FIFTH EDITION

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William E. Boyce
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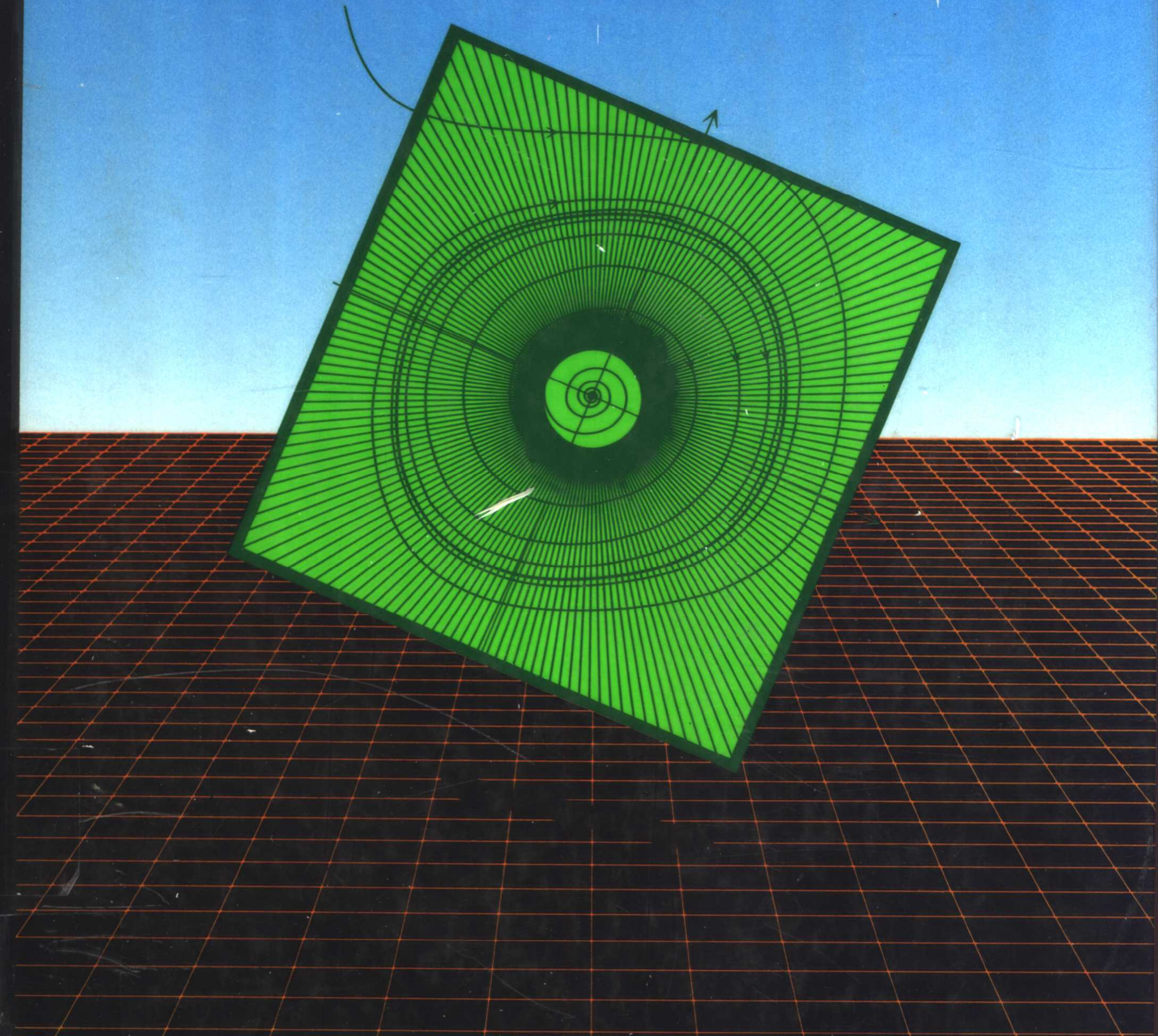

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Elementary
Differential Equations
and Boundary Value Problems

In loving memory of our parents
Ethel and Clyde DiPrima
Marie and Edward Boyce

Preface

A course in elementary differential equations is an excellent vehicle for displaying the interrelations between mathematics and the physical sciences or engineering. Before the engineer or scientist can use differential equations with confidence, he or she must master the techniques of solution and have at least a rudimentary knowledge of the underlying theory. Conversely, the student of mathematics benefits greatly from a knowledge of some of the ways in which the desire to solve specific problems has stimulated work of a more abstract nature.

We wrote this book from the intermediate viewpoint of the applied mathematician, whose interest in differential equations may be both highly theoretical and intensely practical. We have sought to combine a sound and accurate (but not abstract) exposition of the elementary theory of differential equations with considerable material on methods of solution, analysis, and approximation that have proved useful in a wide variety of applications. We have given principal attention to those methods that are capable of broad application and that can be extended to problems beyond the range of our book. We emphasize that these methods have a systematic and orderly structure and are not merely a miscellaneous collection of mathematical tricks. The methods discussed here include not only elementary analytical techniques that lead to exact solutions of certain classes of problems, but also include approximations based on numerical algorithms or series expansions, as well as qualitative or geometrical methods that often lead to a better understanding of the global behavior of solutions. These latter methods are becoming much more accessible to students as a result of their routine use of personal computers or powerful pocket calculators.

Indeed, with the widespread availability of enormous computing power, including versatile symbolic computation packages, it is reasonable to inquire whether elementary analytical methods of solving differential equations remain a worthwhile object of study. We believe that they do, for at least two reasons. First, solving a difficult problem in differential equations often requires the use of a variety of tools, both analytical and numerical. The implementation of an efficient numerical procedure typically rests on a good deal of preliminary analysis — to determine the qualitative features of the solution as a guide to computation, to investigate limiting or special cases, or to discover which ranges of the variables or parameters may require or merit special attention. Second, gaining an understanding of a complex natural process is often accomplished by combining or building upon simpler and more basic models. The latter are often described by differential equations of an elementary type. Thus a thorough knowledge of these equations, their solutions, and the models they represent is the first and indispensable step toward the solution of more complex problems.

One goal of this revision is to encourage students and instructors to exploit the computing power that they now have to enable students to achieve a deeper understanding of differential equations and a keener appreciation of how they can be used to study important problems in the natural sciences or engineering. This edition includes many new computer-generated graphs that help to make clear the qualitative behavior of the often complicated formulas that analytical solution methods produce. At the same time, there is more discussion in the text of the geometrical or asymptotic properties of solutions. There are approximately 275 new problems, many of which call for the student to execute some numerical calculation, to plot (with the help of a suitable graphics package) the graph of a solution, and often to draw appropriate conclusions from such actions. Finally, there are two new sections: one on first order difference equations with an emphasis on the logistic equation, and one on the Lorenz equations. These sections introduce some of the basic ideas associated with bifurcations, chaos, and strange attractors. Besides being fascinating in its own right, this material can be used to demolish the notion that mathematics is a completed subject, rather than one that is constantly growing and regenerating itself.

We have written this book primarily for students who have a knowledge of calculus gained from a normal two- or three-semester course; most of the book is well within the capabilities of such students. In Chapter 7, which deals with systems of first order linear equations, and to some extent also in Chapter 9, experience with matrix algebra will also be helpful. For reference purposes the necessary information about matrices is summarized in two sections at the beginning of Chapter 7. Sections marked with an asterisk probably require greater mathematical sophistication (although, strictly speaking, no more knowledge) than the rest of the book. Some problems are also marked with asterisks, indicating that they are more difficult than most, in some cases going somewhat beyond the material presented in the text itself.

We believe that this book has more than average flexibility for classroom use. Beginning with Chapter 4, the chapters are substantially independent of each other, although Chapter 11 logically follows Chapter 10, and Chapter 9 contains references to Chapter 7. Thus, after the necessary parts of the first three chapters are completed (roughly Sections 1.1, 2.1 through 2.4, and 3.1 through 3.7), the selection of additional topics, and the order and depth in which they are covered, is at the discretion of the instructor. For example, while there is a good deal of material on applications of various kinds, especially in Chapters 2, 3, 9, and 10, most of this material appears in separate sections, so that an instructor can easily choose which applications to include and which to omit. Another possibility is to combine the coverage of second order and of higher order linear equations by going through Chapters 3 and 4 concurrently. Still another possibility is to begin to present the material on numerical methods in Chapter 8 immediately after, or even together with, the material in Chapter 2 on first order initial value problems. Finally, although this revision assumes that students have computers or calculators available, an instructor who wishes not to emphasize this aspect of the subject can do so by exercising a little extra care in the choice of assigned problems.

Each section of the text is followed by a set of problems for the student. These range from the routine to the challenging; some of the latter explore aspects of the theory, or introduce areas of application, not covered in the main text. As mentioned earlier, other problems call for a computer-aided investigation of a differential equation, using numerical or graphical techniques. Answers to nearly all problems are given at the end of the book. There is also a Solutions Manual, compiled by Charles W. Haines of Rochester Institute of Technology, that contains detailed solutions of many problems.

Sections of the book are numbered in decimal form, and theorems and figures are numbered consecutively within each section. Thus Theorem 3.2.4 is the fourth theorem in Section 3.2. General references are given at the end of each chapter, and more specific ones appear occasionally as footnotes.

The scope of the book can be judged from the table of contents, and readers familiar with the previous edition will find that this one follows the same general pattern. Nevertheless, this revision contains many minor changes and the following more significant ones, some of which have already been noted:

1. Corresponding to a trend in the growth of the subject as well as to the development of user-friendly computer graphics packages, this edition includes greater emphasis on the geometrical properties of differential equations and their solutions. Compared to preceding editions, there are more graphs, more discussion of geometrical properties and methods, and more problems that call for the student to construct graphs or to draw conclusions from them.

2. There is more emphasis in the text on, and there are also more problems that call for, conclusions to be drawn from a solution, rather than merely asking for the derivation of the solution itself. This reflects the fact that often the motivation to solve a particular differential equation is the need to understand some natural process or phenomenon that the equation describes.

3. New sections on first order difference equations and on the Lorenz equations have been added, introducing the concepts of bifurcations, chaos, and strange attractors.

4. The basic material on second order linear equations in Chapter 3 has been rewritten to make the presentation more direct and, in particular, to discuss the solution of some simple problems before taking up the general theory.

5. Chapter 4 (higher order linear equations) and Chapter 5 (power series solutions) have been interchanged to make it easier for those instructors who wish to combine their treatment of second order and higher order linear equations.

6. A more thorough discussion of numerical methods for systems of equations has been added to Chapter 8. There are also about 30 new problems in this chapter.

7. Chapter 9, on stability and phase plane analysis, has been considerably expanded. In addition to the new section on the Lorenz equations, there are now two

sections rather than one on the interaction of two populations, as well as a new section on limit cycles, with an emphasis on the van der Pol equation.

It is important to mention that there is an alternate version of this text entitled *Elementary Differential Equations* for courses that do not cover partial differential equations and boundary value problems.

As the subject matter of differential equations continues to grow, as new technologies become commonplace, as old areas of application are expanded, and as new ones appear on the horizon, the content and viewpoint of courses and their textbooks must also evolve. This is the spirit we have sought to express in this book.

William E. Boyce
Troy, New York
June 1991

Acknowledgments

In preparing this revision we have received valuable assistance from a number of individuals. It is a pleasure to express now our warm appreciation to each one for their time and efforts.

While revising the Solutions Manual, Charles W. Haines read the text and checked the answers to many problems. His sharp eye led to the elimination of numerous errors and inconsistencies.

Richard Bagby, Bruce Berndt, Paul Davis, and Thomas Otway reviewed the entire manuscript and offered many perceptive suggestions; as a result, the book is significantly better than it would otherwise have been.

Since the publication of the preceding edition, we have received helpful comments from several users. Of these, R. B. Burckel, Leah Edelstein-Keshet, and Melvin Lax deserve special mention for the detailed and comprehensive nature of their suggestions.

Cathy Caldwell read most of the manuscript, checking the examples and the answers to the new problems. She also was of great assistance in correcting the proof sheets.

There are quite a few new computer-generated figures in this edition. Some of these were originally plotted using PHASER by Hüseyin Koçak, while others were prepared with the help of PHASE PORTRAITS by Herman Gollwitzer.

Finally, and most important of all, I thank my wife Elsa not only for her assistance in such tasks as proofreading and checking calculations, but especially for her never-failing moral support, encouragement, and patience throughout the project.

W. E. B.

Table of Contents

Chapter 1 Introduction

In this brief chapter we try to give perspective to your study of differential equations. First, we indicate several ways of classifying equations, in order to provide organizational structure for the remainder of the book. Later, we outline some of the major figures and trends in the historical development of the subject. The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. Nevertheless, it remains a dynamic field of inquiry today, with many interesting open questions.

1.1 Classification of Differential Equations

Many important and significant problems in engineering, the physical sciences, and the social sciences, when formulated in mathematical terms, require the determination of a function satisfying an equation containing one or more derivatives of the unknown function. Such equations are called **differential equations.** Perhaps the most familiar example is Newton's law

$$m\frac{d^2u(t)}{dt^2} = F\left[t, u(t), \frac{du(t)}{dt}\right] \qquad (1)$$

for the position $u(t)$ of a particle acted on by a force F, which may be a function of time t, the position $u(t)$, and the velocity $du(t)/dt$. To determine the motion of a particle acted on by a given force F it is necessary to find a function u satisfying Eq. (1).

The main purpose of this book is to discuss some of the properties of solutions of differential equations, and to describe some of the methods that have proved effective in finding solutions, or in some cases approximating them. To provide a framework for our presentation we first mention several useful ways of classifying differential equations.

Ordinary and Partial Differential Equations. One of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case only ordinary derivatives appear in the differential equation, and it is said to be an **ordinary differential equation.** In the second case the derivatives are partial derivatives, and the equation is called a **partial differential equation.**

Two examples of ordinary differential equations, in addition to Eq. (1), are

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t), \tag{2}$$

for the charge $Q(t)$ on a condenser in a circuit with capacitance C, resistance R, inductance L, and impressed voltage $E(t)$; and the equation governing the decay with time of an amount $R(t)$ of a radioactive substance, such as radium,

$$\frac{dR(t)}{dt} = -kR(t), \tag{3}$$

where k is a known constant. Typical examples of partial differential equations are the potential equation

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0, \tag{4}$$

the diffusion or heat conduction equation

$$\alpha^2\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u(x, t)}{\partial t}, \tag{5}$$

and the wave equation

$$a^2\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial^2 u(x, t)}{\partial t^2}. \tag{6}$$

Here α^2 and a^2 are certain constants. The potential equation, the diffusion equation, and the wave equation arise in a variety of problems in the fields of electricity and magnetism, elasticity, and fluid mechanics. Each is typical of distinct physical phenomena (note the names), and each is representative of a large class of partial differential equations.

Systems of Differential Equations. Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one equation is sufficient. However, if there are two or more unknown functions, then a system of equations is required. For example, the Lotka-Volterra, or predator–prey, equations are important in ecological modeling. They have the form

$$\begin{aligned} dH/dt &= aH - \alpha HP, \\ dP/dt &= -cP + \gamma HP, \end{aligned} \tag{7}$$

where $H(t)$ and $P(t)$ are the respective populations of the prey and predator species. The constants a, α, c, and γ are based on empirical observations and depend on the particular species that are being studied. Systems of equations are discussed in Chapters 7 and 9; in particular, the Lotka-Volterra equations are examined in Section 9.5.

Order. The **order** of a differential equation is the order of the highest derivative that appears in the equation. Thus Eqs. (1) and (2) are second order ordinary differential equations, and Eq. (3) is a first order ordinary differential equation. Equations (4), (5), and (6) are second order partial differential equations. More generally, the equation

$$F[x, u(x), u'(x), \ldots, u^{(n)}(x)] = 0 \tag{8}$$

is an ordinary differential equation of the nth order. Equation (8) represents a relation between the independent variable x and the values of the function u and its first n derivatives $u', u'', \ldots, u^{(n)}$. It is convenient and customary in differential equations to write y for $u(x)$, with $y', y'', \ldots, y^{(n)}$ standing for $u'(x), u''(x), \ldots, u^{(n)}(x)$. Thus Eq. (8) is written as

$$F(x, y, y', \ldots, y^{(n)}) = 0. \tag{9}$$

For example,

$$y''' + 2e^x y'' + yy' = x^4 \tag{10}$$

is a third order differential equation for $y = u(x)$. Occasionally, other letters will be used instead of y; the meaning will be clear from the context.

We assume that it is always possible to solve a given ordinary differential equation for the highest derivative, obtaining

$$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}). \tag{11}$$

We study only equations of the form (11). This is mainly to avoid the ambiguity that may arise because a single equation of the form (9) may correspond to several equations of the form (11). For example, the equation

$$y'^2 + xy' + 4y = 0$$

leads to the two equations

$$y' = \frac{-x + \sqrt{x^2 - 16y}}{2} \quad \text{or} \quad y' = \frac{-x - \sqrt{x^2 - 16y}}{2}.$$

Solution. A **solution** of the ordinary differential equation (11) on the interval $\alpha < x < \beta$ is a function ϕ such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(x) = f[x, \phi(x), \phi'(x), \ldots, \phi^{(n-1)}(x)] \tag{12}$$

for every x in $\alpha < x < \beta$. Unless stated otherwise, we assume that the function f of Eq. (11) is a real-valued function, and we are interested in obtaining real-valued solutions $y = \phi(x)$.

It is easily verified by direct substitution that the first order equation (3)

$$dR/dt = -kR$$

has the solution

$$R = \phi(t) = ce^{-kt}, \qquad -\infty < t < \infty, \tag{13}$$

where c is an arbitrary constant. Similarly, the functions $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are solutions of

$$y'' + y = 0 \tag{14}$$

for all x. As a somewhat more complicated example, we verify that $\phi_1(x) = x^2 \ln x$ is a solution of

$$x^2y'' - 3xy' + 4y = 0, \qquad x > 0. \tag{15}$$

We have

$$\phi_1(x) = x^2 \ln x,$$

$$\phi_1'(x) = x^2(1/x) + 2x \ln x = x + 2x \ln x,$$

$$\phi_1''(x) = 1 + 2x(1/x) + 2 \ln x = 3 + 2 \ln x.$$

On substituting in the differential equation (15) we obtain

$$\begin{aligned} &x^2(3 + 2 \ln x) - 3x(x + 2x \ln x) + 4(x^2 \ln x) \\ &\quad = 3x^2 - 3x^2 + (2 - 6 + 4)x^2 \ln x = 0, \end{aligned}$$

which verifies that $\phi_1(x) = x^2 \ln x$ is a solution of Eq. (15). It can also be shown that $\phi_2(x) = x^2$ is a solution of Eq. (15); this is left as an exercise.

Although for the equations (3), (14), and (15) we are able to verify that certain simple functions are solutions, in general we do not have such solutions readily available. Thus a fundamental question is the following: Given an equation of the form (11), how can we tell whether it has a solution? This is the question of *existence* of a solution. The fact that we have written down an equation of the form (11) does not necessarily mean that there is a function $y = \phi(x)$ that satisfies it. Indeed, not all differential equations have solutions, nor is the question of existence a purely mathematical one. If a meaningful physical problem is correctly formulated mathematically as a differential equation, then the mathematical problem should have a solution. In this sense an engineer or scientist has some check on the validity of the mathematical formulation.

Second, assuming that a given equation has one solution, does it have other solutions? If so, what type of additional conditions must be specified to single out a particular solution? This is the question of *uniqueness*. Notice that there is an infinity of solutions of the first order equation (3) corresponding to the infinity of possible choices of the constant c in Eq. (13). If R is specified at some time t, this condition will determine a value for c; even so, however, we do not know yet that there may not be other solutions of Eq. (3) which also have the prescribed value of R at the prescribed time t. The questions of existence and uniqueness are difficult questions; they and related questions will be discussed as we proceed.

A third question, a more practical one, is: Given a differential equation of the form (11), how do we actually determine a solution? Note that if we find a solution of the given equation we have at the same time answered the question of the existence of a solution. On the other hand, without knowledge of existence theory we might,