S. Lefschetz

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Graphs and Networks
The Picard-Lefschetz Theory and Feynman Integrals

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Solomon Lefschetz (1884-1972) was one of the great mathematicians of his generation. This volume published posthumously and completed shortly before his death is in his own unique and vigorous style. Were he alive there are many people whom he would thank. Among them are Sandra Spinacci for the careful typing of his manuscript, Mauricio Peixoto for his constant encouragement, and John Mallet-Paret for his careful reading of the manuscript.

January 1975 J. P. LaSalle

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PART I

APPLICATION OF CLASSICAL TOPOLOGY TO GRAPHS AND NETWORKS

PREFACE

This monograph is based, in part, upon lectures given in the Princeton School of Engineering and Applied Science. It presupposes mainly an elementary knowledge of linear algebra and of topology. In topology the limit is dimension two mainly in the latter chapters and questions of topological invariance are carefully avoided.

From the technical viewpoint graphs is our only requirement. However, later, questions notably related to Kuratowski's classical theorem have demanded an easily provided treatment of 2-complexes and surfaces.

January 1972 Solomon Lefschetz

INTRODUCTION

The study of electrical networks rests upon preliminary theory of graphs. In the literature this theory has always been dealt with by special ad hoc methods. My purpose here is to show that actually this theory is nothing else than the first chapter of classical algebraic topology and may be very advantageously treated as such by the well known methods of that science.

Part I of this volume covers the following ground: The first two chapters present, mainly in outline, the needed basic elements of linear algebra. In this part duality is dealt with somewhat more extensively. In Chapter III the merest elements of general topology are discussed. Graph theory proper is covered in Chapters IV and V, first structurally and then as algebra. Chapter VI discusses the applications to networks. In Chapters VII and VIII the elements of the theory of 2-dimensional complexes and surfaces are presented. They are applied in Chapter IX, the last of Part I, to the important question of planar graphs, Kuratowski related theorem, and dual networks.

It is to be noted that in the electrical part, linearity has nowhere been assumed. In general as regards networks, I have been considerably inspired by the splendid paper of Brayton and Moser: A theory of nonlinear networks, Quaterly of Applied Mathematics, Vol. 29, pp. 1-33, 81-104, 1964.

The exposition of the material is new in many parts; moreover in certain parts the material is completely new. This is notably the case in Chapter IX.

CHAPTER I

A RÉSUMÉ OF LINEAR ALGEBRA

Two elements dominate linear algebra: matrices and vectors. One may identify vectors with certain matrices but not vice versa. Thus matrices are the dominant feature. We shall, therefore, first deal with matrices and then with vectors.

As appropriate for a résumé, proofs will rarely be given and for them the reader is referred to any standard text on the subject.

1. Matrices

A matrix is a rectangular array of elements

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & . \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Such an array, known as $m \times n$ matrix is usually abridged as $[a_{jk}]$ or even written a. The standard matrix operations are:

Addition: The sum of two $m \times n$ matrices, a as above and $b = [b_{jk}]$ is the matrix

$$a + b = [a_{jk} + b_{jk}];$$

Product: With a as before and b an $n \times p$ matrix one defines

$$a\,b = \left[\sum_s a_{js}\, b_{sk}\right].$$

The implication is that in both addition and multiplication the operations indicated have a meaning. This is usually clear from the context but one must not be entirely careless about it.

The <u>transpose</u> a' of the $m \times n$ matrix a is the $n \times m$ matrix obtained by permuting the rows and columns of a. Note that if ab has a meaning $(ab)' = b'a'$.

The <u>derivative</u> of a matrix $a(t) = [a_{jk}(t)]$ of elements differentiable functions of t is

$$\dot{a}(t) = [\dot{a}_{jk}(t)].$$

<u>Square matrices</u>. These are the $n \times n$ matrices. The number n is the <u>order</u> of the matrix.

A square numerical, $n \times n$ matrix has a determinant written $|a_{jk}|$ or $|a|$. The matrix is singular if $|a| = 0$, nonsingular otherwise.

The square matrix with diagonal $a_1,\ldots,a_n$ and zeros outside is frequently written $\mathrm{diag}(a_1,\ldots,a_n)$. The unit matrix of order n, written E_n or E (when n is obvious) is $\mathrm{diag}(1,1,\ldots,1)$ (n terms).

A nonsingular matrix a has an inverse a^{-1} characterized by $aa^{-1} = a^{-1}a = E$. If $|a| \neq 0$, $|b| \neq 0$, then $(ab)^{-1} = b^{-1}a^{-1}$.

Recall this important property: <u>inversion and transposition commute</u>. That is $(a^{-1})' = (a')^{-1}$.

Evidently, sums and products of $n \times n$ matrices are $n \times n$ matrices.

<u>Rank of a matrix</u>. The rank ρ of an $m \times n$ numerical matrix a is the largest order of a nonzero determinant formed from the elements of a.

(1.1) <u>Theorem</u>. <u>Let</u> a <u>be an</u> $m \times n$ <u>matrix and</u> b,c <u>nonsingular square matrices of respective order</u> m,n. <u>Then</u> $\mathrm{rank}\ a = \mathrm{rank}\ b\,a\,c$.

It is convenient to note that if $a = [a_{jk}]$ is $m \times n$ and $b = \mathrm{diag}(b_1,\ldots,b_m)$, $c = \mathrm{diag}(c_1,\ldots,c_n)$.

Then

$$ba = [b_j\, a_{jk}], \quad ac = [a_{jk}\, c_k].$$

2. Vectors and Vector Spaces

Vectors are inextricably mixed with a collection of numbers, the scalars, called a field. A field is simply any set of elements obeying the ordinary rules of rational operations, for example all real or all complex numbers. However an interesting field is made up of just two elements 0 and 1 under these rules:

$$0.0 = 0.1 = 1.0 = 1 + 1 = 0; \; 1.1 = 1.$$

In that field, called the field mod 2, $x = -x$, $\frac{1}{x} = x$, $(x \neq 0)$ hence subtraction and division may be forgotten. This is the ideal field in geometric questions in which direction does not occur.

Take now a fixed field F and n elements $A_1,\ldots,A_n$ which obey no special relation (pure symbols). Form all the expressions

$$A = \alpha_1\, A_1 + \cdots + \alpha_n\, A_n$$

with coefficients in F, the obvious rule for addition and the conventions $A = 0$ if every $\alpha_h = 0$, likewise

$$\alpha A = (\alpha\alpha_1)\, A_1 + \cdots + (\alpha\alpha_n)\, A_n$$

for every α in F. The collection of all expressions A is a vector space V, the elements A are the vectors.

The vectors $B_1, B_2,\ldots,B_p$ are linearly dependent if there exists a relation

$$\beta_1\, B_1 + \cdots + \beta_r\, B_r = 0, \; \beta_h \text{ in } F$$

with the β_h not all zero (non-trivial relation). If no such relation exists the B_h are linearly independent (the term "linearly" is often omitted in such statements). The maximum number of linearly independent vectors is the dimension of V

$$\dim V = n. \tag{2.1}$$

Bases. A base for the space V is a set $B_1, \ldots, B_s$ of independent vectors such that every vector C satisfies a relation

$$C = \beta_1 B_1 + \cdots + \beta_s B_s, \quad \beta_h \text{ in } F.$$

(2.2) A base consists exactly of $n (= \dim V)$ elements.

(2.3) Any n independent elements form a base. Hence $A_1, \ldots, A_n$ is a base.

Isomorphism. Two vector spaces V,W over the same field F are isomorphic, written $V \sim W$, if there is a one-one correspondence between their elements preserving the relations of dependence between them. That is if $B_1, \ldots, B_s$ are elements of V and C_h corresponds to B_h then the relations

$$\sum \beta_h B_h = 0, \quad \sum \beta_h C_h = 0$$

imply one another.

(2.4) N.a.s.c. to have $V \sim W$ is that they have the same dimension.

(2.5) If $V \sim W$ one may select for them respective bases $\{B_h\}$, $\{C_h\}$ such that the isomorphism between them associates $\sum \beta_h B_h$ with $\sum \beta_h C_h$.

Change of base. Let $\{B_h\}$, $\{C_h\}$ be two bases for the same vector space V. We have the relations

$$C_h = \sum \gamma_{hj} B_j, \quad B_h = \sum \beta_{hj} C_j$$

with the β,γ in the field F. As a consequence there follow

$$B_h = \sum_s \beta_{hs} \gamma_{sk} B_k, \quad h = 1,2,\ldots,n. \tag{2.6}$$

However, since the B_h are independent these relations must be identically true, that is

$$\sum_s \beta_{hs} \gamma_{sk} = \begin{cases} 1 & \text{if } h = k \\ 0 & \text{otherwise} \end{cases}.$$

This means that the product

$$[\beta_{hj}]\cdot[\gamma_{hj}] = E \tag{2.7}$$

and implies for the determinants

$$|\beta_{hj}|\cdot|\gamma_{hj}| = 1.$$

Consequently, the matrices $[\beta_{hj}]$ and $[\gamma_{hj}]$ are non-singular. Conversely any relation

$$C_h = \sum \gamma_{hj} B_j, \quad |\gamma_{hj}| \neq 0 \tag{2.8}$$

is a change of base from $\{B_h\}$ to $\{C_h\}$ for the space V.

Remark. The important properties of the space V are those which are invariant with respect to changes of base. For the present we only have the dimension, but other properties will appear in the application to graphs.

Direct sum. Let V_1, V_2 be two vector subspaces of V (vector spaces over the same field whose vectors are all in V). We say that V is their direct sum and write

$$V = V_1 \oplus V_2$$