Functional Analysis
An Introduction
(泛函分析导论)

■ Yisheng Huang

Science Press
Beijing

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Preface

Functional analysis is primarily concerned with infinite-dimensional linear (vector) spaces, mainly function spaces whose "points" are functions, and mappings between them, usually called operators or, functionals if the range is on the real line or in the complex plane. It was invented and developed in the last years of the nineteenth century and the first few decades of the twentieth century. During the early period of its development, the original purpose of functional analysis was to use a framework which allows the study of differential and integral equations to be considered in the same formulation (cf. [6]). Later, functional analysis developed rapidly as in-depth study and interconnection on spectral theory of ordinary and partial differential equations, potential theory, Fourier expansions, and applied mathematical techniques, especially, on the influence of mathematical physics and quantum mechanics. Functional analysis handles many problems from different fields in an abstract framework of combination of algebra, analysis and geometry. Today, functional analysis has developed into an area of independent mathematical interest with rich contents and systemic methods, and it provides basic tools and foundation for areas of vital importance such as optimization, boundary value problems, modeling real-world phenomena, finite element methods, variational equations and inequalities, wavelets, image processing, science and engineering, cf. [3],[7],[12],[14],[16],[25],[26],[32],[33],[37].

As a beautiful theory for its richness in applications, functional analysis is now taught to third or fourth year mathematics undergraduates at many universities in the world. Although, there are some excellent textbooks which deal with the subject of functional analysis, few can be regarded as really elementary or introductory partly because of differences of opinion among experts about the level and teaching methodology. In my view, the field of elementary functional analysis is the ideal place in which to strengthen some abstract structural mathematics and to develop analytical technique. Therefore, it is my hope that this book in English version may provide a really introductory, though non-trivial, course on functional analysis for undergraduates, especially for Chinese undergraduates, who have completed basic courses on real and complex variable theory and linear algebra theory. In my view, the field of elementary functional analysis is the ideal place in which to strengthen some abstract structural mathematics and to develop analytical technique.

The book is addressed to be a bilingual textbooks which is suitable for students of mathematics in a one-semester course meeting 72 hours including exercises and discussions. Also it is expected that the approach is basic enough to enable students of physics and engineering to get something of the flavor of the subject.

We now briefly outline the contents of the book. Chapter 1 is devoted to basic results of metric spaces, especially an important fixed point theorem called the Banach contraction mapping theorem, and its applications. Chapter 2 deals with basic definitions, properties and examples related to the normed linear space, especially, Banach spaces. The beautiful theory of inner product space, especially Hilbert spaces, which draws connections between the first two chapters, is introduced and discussed in detail in Chapter 3. In Chapter 4, we describe general properties of linear operators between normed linear spaces, in addition, important results such as the uniform bounded principle, open mapping and closed graph theorems, the Banach theorem, the Hahn-Banach theorem, the representation theorems for some dual spaces and the concept of weak convergence are presented. Chapter 5 is devoted the nice properties of a class of linear operators, compact and self-adjoint operators, acting between Hilbert spaces and their spectral properties. Chapters 1-3 and Chapters 4-5 can serve as the space theory and the operator theory of functional analysis, respectively.

There are over 240 exercises in the book, many of which are straightforward and others quite challenging. It is strongly recommended that students should attempt most of the exercises. This is the way one must do to really learn any branch of mathematics.

The bibliography includes some other excellent books to capable students for further study, for example, [4],[5],[11],[15],[24],[29],[34],[36].

I would like to express my heartfelt thanks for my colleagues, former and present students who have made valuable comments and corrections in writing this book. Considered of my limited ability, mistakes are inevitable. I welcome suggestions of any way in which the presentation of this book can be improved.

Yisheng Huang
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2008.11

List of Symbols

$X \times Y$	Cartesian product of X and Y, 1
$\operatorname{ess\,sup}_E$	norm on $L^\infty(E)$, 63
$W \oplus U$	orthogonal sum of subspaces W and U, 103
$\rightharpoonup$	weak convergence, 177
$\xrightarrow{w^*}$	weak* convergence, 185
$x \perp y$	x is orthogonal to y, 98
$\langle x, y \rangle$	inner product of x and y, 88
$\|x\|$	norm of a vector x, 52
A°	interior of a set A, 20
$\overline{A}$	closure of a set A, 20
∂A	boundary of a set A, 21
$A^\perp$	orthogonal complement of a subspace A of a Hilbert space, 99
$B(a, r)$	open ball center a and radius r, 16
$\mathscr{B}(X, Y)$	space of all bounded linear operators from space X to space Y, 138
$\mathscr{B}(X)$	$= \mathscr{B}(X, X)$, 138
$\operatorname{codim}(N)$	codimension of a linear space N, 79
$\mathbb{C}$	set of complex numbers, 1
c	space of all convergent sequences of numbers, 9
c_0	space of null sequences, 9
$C[a, b]$	space of all continuous functions on $[a, b]$, 8
$C([a, b]; \mathbb{R})$	space of all continuous real valued functions on $[a, b]$, 8
$C([a, b]; \mathbb{C})$	space of all continuous complex valued functions on $[a, b]$, 8
$C^k[a, b]$	space of all k-times continuously differentiable functions on $[a, b]$, 59
$C_0^\infty(\Omega)$	space of all smooth functions on Ω with compact support sets, 95
$\dim(X)$	dimension of a linear space X, 52
$d(x, A)$	$= \inf_{y \in A} d(x, y)$, distance between a point x and a set A, 34
$\mathbb{F}$	either $\mathbb{R}$ or $\mathbb{C}$, 1
F'	set of all accumulation points of an F, 18
$G(T)$	graph of an operator T, 135
$H_0^1(\Omega)$	Sobolev space, 96
$\mathbf{Im}\,(z)$	imaginary part of a complex number z, 1
$\operatorname{Ker}(T)$	kernel of an operator T, 124
ℓ^p	space of all p-power summable sequences of numbers, 64
ℓ^∞	space of all bounded sequences of numbers, 7
$L^p(E)$	space of all p-power integrable functions on E, 60

$L^\infty(E)$	space of all essentially bounded functions on E, 62
$\mathbb{N}$	set of positive integers, 1
$\mathbb{Q}$	set of rational numbers, 1
$\mathbb{R}$	set of real numbers, 1
$\mathscr{R}(T)$	image or range of an operator T, 124
$\mathbf{Re}\,(z)$	real part of a complex number z, 1
s	space of all sequences, 7
S	space of all measurable functions on a set with finite measure, 10
$S(a,r)$	closed ball center a and radius r, 16
$\widetilde{S}(a,r)$	sphere center a and radius r, 16
$\sigma(T)$	spectrum of an operator T, 201
$\sigma_p(T)$	point spectrum of an operator T, 202
$\sigma_c(T)$	continuous spectrum of an operator T, 202
$\sigma_r(T)$	residual spectrum of an operator T, 203
span (M)	linear span of a linear space M, 74
T^*	adjoint (or dual) of an operator T, 188
X^*	$= \mathscr{B}(X, \mathbb{F})$, dual space of X, 138
X^{**}	second dual space (or bidual) of X, 174
X/N	quotient space of X modulo N, 78
$\mathbb{Z}$	set of integer numbers, 1

Contents

Chapter 1

Metric Spaces

1.1 Preliminaries

Throughout this book we usually denote sets by capital letters, such as $X, Y, \cdots$, while elements of sets will be denoted by lower case letters, such as $x, y, \cdots$. We will also use the following usual set operations: $\in, \notin, \subset, \cup, \cap, \varnothing$ (empty set), $X \times Y = \{(x, y) : x \in X, y \in Y\}$ (Cartesian product), $X \backslash Y = \{x \in X : x \notin Y\}$.

In what follows we shall use the following sets, which occur throughout mathematics:

$\mathbb{R}$ = the set of real numbers;

$\mathbb{C}$ = the set of complex numbers;

$\mathbb{Q}$ = the set of rational numbers;

$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$, the set of integer numbers;

$\mathbb{N} = \{1, 2, 3, \cdots\}$, the set of positive integer numbers.

The notation arises as follows: $\mathbb{N}$ for natural numbers, $\mathbb{Z}$ for Zahlen (German for integers), $\mathbb{Q}$ for quotient, $\mathbb{R}$ and $\mathbb{C}$ seem for the initials of real and complex, respectively.

We will simply use the notation $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$ when discussion applies equally well to both.

The real and imaginary parts of a complex number z will be denoted by $\mathbf{Re}\,(z)$ and $\mathbf{Im}\,(z)$, respectively, while the complex conjugate of the complex number z will be denoted by $\bar{z}$.

For each $k \in \mathbb{N}$ we let $\mathbb{F}^k = \mathbb{F} \times \cdots \times \mathbb{F}$. Elements of $\mathbb{F}^k$ will be written in the form $x = (x_1, \cdots, x_k), x_j \in \mathbb{F}, j = 1, \cdots, k$.

Given two sets X and Y, the notation $F : X \to Y$ will denote a mapping from X to Y. The set X is the domain of F and Y is the codomain. If $A \subset X$ and $B \subset Y$, we use the notation

$$F(A) = \{f(x) \in Y : x \in A\}, \quad F^{-1}(B) = \{x \in X : f(x) \in B\},$$

and we call them the image of A for F and the preimage of B for F, respectively.

We now recall some simple but important inequalities which we will frequently use in the succeeding chapters.

(T_1) (Triangle inequality) $\forall\, a, b \in \mathbb{C}$, $|a+b| \leqslant |a| + |b|$.

Proof Since $|\mathbf{Re}\,(z)| \leqslant |z|$ for each $z \in \mathbb{C}$, the result holds by writing

$$|a+b|^2 = (a+b)(\overline{a+b}) = (a+b)(\bar{a}+\bar{b}) \leqslant |a|^2 + |b|^2 + 2|a||b|. \quad \blacksquare$$

(T_2)

$$\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \quad \forall\, a, b \in \mathbb{C}. \tag{1.1}$$

Proof Consider $f(t) = t(1+t)^{-1}$ for $t > -1$. Since $f'(t) = (1+t)^{-2}$, $f(t)$ is an increasing function on $(-1, +\infty)$. It follows by the preceding triangle inequality that $f(|a+b|) \leqslant f(|a| + |b|)$, hence

$$\begin{aligned}\frac{|a+b|}{1+|a+b|} &\leqslant \frac{|a|+|b|}{1+|a|+|b|} = \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \\ &\leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}. \quad \blacksquare\end{aligned}$$

(T_3) (The Young inequality) Let $p > 1$, $1/p + 1/q = 1$, $a \geqslant 0$, $b \geqslant 0$. Then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q},$$

with equality if and only if $a^p = b^q$.

Proof Let $f(t) = 1 - 1/p + t/p - t^{1/p}$, where $t \geqslant 0$. Then $f'(t) < 0$ for $0 < t < 1$ and $f'(t) > 0$ for $t > 1$. Hence $f(t) \geqslant f(1) = 0$, with equality if and only if $t = 1$. Thus we have

$$t^{1/p} \leqslant 1 - 1/p + t/p \tag{1.2}$$

for $t \geqslant 0$. If $b = 0$ then $ab = 0 \leqslant a^p/p$. If $b > 0$ we put $t = a^p b^{-q}$ in (1.2) and obtain the result. ∎

(T_4) (The Hölder inequality) Let $p > 1$, $1/p + 1/q = 1$, $\xi_k, \eta_k \in \mathbb{F}$ for $k = 1, 2, \cdots, n$. Then

$$\sum_{k=1}^{n} |\xi_k \eta_k| \leqslant \Big(\sum_{k=1}^{n} |\xi_k|^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{n} |\eta_k|^q\Big)^{\frac{1}{q}}, \tag{1.3}$$

also,

$$\sum_{k=1}^{n} |\xi_k \eta_k| \leqslant \Big(\sum_{k=1}^{n} |\xi_k|\Big) \max |\eta_k|. \tag{1.4}$$

Proof (1.3) is known as the Hölder inequality. Inequality (1.4) is trivial and may be regarded as the case that $p = 1$ in the Hölder inequality.

To show (1.3) let us write

$$A = \Big(\sum_{k=1}^{n} |\xi_k|^p\Big)^{1/p}, \quad B = \Big(\sum_{k=1}^{n} |\eta_k|^q\Big)^{1/q}.$$

If $AB = 0$ then either $A = 0$ or $B = 0$. In either case we get both sides of (1.3) equal to zero. Now if $AB > 0$, then by the Young inequality above,

$$\frac{|\xi_k|}{A} \frac{|\eta_k|}{B} \leqslant \frac{|\xi_k|^p}{pA^p} + \frac{|\eta_k|^q}{qB^q},$$

whence $\sum\limits_{k=1}^{n} |\xi_k||\eta_k| \leqslant (1/p + 1/q)AB = AB$, which is (1.3). ∎

Remark 1.1.1

(i) It is easy to see from the proof that the quality holds in Hölder's inequality if and only if there exists a constant $M > 0$ such that $\xi_k^p = M\eta_k^q$ for $1 \leqslant k \leqslant n$.

(ii) By using the arguments in the proof of the above inequality (1.3), we have the following two forms of Hölder's inequality (check it! ✍):

$$\sum_{k=1}^{\infty} |\xi_k \eta_k| \leqslant \Big(\sum_{k=1}^{\infty} |\xi_k|^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{\infty} |\eta_k|^q\Big)^{\frac{1}{q}}, \tag{1.5}$$

$\xi_k, \eta_k \in \mathbb{F}$ for $k \in \mathbb{N}$, $p > 1$, $1/p + 1/q = 1$;

$$\int_E |x(t)y(t)|\,\mathrm{d}t \leqslant \Big(\int_E |x(t)|^p\,\mathrm{d}t\Big)^{\frac{1}{p}} \Big(\int_E |y(t)|^q\,\mathrm{d}t\Big)^{\frac{1}{q}}, \tag{1.6}$$

where E is a (Lebesgue) measurable set in $\mathbb{R}$ (or $\mathbb{R}^n$), $x(t)$ and $y(t)$ are (Lebesgue) measurable functions on E, $p > 1$, $1/p + 1/q = 1$.

(iii) When $p = q = 2$, either (1.3) or (1.5) is called the Cauchy inequality.

(T_5) (The Minkowski inequality) Let $p \geqslant 1$, $\xi_k, \eta_k \in \mathbb{F}$ for $k = 1, 2, \cdots, n$. Then

$$\Big(\sum_{k=1}^{n} |\xi_k + \eta_k|^p\Big)^{\frac{1}{p}} \leqslant \Big(\sum_{k=1}^{n} |\xi_k|^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{n} |\eta_k|^p\Big)^{\frac{1}{p}}. \tag{1.7}$$

Proof The case $p = 1$ is trivial. Suppose $p > 1$, then by the triangle inequality and the Hölder inequality (1.3),

$$\begin{aligned}\sum_{k=1}^{n} |\xi_k + \eta_k|^p &\leqslant \sum_{k=1}^{n} |\xi_k||\xi_k + \eta_k|^{p-1} + \sum_{k=1}^{n} |\eta_k||\xi_k + \eta_k|^{p-1} \\ &\leqslant \Big(\sum_{k=1}^{n} |\xi_k|^p\Big)^{1/p} \Big(\sum_{k=1}^{n} |\xi_k + \eta_k|^{q(p-1)}\Big)^{1/q} \\ &\quad + \Big(\sum_{k=1}^{n} |\eta_k|^p\Big)^{1/p} \Big(\sum_{k=1}^{n} |\xi_k + \eta_k|^{q(p-1)}\Big)^{1/q}.\end{aligned}$$

then the Minkowski inequality follows since $q(p-1) = p$. ∎

Remark 1.1.2 The following two forms of the Minkowski inequality still hold:

$$\Big(\sum_{k=1}^{\infty} |\xi_k + \eta_k|^p\Big)^{\frac{1}{p}} \leqslant \Big(\sum_{k=1}^{\infty} |\xi_k|^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{\infty} |\eta_k|^p\Big)^{\frac{1}{p}}, \tag{1.8}$$

where $\xi_k, \eta_k \in \mathbb{F}$, $k \in \mathbb{N}$, $p \geqslant 1$;

$$\Big(\int_E |x(t) + y(t)|^p\,\mathrm{d}t\Big)^{\frac{1}{p}} \leqslant \Big(\int_E |x(t)|^p\,\mathrm{d}t\Big)^{\frac{1}{p}} + \Big(\int_E |y(t)|^p\,\mathrm{d}t\Big)^{\frac{1}{p}}, \tag{1.9}$$

where $x(t)$ and $y(t)$ are measurable functions on a measurable set E in $\mathbb{R}$ (or $\mathbb{R}^n$), $p \geqslant 1$.

Proof We will show (1.8) holds. (1.9) can be proved in the same way (check it! ✍). Indeed, it is trivial if one sum in the right-hand side of (1.8) is the infinity. On the other hand, $\forall\, a, b \in \mathbb{F}$, we have

$$(|a| + |b|)^p \leqslant (2\max(|a|, |b|))^p \leqslant 2^p(|a|^p + |b|^p),$$

which yields

$$\sum_{k=1}^{\infty} |\xi_k + \eta_k|^p \leqslant 2^p\Big(\sum_{k=1}^{\infty} |\xi_k|^p + \sum_{k=1}^{\infty} |\eta_k|^p\Big).$$

It follows that if the sum in the left-hand side of (1.8) is the infinity then at least one sum in the right-hand side of (1.8) is the infinity. Therefore, we may assume all sums are finite. Similarly as in the proof of (1.7), we obtain (1.8). ∎

(T_6) Let $0 < p \leqslant 1$, $\xi_k, \eta_k \in \mathbb{F}$ for $k \in \mathbb{N}$. Then

$$\sum_{k=1}^{m} |\xi_k + \eta_k|^p \leqslant \sum_{k=1}^{m} |\xi_k|^p + \sum_{k=1}^{m} |\eta_k|^p, \tag{1.10}$$

where m is finite or infinite.

Proof It is enough to prove that the inequality $(a+b)^p \leqslant a^p + b^p$ is valid for $0 < p \leqslant 1$, $a \geqslant 0$, $b \geqslant 0$. To prove this we consider

$$f(t) = 1 + t^p - (1+t)^p$$

for $t \geqslant 0$. Taking a derivative we find that $f(t) \geqslant 0$ for $t \geqslant 0$, i.e., $(1+t)^p \leqslant 1 + t^p$. If $b = 0$ then $(a+b)^p = a^p + b^p$, and if $b > 0$ we put $t = a/b$ in $(1+t)^p \leqslant 1 + t^p$, then we get the desired result. ∎

Exercises

✍ **1.1** Let $b \neq 0$. Prove that $|a+b| = |a| + |b|$ if and only if there exists a real constant $k \geqslant 0$ such that $a = kb$.

✍ **1.2** Let $p \geqslant 1$. Prove that $\left(\sum\limits_{k=1}^{n} |a_k|\right)^p \leqslant n^{p-1} \sum\limits_{k=1}^{n} |a_k|^p$.

1.2 Definitions and Examples

The concept of limit is very important one in analysis. We recall it by looking at two very typical examples.

Example 1.2.1 Let $\{x_n\}$ be a sequence of real numbers, $l \in \mathbb{R}$. Suppose that $\{x_n\}$ converges to l as n goes to infinity. Then for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - l| < \varepsilon$ whenever $n > N$.

Example 1.2.2 Let $f(x, y)$ be a function on $\mathbb{R}^2$. Suppose that f converges to a limit $a \in \mathbb{R}$ as (x, y) approaches a point $(x_0, y_0) \in \mathbb{R}^2$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - a| < \varepsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

In Example 1.2.1, we basically say that $\{x_n\}$ can be made arbitrarily close to l if n is big enough. In Example 1.2.2, we basically say that $f(x, y)$ can be made arbitrarily close to a provided that (x, y) is close enough to (x_0, y_0) while remaining different from (x_0, y_0). Here what is important thing is not the algebraic nature of numbers in $\mathbb{R}$ or number pairs in $\mathbb{R}^2$, but the fact that distance from one point to another is well-defined and has certain properties. We now generalize this concept of distance to an arbitrary set.

Definition 1.2.1 *Suppose that X is a nonempty set. By a metric (or distance function) on X, we mean a real valued function $d : X \times X \to \mathbb{R}$ which satisfies the following conditions:*

$(M1)$ $d(x, y) \geqslant 0$ *for all* $x, y \in X$, *and* $d(x, y) = 0$ *if and only if* $x = y$;

$(M2)$ (Symmetry) $d(x, y) = d(y, x)$ *for every* $x, y \in X$;

$(M3)$ (Triangle inequality) *for every* $x, y, z \in X$, *we have* $d(x, z) \leqslant d(x, y) + d(y, z)$.

A metric space is a pair (X, d) which consists of a nonempty set X together with a metric or distance function d.

Remark 1.2.1

(i) It is important to note that the metric d is necessarily nonnegative even if we do not list it in Axiom $(M1)$. Since by the triangle inequality $(M3)$,

$$d(x, y) + d(y, x) \geqslant d(x, x).$$

$(M1)$ and $(M3)$ then give $2d(x, y) \geqslant 0$, so $d(x, y) \geqslant 0$ for all x, y in X.

(ii) It is worth noting that the validity of the inequality

$$|d(x, y) - d(x', y')| \leqslant d(x, x') + d(y, y'), \tag{1.11}$$

which follows easily from the axioms.

We now give some examples of metric spaces.

Example 1.2.3 The space $\mathbb{F}$.

In $\mathbb{F}$, we define a metric $d : \mathbb{F} \times \mathbb{F} \to \mathbb{R}$ by writing $d(x, y) = |x - y|$ for every $x, y \in \mathbb{F}$. It is easy to check that $(M1)$–$(M3)$ are all satisfied. This is called the Euclidean metric or "standard" metric on $\mathbb{F}$ (i.e., $\mathbb{R}$ or $\mathbb{C}$).

Example 1.2.4 The spaces $\mathbb{R}^n$ and $\mathbb{F}^n$.

Suppose that X is the set of all ordered n-tuples $x = (x_1, \cdots, x_n)$ of real numbers x_i, that is, $X = \{(x_1, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \cdots, n\}$. We define a metric $d : X \times X \to \mathbb{R}$ by writing

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

for every $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in X$. It is easy to see that d satisfies $(M1)$ and $(M2)$. $(M3)$ follows from the Minkowski inequality (1.7) with $p = 2$, $\xi_k = x_k - z_k$ and $\eta_k = z_k - y_k$, where $z = (z_i, \cdots, z_n) \in X$. Thus (X, d) is a metric space, we call n-dimensional Euclidean space and write it by $\mathbb{R}^n$ for short. The space $\mathbb{F}^n$ is defined in the same pattern (check it! ✍).

Example 1.2.5 The space s.

Suppose that X is the set of all possible sequences of real or complex numbers. Since X is an amorphous collection there is no obvious candidate for a metric. The following one is most popular:

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\xi_k - \eta_k|}{1 + |\xi_k - \eta_k|}$$

for $x = \{\xi_k\}, y = \{\eta_k\} \in X$. The factor 2^{-k} ensures convergence of the series, we could have k^{-2} or a_k with $a_k > 0$ and $\sum_{k=1}^{\infty} a_k < \infty$ instead. Clearly $(M1)$–$(M2)$ are satisfied and $(M3)$ follows from Inequality (1.1), so (X, d) is a metric space, we denote it by s.

Example 1.2.6 The space ℓ^∞.

This is the space of all bounded infinite sequences $x = \{x_k\}$ of real or complex numbers with standard metric $d(x, y) = \sup_{k \in \mathbb{N}} |x_k - y_k|$.

Example 1.2.7 The space $C[a, b]$.

Suppose that X is the set of all continuous (real or complex valued) functions on $[a, b]$, we define a metric $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \max_{a \leqslant t \leqslant b} |x(t) - y(t)|$$

for every $x(t)$, $y(t) \in X$. Since $|x(t) - y(t)|$ is continuous on $[a,b]$, it attends its maximum on $[a,b]$. It is easy to check that $(M1)$–$(M2)$ are satisfied. To check $(M3)$, let $x(t), y(t), z(t) \in X$ then $\forall\, t \in [a,b]$,

$$\begin{aligned}|x(t) - y(t)| &\leqslant |x(t) - z(t)| + |z(t) - y(t)| \\ &\leqslant \max_{a \leqslant t \leqslant b} |x(t) - z(t)| + \max_{a \leqslant t \leqslant b} |z(t) - y(t)| \\ &\leqslant d(x,z) + d(z,y),\end{aligned}$$

which gives

$$d(x,y) = \max_{a \leqslant t \leqslant b} |x(t) - y(t)| \leqslant d(x,z) + d(z,y).$$

Thus (X,d) is a metric space, we denote it by $C[a,b]$.

Remark 1.2.2 Sometimes we use the notation $C([a,b];\mathbb{R})$ to emphasize that $C[a,b]$ contains only the real valued functions, and use the notation $C([a,b];\mathbb{C})$ to emphasize that $C[a,b]$ contains only complex functions.

Example 1.2.8 The discrete space $\mathcal{D}$.

Suppose that X is a nonempty set. For every $x, y \in X$, define

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

It is easy to check that $(M1)$–$(M2)$ are satisfied. To check $(M3)$, note that the result is trivial if $x = z$. On the other hand, if $x \neq z$, then $d(x,z) = 1$. But then either $y \neq x$ or $y \neq z$, so that $d(x,y) + d(y,z) \geqslant 1$. Hence (X,d) is a metric space, it is known as the discrete metric space and we write it by $\mathcal{D}$.

One could, for example, put the discrete metric on the set $\mathbb{R}$ of real numbers, instead of the Euclidean metric (cf. Example 1.2.3). But then there would be no real variable theory as we know it. The use of the discrete metric is mainly in giving counterexamples to rash assertions and generalizations.

Remark 1.2.3 From the above, we know that we can freely define a metric on a nonempty set. In particular, we can define more than one metrics on the same set (unless it consists of a single point) so that we get different metric spaces. For example, on the set of n-tuples in Example 1.2.4, we may define other metrics by

$$d_1(x,y) = \max_{1\leqslant k\leqslant n} |x_k - y_k|,$$

$$d_2(x,y) = \sum_{k=1}^{n} |x_k - y_k|.$$

Then we obtain two new metric spaces.

For reference we note the inequalities (check them! ✍):

$$\begin{aligned} d_1(x,y) &\leqslant d(x,y) \leqslant \sqrt{n}\, d_1(x,y), \\ d_1(x,y) &\leqslant d_2(x,y) \leqslant n\, d_1(x,y), \\ d(x,y) &\leqslant d_2(x,y) \leqslant \sqrt{n}\, d(x,y). \end{aligned}$$

We can also define another metric on the set of all continuous functions on $[a,b]$ by

$$d_1(x,y) = \int_a^b |x(t) - y(t)|\, \mathrm{d}t,$$

where the integral is the ordinary Riemann integral, thus we get a new metric space different from the space $C[a,b]$ of Example 1.2.7.

Remark 1.2.4 If it is clear what the metric on X is we often simply write "the metric space X", rather than "the metric space (X,d)", but we should always bear in mind that a metric space is really a pair (X,d), not just a set X.

Definition 1.2.2 *Let (X,d) be a metric space and let N be a subset of X. Define $d_N : N \times N \to \mathbb{R}$ by $d_N(x,y) = d(x,y)$ for every $x, y \in N$ (i.e., d_N is the restriction of d to the subset $N \times N$). Then d_N is a metric on N, which we call the metric on N* induced *by d.*

Whenever we consider subsets of metric spaces we will regard them as metric spaces with the induced metrics unless otherwise stated.

Example 1.2.9 The spaces c and c_0.

c is the space of convergent sequences and c_0 is the space of null sequences $(x_k \to 0)$. Both spaces c and c_0 are subspaces of ℓ^∞ with the ℓ^∞ metric. In the space c_0 (but not in c) one may actually use $\max_{k\in\mathbb{N}} |x_k - y_k|$ instead of $\sup_{k\in\mathbb{N}} |x_k - y_k|$ for the metric (why? ✍).

We now give the last example to end this section.