Sets, Sequences, and Mappings:
THE BASIC CONCEPTS OF ANALYSIS

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Preface

A great deal has been said in recent years about the widening gap between calculus and advanced calculus. This gap might more appropriately be said to exist between "mechanical and intuitional" and "rigorous" mathematics. Regardless of terminology, the fact remains that many students who continue in mathematics beyond the calculus sequence find it a shocking experience, and are frustrated by the sudden shift in emphasis from the mechanical to the theoretical, from the concrete to the abstract.

We have written this book in the hope that it may help to bridge this gap. Although we rely on intuitional motivation wherever possible, we have tried to maintain a high degree of precision in the statements of definitions, axioms, and theorems, as well as rigor in the proofs. We have also attempted to handle carefully some of those elusive concepts which in calculus are declared "beyond the scope of this book," but which in advanced calculus and other higher level courses are often introduced casually by a statement such as "We assume that the reader is familiar with . . ." Many examples are used to clarify and illustrate new concepts as they are introduced. More than 300 problems, many of which amplify or supplement material in the text, are included. We have deliberately avoided sketches, simply because they sometimes can be as misleading as they are helpful. (How many students, for example, visualize a set of points on the real line as an interval merely because sets are so often illustrated in this fashion?) However, it is recommended that the student and the instructor use sketches freely in their work, bearing in mind the obvious limitations of such sketches.

The first five chapters consist of a systematic development of many of the important properties of the real number system plus careful treatment of such concepts as mappings, sequences, limits, and continuity. The introduction of open sets in Chapter III permits a simultaneous treatment of these concepts from a topological point of view without, however, mentioning the word "topology" in the first five chapters. Thus an analyst might consider this material as a course in "baby real variables," whereas a topologist might consider it as a development of the topology of the real line. The sixth and final chapter discusses metric spaces, and generalizes many of the earlier concepts and results to arbitrary metric spaces.

For review purposes, an index of axiom and key theorems is provided at the end of the book.

For the last three semesters, we have used this text, in its various stages of development, as the basis of a semester course at Harpur College, and have found it to be highly successful. This course is now a prerequisite for our two-semester advanced calculus sequence for mathematics majors, as well as for topology. Our experience has shown that the first five chapters can be covered in a three-hour course, leaving Chapter VI available for additional independent (or honors) work. The entire text could be covered comfortably in a four-hour course. Although intended primarily for sophomores who have completed the calculus sequence, this material has been given to some advanced undergraduates and beginning graduate students, who have found it both stimulating and challenging.

We would like to acknowledge the contributions, either direct or indirect, of a number of our colleagues, notably Professors Howard Alexander, Robert G. Bartle, Guilford Spencer, and Allen D. Ziebur, all of whom suggested improvements in our original manuscript. The concept of uniformly isolated sequences, used extensively in the text, was developed by Professor Norman Levine in a series of articles appearing in "The American Mathematical Monthly." We have drawn heavily on the material in *Elementary Topology*, by Hall and Spencer, published by John Wiley & Sons, and we wish to acknowledge Professor Spencer's co-authorship of this earlier textbook, as well as his excellent review of the present manuscript. Special thanks are due Professor Bartle for his careful review of our work, which resulted in the corrections of many errors and considerable improvement in the entire book.

The expert may wonder at our apparent preoccupation with sequences, particularly in proofs involving constructions and the Axiom of Choice. The treatment used herein is the culmination of many discussions between the authors and Professor Ziebur. At the latter's suggestion, we have introduced an Axiom of Choice for Sequences, and have made careful use of this axiom in many of our proofs. Some of these proofs seem a little complicated, especially for sophomores, and are starred for possible omission at the instructor's discretion. We are indebted to Professor Ziebur for his many helpful suggestions as well as for several of the proofs which he either created or improved.

We also wish to express our appreciation to the Division of Science and Mathematics, Harpur College, State University of New York, and to the staff of John Wiley & Sons for their tremendous help in the preparation of the manuscript.

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I

Introduction to sets and mappings

1. Sets

Mathematics requires precision in the expression of abstract concepts and in the application of logical processes. This precision is attained through the use of special terminology and symbols which eliminate the normal ambiguity in everyday language. A mastery of this terminology and symbolism is essential to the student who expects to continue in the field of mathematics. For this reason, we have attempted to explain each symbol carefully, and to define each new concept precisely, labeling each as a definition for ease of reference. Many terms in everyday usage have a special and precise meaning in mathematics, and the student is cautioned not to confuse the normal usage with the precise mathematical meaning.

The term *set* is used to designate a collection of objects of some kind. These objects are called the *elements*, or *members*, or *points* of the set.

We generally designate sets by capital letters A, B, C, etc., and members of a set by small letters a, b, c, etc. If a set A consists of the points a, b, c, d, we write $A = \{a, b, c, d\}$. If A consists of just the one point a, we write $A = \{a\}$, thus distinguishing between the *point* a and the *set* $\{a\}$ consisting of only the point a. The notation "$a \in A$" means "a is a member of A," or "a belongs to A"; the notation "$a \notin A$" means "a does not belong to A."

Definition 1.1. The *universe* U is the totality of all points under consideration (during any investigation), and is the source from which we extract sets.

It is evident that the universe is itself a set, but it might be considered as "the master set"; that is, we restrict our horizon to the universe at hand, and do not recognize the existence of any other objects (or sets of objects) except those belonging to our universe. Thus, for example, the equation $x^2 + 1 = 0$ has no solution in the universe consisting of all real numbers. When we studied algebra, we found it necessary to enlarge our universe, and this led to a new universe consisting of all complex numbers.

Before stating our next definition, let us digress for a moment to discuss the term "if and only if," which in the sequel will be abbreviated "iff." Let α and β be two declarative statements. A typical theorem of mathematics is a statement of the form "If α is true, then β is true," which is often shortened to "If α, then β," or "α implies β." Mathematicians consider the following statements as equivalent; that is, any two of these statements have precisely the same meaning:

α implies β
α is a sufficient condition for β
β is a necessary condition for α
if α then β
β if α
α only if β

The typical definition in mathematics is a statement of the form "α is true iff β is true." Such a definition has the following equivalent forms:

α is true iff β is true
α is a necessary and sufficient condition for β
β is a necessary and sufficient condition for α
α implies β *and* β implies α

Mathematicians are pleased when they discover theorems which are "iff" statements, since any such theorem provides two equivalent descriptions of the same concept.

Definition 1.2. A set A is a *subset* of a set B iff every point of A is a point of B.

The statement of Definition 1.2 means that both of the following statements are true:

(1) If the set A is a subset of the set B, then every point of A is a point of B.

(2) If every point of the set A is a point of the set B, then A is a subset of B.

The notation $A \subset B$ (or equivalently, $B \supset A$) means "A is a subset of B," or "the set A is contained in the set B," or "B contains A."

It is clear from Definition 1.2 that for every set A, we must have $A \subset A$.

We have seen that the statement $A \subset B$ means that if $p \in A$, then $p \in B$. Thus, if A is not a subset of B (which we write $A \not\subset B$), then there must exist some point $p \in A$ such that $p \notin B$. Now, for any sets A and B, it is clear that one or the other of these conditions must be satisfied. We express this as follows.

Theorem 1.3. If A and B are sets, then either $A \subset B$ or $A \not\subset B$.

Consider the set $\{1, 2, 3, 4, 5\}$, and let it be our universe; that is, $U = \{1, 2, 3, 4, 5\}$. The subsets of the universe U are determined by taking a Gallup poll. For example, suppose we have a subset $K \subset U$, and we want to determine the members of K. We poll the members of U with the following results:

Is 1 in K?	Yes
Is 2 in K?	No
Is 3 in K?	No
Is 4 in K?	Yes
Is 5 in K?	Yes

Since we have polled our entire universe, we conclude that $K = \{1, 4, 5\}$.

Suppose we have another subset $H \subset U$, and here our Gallup poll yields a "no" from every member of U. We must then conclude that the set H contains no elements. For reasons that will become clear later, we still choose to consider H as a set, which we call the empty set and define as follows.

Definition 1.4. The *empty set* is the set containing no elements, and is denoted by ϕ.

Given any set A, it cannot be true that $\phi \not\subset A$, since there are no points in ϕ. Therefore, by Theorem 1.3, we have the following lemma.

Lemma 1.5. If A is any set, $\phi \subset A$.

Returning to the preceding example, suppose we have a subset $J \subset U$, and here our Gallup poll yields a "yes" from every member of U. We must then conclude that the sets J and U consist of exactly the same points, and are thus indistinguishable; that is, we have merely given the same set two different names. This leads us to the notion of equality of sets.

Definition 1.6. Two sets A and B are *equal* (written $A = B$) iff $A \subset B$ and $B \subset A$.

A reasonable question to ask at this point is: How many subsets can be extracted from a universe consisting of n points, where n is any positive integer? The answer lies buried in the technique of our Gallup poll. We have a total of n members to poll, and each member can give us one of two answers, "yes" or "no." If the answer is "yes," the point is in the subset; if the answer is "no," the point is not in the subset. Thus, in choosing subsets, we have two choices for each member of the universe: We can either *take it* as a member of the subset, or *leave it* out. Since our choice at any particular member is independent of our choice at each of the other members, we find that we have $2 \times 2 \times 2 \times \ldots \times 2$ (n factors) choices, or 2^n possible subsets. In our earlier example, where $U = \{1, 2, 3, 4, 5\}$, we have $2^5 = 32$ subsets. We could arrive at this result in still another way by considering the various subsets as combinations of "yes" answers in our poll, and using the theory of combinations from algebra. Remembering that the number of combinations of n things taken r at a time is given by the formula $C(n, r) = n!/[r!(n - r)!]$, we see that the number of subsets consisting of just one point corresponds to the number of combinations of just one "yes" among the five answers, or $C(5, 1) = 5!/(1!4!) = 5$. Similarly, the number of subsets consisting of two points is $C(5, 2) = 5!/(2!3!) = 10$; three points, $C(5, 3) = 10$; four points, $C(5, 4) = 5$; five points, (recalling that $0! = 1$ by definition), $C(5, 5) = 1$, where this single subset with five points is, of course, the set U itself. Finally, the number of subsets containing no points is $C(5, 0) = 1$, where this single set is the empty set. The total number of subsets is then $5 + 10 + 10 + 5 + 1 + 1 = 32$, which agrees with our preceding result.

Definition 1.7. The *intersection* of two sets A and B (written $A \cap B$) is the set of all points which are in both A and B.

Definition 1.8. The *union* of two sets A and B (written $A \cup B$) is the set of all points which are in at least one of the sets A and B.

To illustrate these concepts, let us consider an example.

Example 1.9. Let the universe be $U = \{1, 2, 3, 4, 5, 6, 7\}$, and suppose we have the following subsets:

$$A = \{1, 2, 5\} \qquad C = \{3, 5, 7\}$$
$$B = \{2, 3, 4\} \qquad D = \{1, 2, 4, 6\}$$

Then

$$A \cap B = \{2\}; \quad A \cap D = \{1, 2\}; \quad B \cap D = \{2, 4\}; \quad C \cap D = \phi$$

Also

$$A \cup B = \{1, 2, 3, 4, 5\}; \quad B \cup C = \{2, 3, 4, 5, 7\}; \quad C \cup D = U$$

We shall make free use of parentheses, much the same as in algebra. For example, Definitions 1.7 and 1.8 extend to more than two sets under the convention:

$$A \cap B \cap C = (A \cap B) \cap C;$$
$$A \cap B \cap C \cap D = (A \cap B \cap C) \cap D$$
$$A \cup B \cup C = (A \cup B) \cup C;$$
$$A \cup B \cup C \cup D = (A \cup B \cup C) \cup D, \text{ etc.}$$

Also, parentheses are essential when unions and intersections are combined, as in $A \cup B \cap C$. Here we may choose either the set $(A \cup B) \cap C$ or the set $A \cup (B \cap C)$, and these two sets are generally not equal. To see this, consider the universe U and the sets A, B, and C defined in Example 1.9. A little calculation (verify this) shows that $(A \cup B) \cap C = \{3, 5\}$, whereas $A \cup (B \cap C) = \{1, 2, 3, 5\}$.

We have seen that a set may be empty, and thus, if we are considering an arbitrary set A, we must allow for the possibility that $A = \phi$. (When we wish to exclude this possibility, we speak of a *nonempty set* A.) This brings up the following natural question: What happens if the empty set ϕ is involved in a union or an intersection? The answer follows immediately from Definitions 1.7 and 1.8, but because of its importance we state it for easy reference as a lemma.

Lemma 1.10. If A is any set (including $A = \phi$), then $A \cap \phi = \phi$ and $A \cup \phi = A$.

One of the important techniques of set theory is that of proving set equalities. Suppose, for instance, that we want to prove the following:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

We might first test it by trying it on some particular sets. Let us again use Example 1.9. We see that $A = \{1, 2, 5\}$, and $B \cup C = \{2, 3, 4, 5, 7\}$, so that $A \cap (B \cup C) = \{2, 5\}$. On the other hand, $A \cap B = \{2\}$, and $A \cap C = \{5\}$, so that

$$(A \cap B) \cup (A \cap C) = \{2, 5\}$$

We have thus *verified* the equality for our particular choice of sets. Note, however, that we have not proved the statement, any more than we can prove the trigonometric identity $\sin 2x = 2 \sin x \cos x$ merely by verifying it for the particular choice $x = 0$. However, we cannot be sure that the statement is wrong either. Therefore it seems reasonable to try to prove it. The general method for proving equality of sets is indicated by Definition 1.6; that is, we show that each set is contained in the other. For simplicity of notation, let us designate the left-hand set above by L, and the right-hand set by R, so that we wish to prove $L = R$.

We choose an arbitrary point $p \in L$, and show that $p \in R$. Since the choice of p is arbitrary, we conclude that every point of L is also a point of R; that is, $L \subset R$. Similarly, we choose an arbitrary point $p \in R$, show that $p \in L$, and conclude that $R \subset L$; hence $L = R$. We now state the set equality as a theorem, and give a formal proof, following the preceding method. In this first proof, we have numbered each step, and we have also supplied reasons for each conclusion in the first half of the proof. This procedure is employed merely to clarify the technique, and is not continued in subsequent proofs.

Theorem 1.11 (The Distributive Law for Intersections over Unions). If A, B, C are sets, then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof. We prove first: $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

(1) If $A \cap (B \cup C) = \phi$, the result is trivial. (by Lemma 1.5)

(2) If $A \cap (B \cup C) \neq \phi$, let $p \in A \cap (B \cup C)$.

(3) Then $p \in A$ and $p \in (B \cup C)$. (by Definition 1.7)

(4) But $p \in (B \cup C)$ means $p \in B$ or $p \in C$. (by Definition 1.8)

(5) ***Case 1.*** If $p \in B$, then $p \in (A \cap B)$. (by Definition 1.7 and (3))

Therefore $p \in (A \cap B) \cup (A \cap C)$ (by Definition 1.8)

Case 2. If $p \in C$, then $p \in (A \cap C)$. (by Definition 1.7 and (3))

Therefore $p \in (A \cap B) \cup (A \cap C)$. (by Definition 1.8)

(6) Consequently $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. (by steps (2) to (5) and Definition 1.2)

To complete the proof of the theorem, we must show that

$$(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$$

(1) If $(A \cap B) \cup (A \cap C) = \phi$, the result is trivial.

(2) If $(A \cap B) \cup (A \cap C) \neq \phi$, let $p \in (A \cap B) \cup (A \cap C)$.

(3) Then, $p \in (A \cap B)$ or $p \in (A \cap C)$.

(4) ***Case 1.*** If $p \in (A \cap B)$, then $p \in A$ and $p \in B$.
Now $p \in B$ implies $p \in (B \cup C)$.
Therefore $p \in A \cap (B \cup C)$.

Case 2. If $p \in (A \cap C)$, then $p \in A$ and $p \in C$.
Now, $p \in C$ implies $p \in (B \cup C)$.
Therefore $p \in A \cap (B \cup C)$.

(5) Hence $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Consequently $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. QED

The second half of Theorem 1.11 can also be established in the following neat way, using Problem 8. We see at once that

$$B \subset B \cup C \quad \text{and} \quad C \subset B \cup C$$

By Problem 8,

$$A \cap B \subset A \cap (B \cup C) \quad \text{and} \quad A \cap C \subset A \cap (B \cup C)$$

Therefore $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. QED

Theorem 1.12. Given any set A in the universe U, there is exactly one set X such that both of the following conditions hold:

$$(a)\ A \cup X = U \qquad (b)\ A \cap X = \phi$$

Proof. Suppose that there exist two such sets X and Y satisfying these conditions. Then

$$A \cup X = U, \qquad A \cap X = \phi, \qquad A \cup Y = U, \qquad A \cap Y = \phi$$

Consequently

$$Y = Y \cap U = Y \cap (A \cup X) = (Y \cap A) \cup (Y \cap X) = Y \cap X$$

Therefore, by Problem 3, $Y \subset X$. By a similar argument, $X \subset Y$. Therefore $X = Y$, and we have proven that there cannot be two sets satisfying the given conditions. There is at least one set satisfying these conditions, the set X defined as all points of U not in A. This completes the proof of the theorem. QED

We define the complement of a set A in the universe U to be the unique set X defined in Theorem 1.12. The word "unique" as used in mathematics means "one and only one."

Definition 1.13. The *complement* of a set A in the universe U is the unique subset X of U satisfying the two conditions

$$A \cup X = U \qquad A \cap X = \phi$$

We denote the complement of A by $C(A)$, and observe that $C(A)$ consists of all elements of U which are not in A.

It follows from Definition 1.13 that if U is the universe, then $C(U) = \phi$ and $C(\phi) = U$.

In general, when we are considering arbitrary sets, we may tacitly assume that some universe U is given, and that all the sets we are considering are subsets of U. Similarly, we may discuss the complement of a set, again assuming that we have taken the complement with respect to our universe U.

Theorem 1.14. For any set A, $C[C(A)] = A$.

Proof. We see from Definition 1.13 that $C[C(A)]$ is the unique set X satisfying both of the equations

$$C(A) \cup X = U \qquad C(A) \cap X = \phi$$

Since $X = A$ satisfies both these equations, we have at once $A = C[C(A)]$ QED

Theorem 1.15. If A and B are sets, then $A \subset B$ iff $C(B) \subset C(A)$.

Proof. Suppose first that $A \subset B$. By Problem 8,

$$A \cap C(B) \subset B \cap C(B) = \phi$$

so that $A \cap C(B) = \phi$. Consequently

$$\begin{aligned} C(B) &= C(B) \cap [A \cup C(A)] \\ &= [C(B) \cap A] \cup [C(B) \cap C(A)] = C(B) \cap C(A) \end{aligned}$$

Therefore, by Problem 3, $C(B) \subset C(A)$.

For the converse, suppose that $C(B) \subset C(A)$. Then by the preceding paragraph, $C[C(A)] \subset C[C(B)]$, which, by Theorem 1.14, can be written in the form $A \subset B$. This completes the proof.

QED

The next theorem states some important relationships among complements, unions, and intersections.

Theorem 1.16 (DeMorgan's Laws). If A and B are sets, then

(*a*) $C(A \cup B) = C(A) \cap C(B)$; (*b*) $C(A \cap B) = C(A) \cup C(B)$

Proof of (*a*). We prove that $C(A \cup B) \subset C(A) \cap C(B)$. It is easily seen that $A \subset A \cup B$ and $B \subset A \cup B$. Therefore, by Theorem 1.15, $C(A \cup B) \subset C(A)$ and $C(A \cup B) \subset C(B)$. Consequently

$$C(A \cup B) \subset C(A) \cap C(B)$$

To complete the proof of (*a*), we must show that

$$C(A) \cap C(B) \subset C(A \cup B)$$

By Lemma 1.6, we lose no generality if we assume that the left-hand set is non-empty. Let $p \in C(A) \cap C(B)$. Then $p \in C(A)$ and $p \in C(B)$, so $p \notin A$ and $p \notin B$. Thus $p \notin (A \cup B)$, and hence it follows that $p \in C(A \cup B)$. Therefore

$$C(A) \cap C(B) \subset C(A \cup B)$$

and (*a*) is proved.

The proof of (*b*) is left as an exercise.

Let us now consider an application of set theory where the universe does not consist of numbers. Let a and b be two statements, each of which in a given situation is either true or false. Let S be the set of all logical situations in which a and b occur;

that is, S is the universe in this case. We may then define the following sets:

$$
\begin{aligned}
A &= \text{set of all situations in which } a \text{ is true.}\\
C(A) &= \text{set of all situations in which } a \text{ is false.}\\
B &= \text{set of all situations in which } b \text{ is true.}\\
C(B) &= \text{set of all situations in which } b \text{ is false.}
\end{aligned}
$$

Now suppose we make the following *direct statement*, which we may assert as a theorem: "If a is true, then b is true." This says that, of all the situations in S, those for which a is true must be included among those for which b is true; or, in terms of the sets just defined, $A \subset B$. We can now apply Theorem 1.14 to obtain $C(B) \subset C(A)$. Using the same definitions, this is the same as the assertion: "If b is false, then a is false." This latter statement is the *contrapositive* of the given direct statement, and we have just indicated by means of set theory that *any direct statement is logically equivalent to its contrapositive statement.*

We now prove a simple theorem from number theory to illustrate the contrapositive technique of proof.

Theorem 1.17. If the square of a positive integer is even, then the positive integer is even.

Proof. The contrapositive of the given statement is as follows: If a positive integer is odd (that is, not even), then its square is odd. To prove this, we note that any positive integer which is odd can be written in the form $2k - 1$, where k is a positive integer. The square of this integer is then

$$(2k - 1)^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k) + 1$$

which is clearly an odd integer. QED

PROBLEMS

1. If $U = \{a, b, c, d\}$, find all the subsets of U.

2. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and let A, B, and H be defined as follows: $A = \{6, 8, 9\}$, $B = \{1, 3, 7, 8, 9\}$, $H = \{2, 6, 8, 9\}$. Then

(*a*) Find $A \cup B$ and $A \cap H$.

(*b*) Find $(A \cap B) \cup H$.

(*c*) Find $C(B)$.

(*d*) Find the set K defined as follows: The point x is in K iff $x + 3$ is in A.