Viscoelastic Fracture Mechanics

(黏弹性断裂力学)

Zhang Chunyuan

Science Press
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FOREWORD

Fracture mechanics provides an important tool for the study of the influence of sharp flows, such as cracks in a structure on its mechanical performance under external loading. The sharp flows result in complicated stress and displacement fields even when the structure is made of a linear elastic material.

The situation is further complicated by non-linear and time-dependent response of the material. Although linear elastic fracture mechanics is now well endowed with several quality textbooks, viscoelastic fracture mechanics is not so fortunate. I am not aware of more than a handful of books on this topic in English and Russian languages.

Professor Zhang Chunyuan's contribution to the field of viscoelastic fracture mechanics is therefore timely and most welcome. I understand it is the first book of its kind in Chinese.

From the list of contents of the book, which Professor Zhang translated into English for my benefit, I see that it is going to be a self-contained treatise on the subject of linear, non-linear and thermoviscoelastic fracture mechanics. Professor Zhang is eminently qualified to write such a treatise, and I have not doubt that it will prove popular with the Chinese student and research community.

I shall eagerly look forward to the appearance of an English and/or Russian language translation of the book.

Professor B L Karihaloo
School of Civil and Mining Engineering
The University of Sydney
Sydney, January 1992

PREFACE

The emergence and development of Viscoelastic Fracture Mechanics are closely related with the wide application of new viscoelastic materials and the employment of a variety of materials in extreme conditions, such as high temperature, high stress level, etc. People find that a serious event of fracture usually occurs not immediately after the load application, but suddenly occurs under the application of the load after undergoing a period of time. We call this type of fracture as delayed fracture. A crack of buttressed concrete dam may occur with delayed instability. A vane of turbine may be suddenly fractured under high temperature after running a period of time. Delayed fracture is as danger as fatigue and usually is very difficult to prevent. Neither elastic nor elastic-plastic fracture mechanics can treat such kind of fracture problems that involving time-effects. Many engineering materials, such as polymers, composites, non-ferrous metals, rocks, concrete and so on, exhibit noticeable time-effect, this compel us to use new rheological models when we research crack problems. Most metals behave as linear elastic bodies at room temperature and in infinitesimal cases. Nevertheless, the vibration of a metal reed will still quickly attenuate in vacuum. This shows that there is viscous resistance in the internal of the material. The phenomenon that the strain increases with time under a constant load is referred to as creep. The phenomenon that the stress decreases with time under a constant strain is referred to as relaxation. The phenomenon, that the increasing curve does not coincide with the decreasing one but above it in a stress-strain diagram, is referred to as elastic hysteresis. The loop thus formed is called hysteresis loop. Actual materials exhibit the phenomena of creep, relaxation and hysteresis to a certain extent. Such materials that possess viscous and elastic properties simultaneously are called viscoelastic materials. The constitutive equation of viscoelastic body is no longer an algebraic equation for which the strain is proportional to the stress, but a differential or an integral equation that involves a time variable. In non-isothermal cases, the constitutive equation will involve also the heat flux, entropy and free energy, etc.

Viscoelastic Fracture Mechanics is a newly rising course that lies among viscoelasticity, fracture mechanics, rational continuum mechanics and irreversible thermodynamics. It applies the viscoelastic constitutive equations to research the initiation, propagation and rest laws of cracks. It provides the theoretical basis of the strength analysis of the machines and structures and provides the scientific basis for predict the service-life of the engineering components.

The author had given a lecture on "Viscoelasticity and Viscoelastic Fracture Mechanics" for "The National Symposium on Basic Theory of Rheological Mechanics" in 1983. The notes of this lecture had used in Xiangtan University as a textbook for postgraduate students who have the specialities of Rheology, Solid Mechanics and Metal Materials for many terms. This book is the revision, the modification and the development of those notes.

This book includes two parts, namely Viscoelasticity and Viscoelastic Fracture Mechanics. It is divided into seven chapters. Chapters 1, 2, 4 and 6, and Section 7.8 to 7.10 deal with the basic theory of viscoelasticity, which includes linear viscoelasticity, non-linear viscoelasticity and thermo-viscoelasticity. Chapters 3, 5 and 7 deal with linear viscoelastic fracture mechanics, viscoelastic fracture mechanics that taking into account the non-linear effects in the failure zone at the crack-tip and non-linear thermoviscoelastic fracture mechanics. Many of the contents are the summation of the basic theory established in a long period when I was researching the crack instability problem of Zhexi diamond-head buttressed concrete dam in Hunan Province of China.

When writing this book, I made every effort to explain the abstract mathematical ideas in clear physical languages and to express theoretical results in a form that is convenient for engineering application. I also made every effort to describe the contents in a way that I explain the profound conceptions in a simple way, proceed in an orderly way and step by step, deduce rigorously and indicate the source clearly. For reading convenience, an introduction to related modern mathematical and mechanical bases is included in the Appendices. Examples and exercises are also included in the end of every chapter. This book can be used as a textbook for senior graduate or postgraduate students. It also can be used as a reference book for research workers, engineers and teachers of colleges and universities on related specialities.

This book was supported by the foundation of Hunan Education Committee, the foundation of Xiangtan University and Key Science of Mechanics in Xiangtan University. I am indebted to many authors whose writings are classics in this field. To Christenson, Eringen, Flügge, Kaminskii, Rabotnov, Schapery, Williams, Wnuk, Griffith, Knauss, Nuismer, Liebowitz, Eftis, Coleman, Graham, Guo Zhongheng and many others, I am especially indebted. I own so much to so many of my colleagues, friends, and students. To Prof. Karihaloo, whom I work with when I visited the University of Sydney, I am grateful for his discussions, comments and gentle encouragement. I gratefully acknowledge Academician Sun Jun, Prof. Liu Shuyi, Prof. Yang Tingqing, Prof. Lee Hao, Prof. Zhou Yichun, Prof. Zhang Ping, Prof. Shen Wei and many others for their comments and helps. I also wish to thank many of my students, without their discussions the book would not have taken this form. Finally, I am extremely grateful to my wife, Prof. Ning Yaqin and my parents for their encouragements, discussions and helps.

Zhang Chunyuan
Xiangtan, Hunan
February 27, 2004

COMMON NOTATIONS

MATHEMATICAL NOTATIONS

$\in$	belongs to
$\cap$, $\cup$	intersection, union
$\times$	Cartesian product
$\subseteq$, $\supseteq$	included, include
Σ, Π	summation of, production of
sup	least upper bound
Res	residual
$a \overset{\text{def}}{=} b$	a is defined by b
$\exp x$	e^x
$*$	convolution
δ_{ik}	Kronecker δ function
$\Gamma(\ )$	Γ function
$\mathrm{E}(k)$	full elliptic integral of the second kind
$\theta(t)$	Heaviside unit-step function
$\delta(t)$	Dirac δ unit-impulse function
$I_\alpha(t)$	Abel function
$E_\alpha(\beta, t)$	Rabotnov fractional function
$O(\varepsilon)$	infinite small quantity that has the same order as ε
$\varnothing$	empty set
$\mathscr{L}(\ )$	Laplace transform
$\mathscr{L}^{-1}(\ )$	inverse Laplace transform
$f * \mathrm{d}g$	Stieltjes convolution
tr ( )	trace

PHYSICAL NOTATIONS

t	time
x_i	components of Descartes co-ordinates
u_i	components of the displacement
ε_{ij}	components of the strain
σ_{ij}	components of the stress
e_{ij}	deviatoric components of the strain
s_{ij}	deviatoric components of the stress
f_i	body force per unit mass
T	local thermodynamic temperature
ρ	mass density
G	shear modulus of elasticity
E	Young's modulus of elasticity
K	bulk modulus of elasticity
ν	Poisson's ratio
λ	Lamé's constant
K_{I}, K_{II}, K_{III}	stress intensity factors
G_{I}, G_{II}, G_{III}	energy release rates
K_{Ic}	critical value of the stress intensity factor K_{I}
G_{Ic}	critical value of the energy release rate G_{I}
δ_c	critical value of the opening crack displacement
$a(T)$	shift function
t_ξ	reduced time

CONTENTS

1. CONSTITUTIVE EQUATIONS FOR VISCOELASTIC BODIES

Notations

$D=\dfrac{d}{dt}$	differential operator	$j(\tau)$, $L(\log\tau)$	creep spectrum
$\dot{e}$	$\dfrac{de}{dt}$	$g(\tau)$, $H(\log\tau)$	relaxation spectrum
$u_{i,j}$	$\dfrac{\partial u_i}{\partial x_j}$	$G(t)$	relaxation modulus
s	stress deviator	$J(t)$	creep compliance
e	strain deviator	$E(t)$, $G_1(t)$, $G_2(t)$	uniaxial, shear, bulk relaxation modulus
η	coefficient of viscosity	$D(t)$, $J_1(t)$, $J_2(t)$	uniaxial, shear, bulk creep, compliance
τ_m	mean relaxation time,	$\psi(t)$, $\psi_1(t)$, $\psi_2(t)$	uniaxial, shear, bulk relaxation kernel
τ_m'	mean retardation time	$a(T)$	shift function
τ	relaxation time	$\phi(t)$, $\phi_1(t)$, $\phi_2(t)$	uniaxial, shear, bulk creep kernel
τ'	retardation time		

1.1 Introduction

This Chapter deals with the constitutive equation of a special kind of material, viscoelastic bodies. *Constitutive equation* is an equation that describes the special property of a material. By means of which, we can distinguish one material from the others. It describes the relationship among stress, rate of stress, strain and rate of strain, etc. Generally, it still involves thermodynamic variables, such as temperature, entropy, etc. Nevertheless, this Chapter deals only with isothermal constitutive equations of linear viscoelastic bodies. Because of the complex nature of a true material, mathematical abstraction can only reflect some main properties in a certain extent. Thus, any constitutive equation can only define an "*idealized material*". Properly select idealized material can obtain a good approximation of a true material. In this Chapter, we do not decide to discuss constitutive equations from the basic axiom like rheology, but focus our attention on viscoelastic bodies. We give emphasis to introduce constitutive equations of differential type from model theories, and to introduce constitutive equations of integral type. Since the difference of governing equations between viscoelastic bodies and elastic bodies lies in the constitutive equation, the key step to solve viscoelastic problem is to research the constitutive equation. Although the constitutive equation of viscoelastic bodies and elastic bodies is not the same, the analogy between them is apparent. This analogy is just the basis for solving viscoelastic problems.

We review first the well-known constitutive equation of linear elastic bodies, the generalized Hooke's law, and then express it in a form that easy to be compared. For convenience, we use tensor notations. We write Cartesian co-ordinates $\{x, y, z\}$, displacement components (u, v, w), strain components $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \frac{1}{2}\gamma_{xy}, \frac{1}{2}\gamma_{yz}, \frac{1}{2}\gamma_{zx})$ and stress components † $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx})$ as $\{x: x_1, x_2, x_3\}$, (u_1, u_2, u_3),$(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31})$, $(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})$. Isotropic, linear, elastic constitutive equation can be written as

$$
\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}),\\
\varepsilon_{22} &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{33} + \sigma_{11})] = \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}),\\
\varepsilon_{22} &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] = \frac{1+\nu}{E}\sigma_{33} - \frac{\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}),\\
\varepsilon_{12} &= \frac{\gamma_{12}}{2} = \frac{\sigma_{12}}{2G} = \frac{1+\nu}{E}\sigma_{12}, \qquad \varepsilon_{21} = \frac{1+\nu}{E}\sigma_{21},\\
\varepsilon_{23} &= \frac{\gamma_{23}}{2} = \frac{\sigma_{23}}{2G} = \frac{1+\nu}{E}\sigma_{23}, \qquad \varepsilon_{32} = \frac{1+\nu}{E}\sigma_{32},\\
\varepsilon_{31} &= \frac{\gamma_{31}}{2} = \frac{\sigma_{31}}{2G} = \frac{1+\nu}{E}\sigma_{31}, \qquad \varepsilon_{13} = \frac{1+\nu}{E}\sigma_{13},
\end{aligned}
\tag{1.1-1}
$$

where E is elastic modulus, ν is Poisson's ratio and G is elastic modulus in shear. (1.1-1) can be written in an abridged form (Fung[1]).

$$
\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \qquad (i, j = 1,2,3). \tag{1.1-2}
$$

We have used the *Einstein's summation convention* in (1.1-2). The convention is as follows: *An index (whether superscript or subscript) repeats* ***once*** *and* ***only once*** *in a term will denote a summation with respect to that index over its range.* The range of an index i is the set of n integer values 1 to n (usually, 1 to 3). For example: $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$. An index that is summed over is called *a dummy index*; one that is not summed out is called a *free index.* Since a dummy index just indicates summation, it is immaterial which symbol is used. Thus, $\sigma_{KK} = \sigma_{mm}$ and $a_i x^i = a_m x^m$. We have also used the *Kronecker δ symbol.* In δ_{ij}, when $i = j$, it equals unit and when $i \neq j$, it equals zero:

† Note that in this book the subscripts in σ_{ij} indicate that the stress is acting on a plane normal to the x_j-axis and the stress component is in the x_i-direction. This is notation is not the same as some of the other books.

$$\delta_{ik} = \begin{cases} 1, & \text{when } i = k, \\ 0, & \text{when } i \neq k. \end{cases} \tag{1.1-3}$$

In view of the above convention, (1.1-2) represents nine equations, that is (1.1-1). This equation can be solved in the following form:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij}, \tag{1.1-4}$$

where, λ denotes Lamé's constant. We shall also use bulk modulus K. Since isotropic materials have only two independent material constants, there are certain relationships between elastic constants that we shall use later. We list all these relationships in Table 1.1-1 (see, for example, Fung [1]).

Table1.1-1 Relationships between G, E, K, v and λ

	G, E	G, K	G, v	G, λ	E, K	E, v	K, v	K, λ	v, λ
G	—	—	—	—	$\frac{3EK}{9K-E}$	$\frac{E}{2(1+v)}$	$\frac{3K(1-2v)}{2(1+v)}$	$\frac{3}{2}(K-\lambda)$	$\frac{\lambda(1-2v)}{2v}$
E	—	$\frac{9GK}{3K+G}$	$2G(1+v)$	$\frac{G(3\lambda+2G)}{\lambda+G}$	—	—	$3K(1-2v)$	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{\lambda(1+v)(1-2v)}{v}$
K	$\frac{EG}{3(3G-E)}$	—	$\frac{2G(1+v)}{3(1-2v)}$	$\lambda+\frac{2}{3}G$	—	$\frac{E}{3(1-2v)}$	—	—	$\frac{\lambda(1+v)}{3v}$
v	$\frac{E}{2G}-1$	$\frac{3K-2G}{2(3K+G)}$	—	$\frac{\lambda}{2(\lambda+G)}$	$\frac{1}{2}(1-\frac{E}{3K})$	—	—	$\frac{\lambda}{3K-\lambda}$	—
λ	$\frac{G(E-2G)}{3G-E}$	$K-\frac{2}{3}G$	$\frac{2Gv}{1-2v}$	—	$\frac{3K(3K-E)}{9K-E}$	$\frac{Ev}{(1+v)(1-2v)}$	$\frac{3Kv}{1+v}$	—	—

We may add two useful relationships between elastic constants to this table:

$$\frac{G}{\lambda+G} = 1-2v, \qquad \frac{\lambda}{\lambda+2G} = \frac{v}{1+v}. \tag{1.1-5}$$

For usual materials, $0 \leqslant v \leqslant 1/2$, for plastics, $1/3 < v < 1/2$, $8/3 < K/G < \infty$; for rock, $K/G \approx 1.7$; for glass, $K/G \approx 2.7$; for rigid body, $G \to \infty$, $K \to \infty$. If $v = 1/4$, we have $\lambda = G$. This simplifies the equations of elasticity considerably. The special value $v = 1/2$ implies that $G = E/3$, $1/K = 0$, and $\varepsilon_{kk} = 0$.

From (1.1-2) or (1.1-4) we can see that the strain tensor is not proportional to the stress tensor. To reduce the strain-stress relation to a regular form, we may resolve the strain tensor and stress tensor into a *deviatoric tensor* and a *spherical tensor*. The former is related to the distortional change, and the later reflects the volume change of the body.

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} - \frac{\varepsilon_{kk}}{3} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \frac{\varepsilon_{kk}}{3} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \frac{\varepsilon_{kk}}{3} \end{bmatrix} + \begin{bmatrix} \frac{\varepsilon_{kk}}{3} & 0 & 0 \\ 0 & \frac{\varepsilon_{kk}}{3} & 0 \\ 0 & 0 & \frac{\varepsilon_{kk}}{3} \end{bmatrix},$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \varepsilon_{23} \\ \sigma_{31} & \sigma_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} - \frac{\varepsilon_{kk}}{3} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \frac{\varepsilon_{kk}}{3} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \frac{\varepsilon_{kk}}{3} \end{bmatrix} + \begin{bmatrix} \frac{\varepsilon_{kk}}{3} & 0 & 0 \\ 0 & \frac{\varepsilon_{kk}}{3} & 0 \\ 0 & 0 & \frac{\varepsilon_{kk}}{3} \end{bmatrix}. \tag{1.1-6}$$

The components of strain and stress deviators are defined as follows:

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij},$$
$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}. \tag{1.1-7}$$

It is easy to show (see, for example, Reiner [2]) that (1.1-2) or (1.1-4) is equivalent to the distortional equation and the dilatational equation:

$$e_{ij} = s_{ij}/2G \quad \text{or} \quad s_{ij} = 2Ge_{ij},$$
$$\varepsilon_{kk} = \sigma_{kk}/3K \quad \text{or} \quad \sigma_{kk} = 3K\varepsilon_{kk}. \tag{1.1-8}$$

We see that the constitutive equations for isotropic, elastic materials can be resolved into two groups. The deviatoric equation expresses the proportionality between the deviatoric strain tensor and the deviatoric stress tensor. The volume change equation expresses the proportionality between spherical strain tensor and the spherical stress tensor. We shall see later that the constitutive equation for isotropic, viscoelastic material can also be resolved into two parts. That is deviatoric equation and the volume change equation. If we take the Laplace transform of the both sides of the constitutive equation for viscoelastic bodies, we may see that it has the same form as the elastic constitutive equation. This motivates us to solve viscoelastic problem by taking Laplace transform of the governing equations for viscoelastic bodies. In the transformed mapping space, it reduced to the usual elastic problem. If this problem has a solution, the solution of the original viscoelastic problem can be obtained by an inversion.

Before Section 6.8, we assume that the temperature is independent of time and the space place and that displacements and strains are infinitesimal. We begin our discussion from the simple shearing case for viscoelastic bodies and only discuss deviatoric equation. The volume change equation and the uniaxial case can be discussed similarly.

1.2 Stress Relaxation

The phenomenon that the stress in a body reduces with time under constant strain is called *stress relaxation.* We now consider the behavior of a slab of material in simple shearing motion (Fig. 1.2-1). Assume that it can be regarded as homogeneously deformed, with the amount of shear $e(t)$ variable in time. Let $s(t)$ be the shearing stress. Since in this case the only existing component of strain or stress is $e_{xy}(t) = \varepsilon_{xy}(t)$ or $s_{xy}(t) = \sigma_{xy}(t)$, we have ignored the subscript for convenience and simply call it strain or stress.

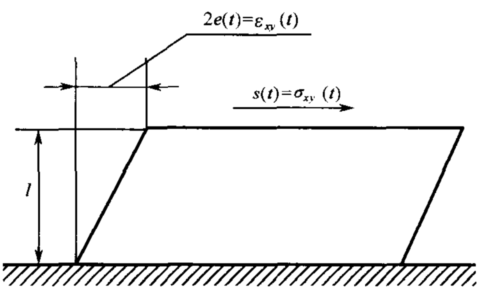

Fig. 1.2-1 Simple shearing history

Consider the stress varying law under the action of the step shearing history $e(t) = e_0\theta(t)$ (at the instant $t = 0$ strain e_0 is applied suddenly and then holds constantly, see Fig. 1.2-2(a)). Here the *Heaviside unit step function* $\theta(t)$ is defined as

$$\theta(t)=\begin{cases}0, & \text{when } t<0,\\ 1, & \text{when } t\geq 0,\end{cases} \tag{1.2-1}$$

In physics, we often use $\theta(t)$ to describe a process of a variable that suddenly change to unit at instant $t = 0$, and then holds constantly. If we multiply it by a continuous function, this implies that the function at the negative semi-axial has been cut out and at $t = 0$ has sudden jump of an amount $f(0)$.

There are two limit cases: First, if the material is an idealized linear-elastic solid, substituting $e(t) = e_0\theta(t)$ into the Hooke's law $s = 2Ge$, we have

$$s(t)=2Ge_0\theta(t)=s_0\theta(t), \tag{1.2-2}$$

The stress suddenly increases from 0 to s_0, and then holds constantly. It will never relax forever (see Fig. 1.2-2(b)).

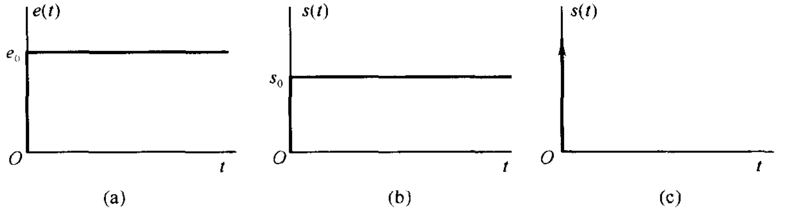

Fig. 1.2-2 Stress relaxation under step-strain history
(a) Step-strain history (b) Elastic bodies (c) Viscous bodies

Second, if the material is an idealized viscous fluid, substituting $e(t) = e_0\theta(t)$ into the *Newton's law* $s = 2\eta\,(\mathrm{d}/\mathrm{d}t)e$, here η is the viscosity coefficient, we have

$$s(t)=2\eta e_0\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}=2\eta e_0\delta(t), \tag{1.2-3}$$

(see Fig. 1.2-2(c)). Here $\delta(t)$ is the *Dirac-delta function*, which is defined as a function with a singularity at the origin:

$$\delta(t)=\begin{cases}\infty, & \text{when } t=0,\\ 0, & \text{when } t\neq 0,\end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty}\delta(t)\mathrm{d}t=\int_{I}\delta(t)\mathrm{d}t=1. \tag{1.2-4}$$

Here I is any segment that includes the point $t = 0$. Equation (1.2-3) indicates that for idealized viscous fluid, $s(t)$ tends to infinite under unit-step shearing strain at $t = 0$. This is due to the rate of strain tends to infinite. When $t > 0$, $s(t)$ completely relaxes to zero. In physics, we often use $\delta(t)$ to denote the distributive intensity function of a concentrated quantity. This quantity often applies at a point, such as a concentrated force, a point electric charge, mass of a mass point, etc. In this manner, we can treat the continuous quantities and the concentrated quantities in a unified form. The $\delta(t)$ function is a kind of the generalized function, which have the following properties:

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t)=\delta(t), \qquad \int_{-\infty}^{t}\delta(t)\mathrm{d}t=\theta(t). \tag{1.2-5}$$

$$\int_{-\infty}^{+\infty} f(t)\delta(t)\mathrm{d}t = f(0), \quad \int_I f(t)\delta(t-t_0)\mathrm{d}t = \begin{cases} f(t_0), & \text{when } I \text{ includes } t=t_0 \\ 0, & \text{when } I \text{ does not include } t=t_0 \end{cases} \tag{1.2-6}$$

where $f(t)$ is a continuous function.

Close observation of real materials shows that neither of these idealizations is quite accurate. Real material usually has both elastic and viscous properties. Under step-shearing strain, the stress usually decreases from its initial value quite rapidly at first, and late more gradually approaching some limiting value. If this limiting value is not zero or is zero, we may call it a solid (Fig. 1.2-3 (a)) or a fluid (Fig. 1.2-3 (b)). Since these limits depend on one's subjective judgment and any body could not wait so long a time, some times it is different to absolutely distinguish a solid and a fluid (Fig. 1.2-3 (c)).

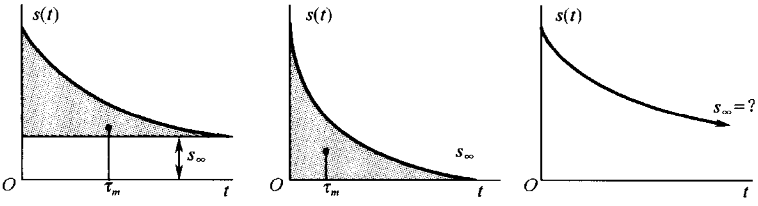

Fig. 1.2-3 Stress relaxation of real materials

We now define the abscissa τ_m of the centroid of the shaded area of Fig. 1.2-3 (a), (b) as the *mean relaxation time*

$$\tau_m = \int_0^\infty t[s(t)-s_\infty]\mathrm{d}t \bigg/ \int_0^\infty [s(t)-s_\infty]\mathrm{d}t. \tag{1.2-7}$$

We use it as an order-of-magnitude estimate of the time required for stress relaxation to approach completion. The smaller the τ_m, the shorter the time required for completing the same percentage of the total quantity of stress relaxation $\{[s_0-s(t)]/[s_0-s_\infty]\}\times100\%$. The distinction between solid and fluid is usually based on a subjective comparison of the relaxation time and the time of observation (Pipkin [3]). We usually call the material a solid or a fluid according to the ratio of the relaxation time over the observation time is relatively long or short. If you are hit on the head by silly putty, you will think that it is a solid even if it can flow like a viscous body. This is due to the application time is so short by comparison with the relaxation time that the process being complete before there is time for much stress relaxation and it reveals elasticity. A tile can jump up and down when you throw it upon the water is another example of fluid can reveal elasticity. On the contrary, if you can dabble your finger in a material, you may probably call it a fluid even though it may return exactly to its initial shape after a long time. That's just the point that we must base on a relatively long observation time when we select rheological models.