, exponentials, logarithms, and the trigonometric functions. Besides the behavior of these functions, we will explore ways of handling the graphs, tables, and formulas that represent them.