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A. N. Parshin　I. R. Shafarevich (Eds.)

Algebraic Geometry IV
Linear Algebraic Groups, Invariant Theory

代数几何 IV
线性代数群，不变量理论

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

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I. Linear Algebraic Groups

T.A. Springer

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Introduction

A linear algebraic group over an algebraically closed field k is a subgroup of a group $GL_n(k)$ of invertible $n \times n$-matrices with entries in k, whose elements are precisely the solutions of a set of polynomial equations in the matrix coordinates. The present article contains a review of the theory of linear algebraic groups.

To develop the theory one needs tools from algebraic geometry. The reader is assumed to have some familiarity with that subject.

Chapter 1 of the article reviews the basic facts from the theory of linear algebraic groups over an algebraically closed field k. This theory culminates in a classification of simple linear algebraic groups. I have tried to explain carefully the fundamental notions and results, to illustrate them with concrete examples, and to give some idea of the methods of proof.

There are several monographs about the material of this chapter ([B2], [Hu], [Sp3]), where the interested reader can find more details about this material.

Chapter 2 discusses the relative theory, where a field of definition comes into play. This is, roughly, a subfield F of k such that the polynomial equations of the first line can be taken to have coefficients in F. This relative theory is required, for example, if one wishes to deal with arithmetical questions involving algebraic groups.

At the moment there do not exist monographs covering this theory, which makes it less accesible. I have tried to present a coherent picture, following the same lines as in Chapter 1.

In Chapter 3 special features are discussed of the relative theory, for particular fields of definition F, notably finite, local and global fields. The aim of the chapter is to show how the theory of algebraic groups is used in questions about such special fields.

There is a great abundance of material. Because of limitations of space I have sometimes been quite sketchy.[1]

The references at the end of the article do not have the pretension of being complete. But I hope that with the help of them a reader will be able to trace in the literature further details, of he wishes to do so.

A reference in the article to I, 2.3.4 (resp. 2.3.4) refers to no. 2.3.4 of Chapter 1 (resp. of the same Chapter).

Historical Comments

By way of introduction to the subject of linear algebraic groups there follows a brief review of anterior developments which have been incorporated, in some

[1] This article was written in 1988. Today (in 1993) I would perhaps have written some parts differently. But I have not tried to rewrite the article. I only made some necessary adjustments.

way or another, in the theory of linear algebraic groups, or which have influenced that theory.

First there is the study of concrete linear groups. Galois already introduced the group $\mathrm{PGL}_2(\mathbb{F}_p)$ of fractional invertible linear maps $(z \mapsto (az + b)/(cz + d)^{-1})$ of the prime field $\mathbb{F}_p$. An extensive study of the general linear groups over such a field (in any dimension) and related "classical" groups (like orthogonal ones) was made by C. Jordan in 1870 (in his book "Traité des substitutions"). This was continued by L.E. Dickson around 1910 and by J. Diedonné around 1950. These authors study group-theoretical questions, such as the determination of all normal subgroups, for classical groups.

A landmark in this development is C. Chevalley's paper "Sur certains groupes simples" (Tôhoku Math. J., 1955, 14–66), in which Lie theory makes its appearance. He constructs, for any simple Lie algebra over the complex field, a corresponding linear group over any field F and he discusses their group-theoretical properties. The standard classical groups are special cases.

Incidentally, Jordan's book – mentioned above – contains a version of Jordan's normal form of matrices. The Jordan decomposition in linear algebraic groups (see I, 3.1) is a descendant.

Linear algebraic groups over the field of complex numbers appear in E. Picard's work on Galois theory of linear differential equations (around 1885, see his paper "Equations différentielles linéaires et les groupes algébriques de transformations", Oeuvres II, 117–131). An example of the questions studied by him is the following. Consider an n^{th} order homogeneous linear differential equation in the complex plane

$$\frac{d^n f}{dz^n} + a_{n-1}(z)\frac{d^{n-1}f}{dz^{n-1}} + \cdots + a_0(z)f = 0,$$

with polynomial coefficients a_i. One knows that the everywhere holomorphic solutions form an n-dimensional complex vector space, let $(f_1, \ldots, f_n)$ be a basis. Let L be the subfield of the field of meromorphic functions obtained by adjoining to the field $\mathbb{C}(z)$ of rational functions the f_i and all their derivatives. Denote by G the group of $\mathbb{C}(z)$-linear automorphisms of L which commute with derivation. This is the Galois group of the equation. Picard's aim is to develop a Galois theory. If $g \in G$ there exist complex numbers $(x_{ij}(g))$ such that

$$g \cdot f_i = \sum_{j=1}^{n} x_{ji}(g) f_j$$

and Picard shows that the matrices $(x_{ij}(g)) \in \mathrm{GL}_n(\mathbb{C})$ form a linear algebraic group over $\mathbb{C}$, isomorphic to G. He seems to be the first to use a name like "algebraic group".

This Galois theory was later algebraized and further developed by Ritt (around 1930) and Kolchin. The work of the latter of 1948 (see his paper "On certain concepts in the theory of algebraic matric groups", Ann. of Math. 49, 771–789) contains results which are now basic ones in the theory of linear algebraic groups, such as the properties of the identity component (I, 2.2.2) and the Lie-Kolchin theorem (I, 3.4.1) which states that a connected solvable linear algebraic group can be triangulized. This extends a result of Lie for complex solvable Lie algebras. In contrast to the latter result, the Lie-Kolchin theorem is true in any characteristic.

A. Weil's work on Jacobians of algebraic curves (see his book "Variétés abéliennes et courbes algébriques", 1948) led him to a study of general algebraic groups, i.e. algebraic varieties with a group structure given by morphisms in the sense of algebraic geometry. His interest was primarily in abelian varieties, i.e. connected algebraic groups which are projective varieties (in which the group structure is automatically commutative). Classically, abelian varieties over $\mathbb{C}$ were studied by transcendental methods which go back to Riemann.

Weil (and others) established somewhat later basic general facts about quotients of an arbitrary algebraic group by an algebraic subgroup. They are indispensable ingredients for the theory of linear algebraic groups.

The theory of linear algebraic groups was founded by A. Borel in 1956 ("Groupes linéaires algébriques", Ann. of Math. 64, 20–82). His work was completed by Chevalley ("Classification des groupes de Lie algébriques", Séminaire Ecole Normale Supérieure, 1956–1958). In Borel's work the influence of Kolchin's work, alluded to above, is clearly visible. Another essential element is the analogy with the theory of Lie groups, in its "global" form. An infinitesimal approach to linear algebraic groups via Lie algebras is unsuitable in characteristic $p > 0$.

Using the global methods of algebraic geometry, Borel established basic results, such as conjugacy theorems for maximal tori and Borel subgroups (I, 3.5.3, I. 3.5.1). To obtain these he proves a fixed point theorem (I, 3.4.3), which generalizes the Lie-Kolchin theorem, mentioned before. Chevalley showed that analogues of results established in Lie theory with the help of the Lie algebra can be obtained with global methods (for example results about radicals, see I. 4.2.6). The main result of his Séminaire is that the classification of simple linear algebraic groups over an algebraically closed field of any characteristic, is completely analogous to the classification of simple Lie algebras over the field of complex numbers.

In the work of Borel and Chevalley the influence of ideas and results from the theory of Lie groups has been considerable. Grosso modo, Chapter I of the article is a review of the work of Borel and Chevalley.

Finally, mention should be made of some generalizations of algebraic groups, which we have not – or hardly – touched upon. First there are the group schemes, studied extensively by Grothendieck and his collaborators (M. Demazure and A. Grothendieck, Schémas en groupes, Lect. Notes in Math. nos. 151, 152, 153, 1970). In this article they appear in only a few places. More recent generalizations are the quantum groups, which are algebraic groups in "non-commutative geometry". We have only given the definition (in 2.1.6). We have not said anything about "infinite dimensional" algebraic groups, such as Kac-Moody groups.

Chapter 1
Linear Algebraic Groups over an Algebraically Closed Field

§ 1. Recollections from Algebraic Geometry

Some familiarity with algebraic geometry is assumed. We shall recall a number of basic notions and results. For more details see [H], [Mu] or [Sp3].

1.1. Affine Varieties. Let k be an algebraically closed field. An *affine algebraic variety* X over k is determined by its *algebra of regular functions* $k[X]$, a k-algebra of finite type, which is reduced i.e. without non-zero nilpotent elements. Such k-algebras are called *affine*. X is the set of k-algebra homomorphisms $k[X] \to k$. For each ideal I of $k[X]$, let $\mathscr{V}(I)$ be the set of $x \in X$ such that $x(I) = 0$. The sets $\mathscr{V}(I)$ are the closed sets for a topology on X, the *Zariski topology*.

The elements of $k[X]$ define k-valued functions on X, the *regular* functions.

The affine variety defined by the polynomial algebra $k[T_1, \ldots, T_n]$ is affine n-space $\mathbb{A}^n$, also denoted k^n.

1.2. Morphisms

1.2.1. If X and Y are affine varieties, a homomorphism of k-algebras φ^*: $k[X] \to k[Y]$ defines a map φ: $Y \to X$, which is continuous. Such maps are the *morphisms* of affine k-varieties.

1.2.2. A closed subset Y of the affine variety X has a canonical structure of affine variety, with algebra $k[Y] = k[X]/I$, where I is the ideal of functions vanishing on Y. Such a variety is a *closed subvariety* of X. The corresponding morphism is a closed immersion.

1.2.3. Next let $f \in k[X] - \{0\}$ and take $k[Y] = k[X]_f = k[X][T]/(1 - fT)$, a localization of $k[X]$, with φ^* the canonical homomorphism. Then Y can be viewed as the open subset $D(f) = \{x \in X \mid f(x) \neq 0\}$ of X. Such a set $D(f)$, provided with the k-algebra $k[X]_f$, is an affine variety. Any open subset of X is a union of finitely many open sets of the form $D(f)$.

Example. Let $X = \mathbb{M}_n(k)$, the space of $n \times n$-matrices with entries in k, which is isomorphic to $\mathbb{A}^{n^2}$. Let $d(X) = \det(X)$ be the determinant function. The open set X_d is the set of all invertible $n \times n$-matrices.

1.2.4. If X and Y are affine varieties, there exists a product variety $X \times Y$, with $k[X \times Y] = k[X] \otimes_k k[Y]$.

1.2.5. Affine algebraic varieties and their morphisms make up a category, which is the opposite of the category of affine k-algebras.

1.3. Some Topological Properties. Let X be an affine variety. It has the noetherian property: any family of closed subsets contains a minimal element (for inclusion).

X is *reducible* if it is a union of two non-empty proper closed subsets. Otherwise X is *irreducible*. Irreducibility is equivalent to: a non-empty open subset of X is dense.

Also, X is irreducible if and only if the algebra $k[X]$ is an integral domain. In that case the quotient field of $k[X]$ is denoted by $k(X)$. Its transcendence degree over k is the *dimension* $\dim X$ of X.

Any affine variety is the union of finitely many irreducible closed subsets, its *irreducible components*, which are unique.

1.4. Tangent Spaces. If x is a point of the affine variety X, the homomorphism x: $k[X] \to k$ defines a $k[X]$-module k_x with underlying vector space k. The *tangent space* $T_x X$ of X at x is the k-vector space of k-derivations of $k[X]$ in k_x, i.e. linear maps D: $k[X] \to k$ such that $D(fg) = f(x)(Dg) + (Df)g(x)$. If M_x is the maximal ideal $\operatorname{Ker} x$ of $k[X]$ then $T_x X$ is isomorphic to the dual of $M_x/(M_x)^2$. A morphism φ: $Y \to X$ defines a map of tangent spaces $(d\varphi)_y$: $T_y Y \to T_x X$, the *differential* of at y.

If X is irreducible then $x \in X$ is *smooth* (or simple, or non-singular) if $\dim T_x X = \dim X$. The smooth points of X form a non-empty open subset. We say that X is *smooth* if all its points are smooth.

1.5. Properties of Morphisms

1.5.1. Let X and Y be irreducible affine varieties and φ: $Y \to X$ a morphism. It is said to be *dominant* if φY is dense in X. In that case φY contains a non-empty open subset of X.

1.5.2. If φ is dominant the defining homomorphism φ^*: $k[X] \to k[Y]$ is injective. Then φ is *separable* if the field $k(Y)$ is a separable extension of $\varphi^* k(X)$. This is so if and only if there exists a simple point $y \in Y$ such that φy is simple and that $(d\varphi)_y$: $T_y Y \to T_{\varphi y} X$ is surjective. The set of such points of Y is open.

1.5.3. If φ is dominant there is a non-empty open subset U of Y such that the restriction of φ to U is an open map (i.e. the image of an open set is open). Moreover, U can be chosen such that for any closed irreducible subvariety X' of X and any irreducible component Y' of $\varphi^{-1} X'$ such that $Y' \cap U \neq \varnothing$ we have $\dim Y' - \dim X' = \dim Y - \dim X$.

1.5.4. If φ is dominant and if for some $y \in Y$ the fiber $\varphi^{-1}(\varphi y)$ is finite then $\dim Y = \dim X$.

1.6. Non-Affine Varieties

1.6.1. Let X be an affine variety and U an open subset. A k-valued function f on U is regular if for any $x \in U$ there is an open neighborhood $D(g)$ of x in U

such that the restriction of f to $D(g)$ lies in $k[D(g)]$ (see 1.2.3). Let $\mathcal{O}_X(U)$ be the k-algebra of these functions. We say that U is an *affine open subset* if $\mathcal{O}_X(U)$ is an affine k-algebra, whose homomorphisms in k are precisely the evaluation maps $f \mapsto f(x)$ for $x \in U$.

The intersection of two affine open subsets U, V is also affine open. We have $k[U \cap V] = k[U] \otimes_{k[X]} k[V]$.

The $\mathcal{O}_X(U)$ for U open in X define a sheaf $\mathcal{O}_X$ of k-algebras on X, which defines a ringed space $(X, \mathcal{O}_X)$.

1.6.2. A ringed space $(X, \mathcal{O}_X)$ is called an algebraic variety (non necessarily affine) if X has a finite covering by open subsets $(X_i)_{i \in I}$ such that (a) for each i the restriction ringed space $(X_i, \mathcal{O}_X|_{X_i})$ is isomorphic to one of the kind described in 1.6.1, (b) for each pair (i, j) the intersection $X_i \cap X_j$ is an affine open subset of X_i, for the structure of (a) and the k-algebra $\mathcal{O}_X(X_i \cap X_j)$ is generated by $\mathcal{O}_X(X_i)$ and $\mathcal{O}_X(X_j)$.

Morphisms of algebraic varieties are defined in an obvious way. The notions and results reviewed above for affine varieties carry over, as far as this makes sense.

We have the notions of open resp. closed subvariety of an algebraic variety X. A *locally closed subvariety* of X is an open subvariety of a closed subvariety of X.

1.6.3. Example. Projective n-space $\mathbb{P}^n$. Here the underlying set X is the set of all lines in k^{n+1}. Let X_i be the set of those lines which have a basis vector $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$. Then X_i can be given a structure of affine algebraic variety isomorphic to $\mathbb{A}^n$. These structures can be glued together to give a structure of algebraic variety on X. If V is a finite dimensional vector space over k the set of lines in V has a structure of projective variety $\mathbb{P}(V)$, isomorphic to $\mathbb{P}^n$, where $n + 1 = \dim V$.

A *projective variety* is one which is isomorphic to a closed subvariety of some $\mathbb{P}^n$. A *quasi-projective variety* is an open subvariety of a projective variety.

Projective n-space can also be defined as the set of homomorphisms of $k[T_0, T_1, \ldots, T_n]$ to $k[T]$ which are homogeneous for the standard gradings, closed sets being those sets of homomorphisms which annihilate homogeneous ideals in the first algebra.

1.6.4. A variety X is *complete* if it has the following property: for any variety Y the projection map $X \times Y \to Y$ is closed. Projective varieties are complete. Affine varieties with infinitely many points are not complete.

§2. Linear Algebraic Groups, Basic Definitions and Properties

2.1. The Definition of a Linear Algebraic Group

2.1.1. The most direct definition of the notion of a linear algebraic group – which however is non-intrinsic – is as follows. For each integer $n \geqslant 1$ the group GL_n of non-singular $n \times n$-matrices is an affine algebraic variety (1.2.3).

Definition. *A linear algebraic group G* over k is a subgroup of some GL_n which is a closed subset of GL_n.

2.1.2. Examples of Linear Algebraic Groups

(a) GL_n. According to 1.2.3 we have $k[GL_n] = k[T_{ij}, d^{-1}]$, where $d = \det(T_{ij})$, the T_{ij} being matrix variables. GL_n is a *general linear group*.

We write $GL_1 = G_m$. This is the *multiplicative group*, also written k^*. We have $k[G_m] = k[T, T^{-1}]$, the algebra of Laurent polynomials over k.

We write $SL_n = \{X \in GL_n | d(X) = 1\}$, this is the *special linear group*. We have $k[SL_n] = k[GL_n]/(d - 1) \cong k[T_{ij}]/(d - 1)$.

(b) The subgroup of SL_2 consisting of the matrices

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with $x \in k$ is the *additive group* G_a, also written k. We have $k[G_a] = k[T]$.

(c) Let $S \in GL_n$. The $X \in GL_n$ with $XS({}^tX) = S$ form a linear algebraic group. Instances are the various classical groups:

(i) If $n = 2m$ is even and

$$S = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$$

we obtain the *symplectic group* Sp_n.

(ii) If $\mathrm{char}(k) \neq 2$ and $S = 1_n$ then G is the *orthogonal group*. The *special orthogonal group* is the intersection with SL_n, it is a normal subgroup of index 2.

We shall prefer to use another description of the orthogonal groups. If $n = 2m$ is even we denote by O_n the group defined above, with

$$S = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix}$$

and if $n = 2m + 1$ is odd we denote by O_n the group obtained from

$$S = \begin{bmatrix} 0 & 1_m & 0 \\ 1_m & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then O_n is conjugate in GL_n with the orthogonal group defined first. We write $SO_n = O_n \cap SL_n$.

(iii) If $\mathrm{char}(k) = 2$ orthogonal groups are defined in another way. If $n = 2m$ is even put

$$T = \begin{pmatrix} 0 & 1_m \\ 0 & 0 \end{pmatrix}$$

and define O_n to be the subgroup of GL_n consisting of the matrices X such that $XT({}^tX) + X$ is skew, i.e. symmetric with diagonal elements zero. This is again an algebraic group. If $n = 2m + 1$ is odd the definition is similar, replacing T by a larger matrix, as before.

(d) The group of diagonal matrices in GL_n (resp. upper triangular matrices, resp. upper triangular matrices with diagonal elements one) is a linear algebraic group.

(e) Any finite subgroup of GL_n is a linear algebraic group.