

S. Lefschetz

**Applied
Mathematical
Sciences
16**

Applications of Algebraic Topology



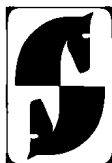
**Springer-verlag
New York Heidelberg Berlin**

S. Lefschetz

Applications of Algebraic Topology

Graphs and Networks
The Picard-Lefschetz Theory
and Feynman Integrals

With 52 Illustrations



Springer-Verlag New York · Heidelberg · Berlin
1975

S. Lefschetz

Formerly of Princeton University

AMS Classifications: 55-01, 55A15, 81A15

Library of Congress Cataloging in Publication Data

Lefschetz, Solomon, 1884-1972.

Applications of algebraic topology.

(Applied mathematical sciences; v. 16)

Bibliography: p.

Includes index.

1. Algebraic topology. 2. Graph theory. 3. Electric networks. 4. Feynman integrals. I. Title.

II. Series.

QA1.A647 vol. 16 [QA611] 510'.8 [514'.2] 75-6924

ISBN 0-387-90137-X

All rights reserved.

No part of this book may be translated or reproduced in any form without written permission from Springer-Verlag.

© 1975 by Springer-Verlag New York Inc.

Printed in the United States of America.

ISBN 0-387-90137-X Springer-Verlag New York • Heidelberg • Berlin

ISBN 0-540-90137-X Springer-Verlag Berlin • Heidelberg • New York

Applied Mathematical Sciences | Volume 16

Solomon Lefschetz (1884-1972) was one of the great mathematicians of his generation. This volume published posthumously and completed shortly before his death is in his own unique and vigorous style. Were he alive there are many people whom he would thank. Among them are Sandra Spinacci for the careful typing of his manuscript, Mauricio Peixoto for his constant encouragement, and John Mallet-Paret for his careful reading of the manuscript.

January 1975

J. P. LaSalle

TABLE OF CONTENTS

PART I

APPLICATION OF CLASSICAL TOPOLOGY
TO GRAPHS AND NETWORKS

	<u>Page</u>
INTRODUCTION	4
CHAPTER I. A RÉSUMÉ OF LINEAR ALGEBRA	5
1. Matrices	5
2. Vector and Vector Spaces	7
3. Column Vectors and Row Vectors	10
4. Application to Linear Equations	11
CHAPTER II. DUALITY IN VECTOR SPACES	13
1. General Remarks on Duality	13
2. Questions of Nomenclature	14
3. Linear Functions on Vector Spaces.	15
Multiplication	15
4. Linear Transformations. Duality	16
5. Vector Space Sequence of Walter Mayer	18
CHAPTER III. TOPOLOGICAL PRELIMINARIES	22
1. First Intuitive Notions of Topology	22
2. Affine and Euclidean Spaces	25
3. Continuity, Mapping, Homeomorphism	26
4. General Sets and Their Combinations	27
5. Some Important Subsets of a Space	28
6. Connectedness	29
7. Theorem of Jordan-Schoenflies	30
CHAPTER IV. GRAPHS. GEOMETRIC STRUCTURE	34
1. Structure of Graphs	34
2. Subdivision. Characteristic Betti Number	37
CHAPTER V. GRAPH ALGEBRA	43
1. Preliminaries	43
2. Dimensional Calculations	46
3. Space Duality. Co-theory	48
CHAPTER VI. ELECTRICAL NETWORKS	51
1. Kirchoff's Laws	51
2. Different Types of Elements in the Branches	53
3. A Structural Property	54
4. Differential Equations of an	56
Electrical Network	56
CHAPTER VII. COMPLEXES	61
1. Complexes	61
2. Subdivision	64
3. Complex Algebra	66
4. Subdivision Invariance	68
CHAPTER VIII. SURFACES	71
1. Definition of Surfaces	71
2. Orientable and Nonorientable Surfaces	72
3. Cuts	76
4. A Property of the Sphere	79

Page

	5. Reduction of Orientable Surfaces to a Normal Form	83
	6. Reduction of Nonorientable Surfaces to a Normal Form	84
	7. Duality in Surfaces	86
CHAPTER IX.	PLANAR GRAPHS	89
	1. Preliminaries	89
	2. Statement and Solution of the Spherical Graph Problem	90
	3. Generalization	95
	4. Direct Characterization of Planar Graphs by Kuratowski	96
	5. Reciprocal Networks	103
	6. Duality of Electrical Networks	104
PART II		
THE PICARD-LEFSCHETZ THEORY AND FEYNMAN INTEGRALS		
INTRODUCTION		113
CHAPTER I.	TOPOLOGICAL AND ALGEBRAIC CONSIDERATIONS	119
	1. Complex Analytic and Projective Spaces	119
	2. Application to Complex Projective n-space \mathcal{P}^n	119
	3. Algebraic Varieties	121
	4. A Résumé of Standard Notions of Algebraic Topology	124
	5. Homotopy. Simplicial Mappings	128
	6. Singular Theory	129
	7. The Poincaré Group of Paths	130
	8. Intersection Properties for Orientable M^{2n} Complex	131
	9. Real Manifolds	133
CHAPTER II.	THE PICARD-LEFSCHETZ THEORY	135
	1. Genesis of the Problem	135
	2. Method	136
	3. Construction of the Lacets of Surface ϕ_z	138
	4. Cycles of ϕ_z . Variations of Integrals Taken On ϕ_z	140
	5. An Alternate Proof of the Picard-Lefschetz Theorem	140
	6. The Λ_1 -manifold M . Its Cycles and Their Relation to Variations	146
CHAPTER III.	EXTENSION TO HIGHER VARIETIES	149
	1. Preliminary Remarks	149
	2. First Application	150
	3. Extension to Multiple Integrals	151

	<u>Page</u>
4. The 2-Cycles of an Algebraic Surface	152
CHAPTER IV. FEYNMAN INTEGRALS	154
1. On Graphs	154
2. Algebraic Properties	156
3. Feynman Graphs	160
4. Feynman Integrals	162
5. Singularities	163
6. Polar Loci	164
7. More General Singularities	168
8. On the Loop-Complex	170
9. Some Complements	170
10. Examples	171
11. Calculation of an Integral	174
12. A Final Observation	175
CHAPTER V. FEYNMAN INTEGRALS. B.	177
1. Introduction	177
2. General Theory	177
3. Relative Theory	178
4. Application to Graphs	178
5. On Certain Transformations	180
BIBLIOGRAPHY	181
SUBJECT INDEX PART I	183
SUBJECT INDEX PART II	187

PART I

APPLICATION OF CLASSICAL TOPOLOGY
TO GRAPHS AND NETWORKS

PREFACE

This monograph is based, in part, upon lectures given in the Princeton School of Engineering and Applied Science. It presupposes mainly an elementary knowledge of linear algebra and of topology. In topology the limit is dimension two mainly in the latter chapters and questions of topological invariance are carefully avoided.

From the technical viewpoint graphs is our only requirement. However, later, questions notably related to Kuratowski's classical theorem have demanded an easily provided treatment of 2-complexes and surfaces.

January 1972

Solomon Lefschetz

INTRODUCTION

The study of electrical networks rests upon preliminary theory of graphs. In the literature this theory has always been dealt with by special ad hoc methods. My purpose here is to show that actually this theory is nothing else than the first chapter of classical algebraic topology and may be very advantageously treated as such by the well known methods of that science.

Part I of this volume covers the following ground: The first two chapters present, mainly in outline, the needed basic elements of linear algebra. In this part duality is dealt with somewhat more extensively. In Chapter III the merest elements of general topology are discussed. Graph theory proper is covered in Chapters IV and V, first structurally and then as algebra. Chapter VI discusses the applications to networks. In Chapters VII and VIII the elements of the theory of 2-dimensional complexes and surfaces are presented. They are applied in Chapter IX, the last of Part I, to the important question of planar graphs, Kuratowski related theorem, and dual networks.

It is to be noted that in the electrical part, linearity has nowhere been assumed. In general as regards networks, I have been considerably inspired by the splendid paper of Brayton and Moser: A theory of nonlinear networks, Quarterly of Applied Mathematics, Vol. 29, pp. 1-33, 81-104, 1964.

The exposition of the material is new in many parts; moreover in certain parts the material is completely new. This is notably the case in Chapter IX.

CHAPTER I

A RÉSUMÉ OF LINEAR ALGEBRA

Two elements dominate linear algebra: matrices and vectors. One may identify vectors with certain matrices but not vice versa. Thus matrices are the dominant feature. We shall, therefore, first deal with matrices and then with vectors.

As appropriate for a résumé, proofs will rarely be given and for them the reader is referred to any standard text on the subject.

1. Matrices

A matrix is a rectangular array of elements

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \cdot \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Such an array, known as $m \times n$ matrix is usually abridged as $[a_{jk}]$ or even written a . The standard matrix operations are:

Addition: The sum of two $m \times n$ matrices, a as above and $b = [b_{jk}]$ is the matrix

$$a + b = [a_{jk} + b_{jk}];$$

Product: With a as before and b an $n \times p$ matrix one defines

$$a b = \left[\sum_s a_{js} b_{sk} \right].$$

The implication is that in both addition and multiplication the operations indicated have a meaning. This is usually clear from the context but one must not be entirely careless about it.

The transpose a' of the $m \times n$ matrix a is the $n \times m$ matrix obtained by permuting the rows and columns of a . Note that if ab has a meaning $(ab)' = b'a'$.

The derivative of a matrix $a(t) = [a_{jk}(t)]$ of elements differentiable functions of t is

$$\dot{a}(t) = [\dot{a}_{jk}(t)].$$

Square matrices. These are the $n \times n$ matrices. The number n is the order of the matrix.

A square numerical, $n \times n$ matrix has a determinant written $|a_{jk}|$ or $|a|$. The matrix is singular if $|a| = 0$, nonsingular otherwise.

The square matrix with diagonal a_1, \dots, a_n and zeros outside is frequently written $\text{diag}(a_1, \dots, a_n)$. The unit matrix of order n , written E_n or E (when n is obvious) is $\text{diag}(1, 1, \dots, 1)$ (n terms).

A nonsingular matrix a has an inverse a^{-1} characterized by $aa^{-1} = a^{-1}a = E$. If $|a| \neq 0$, $|b| \neq 0$, then $(ab)^{-1} = b^{-1}a^{-1}$.

Recall this important property: inversion and transposition commute. That is $(a^{-1})' = (a')^{-1}$.

Evidently, sums and products of $n \times n$ matrices are $n \times n$ matrices.

Rank of a matrix. The rank ρ of an $m \times n$ numerical matrix a is the largest order of a nonzero determinant formed from the elements of a .

(1.1) Theorem. Let a be an $m \times n$ matrix and b, c nonsingular square matrices of respective order m, n . Then
 $\text{rank } a = \text{rank } b a c$.

It is convenient to note that if $a = [a_{jk}]$ is $m \times n$ and $b = \text{diag}(b_1, \dots, b_m)$, $c = \text{diag}(c_1, \dots, c_n)$.

Then

$$ba = [b_j a_{jk}], \quad ac = [a_{jk} c_k].$$

2. Vectors and Vector Spaces

Vectors are inextricably mixed with a collection of numbers, the scalars, called a field. A field is simply any set of elements obeying the ordinary rules of rational operations, for example all real or all complex numbers. However an interesting field is made up of just two elements 0 and 1 under these rules:

$$0.0 = 0.1 = 1.0 = 1 + 1 = 0; \quad 1.1 = 1.$$

In that field, called the field mod 2, $x = -x$, $\frac{1}{x} = x$, ($x \neq 0$) hence subtraction and division may be forgotten. This is the ideal field in geometric questions in which direction does not occur.

Take now a fixed field F and n elements A_1, \dots, A_n which obey no special relation (pure symbols). Form all the expressions

$$A = \alpha_1 A_1 + \dots + \alpha_n A_n$$

with coefficients in F , the obvious rule for addition and the conventions $A = 0$ if every $\alpha_h = 0$, likewise

$$\alpha A = (\alpha \alpha_1) A_1 + \dots + (\alpha \alpha_n) A_n$$

for every α in F . The collection of all expressions A is a vector space V , the elements A are the vectors.

The vectors B_1, B_2, \dots, B_p are linearly dependent if there exists a relation

$$\beta_1 B_1 + \dots + \beta_r B_r = 0, \quad \beta_h \text{ in } F$$

with the β_h not all zero (non-trivial relation). If no such relation exists the B_h are linearly independent (the term "linearly" is often omitted in such statements). The maximum number of linearly independent vectors is the dimension of V

$$\dim V = n. \quad (2.1)$$

Bases. A base for the space V is a set B_1, \dots, B_s of independent vectors such that every vector C satisfies a relation

$$C = \beta_1 B_1 + \dots + \beta_s B_s, \quad \beta_h \text{ in } F.$$

(2.2) A base consists exactly of $n (= \dim V)$ elements.

(2.3) Any n independent elements form a base. Hence A_1, \dots, A_n is a base.

Isomorphism. Two vector spaces V, W over the same field F are isomorphic, written $V \sim W$, if there is a one-one correspondence between their elements preserving the relations of dependence between them. That is if B_1, \dots, B_s are elements of V and C_h corresponds to B_h then the relations

$$\sum \beta_h B_h = 0, \quad \sum \beta_h C_h = 0$$

imply one another.

(2.4) N.a.s.c. to have $V \sim W$ is that they have the same dimension.

(2.5) If $V \sim W$ one may select for them respective bases $\{B_h\}, \{C_h\}$ such that the isomorphism between them associates $\sum \beta_h B_h$ with $\sum \beta_h C_h$.

Change of base. Let $\{B_h\}, \{C_h\}$ be two bases for the same vector space V . We have the relations

$$C_h = \sum \gamma_{hj} B_j, \quad B_h = \sum \beta_{hj} C_j$$

with the β, γ in the field F . As a consequence there follow

$$B_h = \sum_s \beta_{hs} \gamma_{sk} B_k, \quad h = 1, 2, \dots, n. \quad (2.6)$$

However, since the B_h are independent these relations must be identically true, that is

$$\sum_s \beta_{hs} \gamma_{sk} = \begin{cases} 1 & \text{if } h = k \\ 0 & \text{otherwise} \end{cases}.$$

This means that the product

$$[\beta_{hj}] \cdot [\gamma_{hj}] = E \quad (2.7)$$

and implies for the determinants

$$|\beta_{hj}| \cdot |\gamma_{hj}| = 1.$$

Consequently, the matrices $[\beta_{hj}]$ and $[\gamma_{hj}]$ are non-singular. Conversely any relation

$$C_h = \sum \gamma_{hj} B_j, \quad |\gamma_{hj}| \neq 0 \quad (2.8)$$

is a change of base from $\{B_h\}$ to $\{C_h\}$ for the space V .

Remark. The important properties of the space V are those which are invariant with respect to changes of base. For the present we only have the dimension, but other properties will appear in the application to graphs.

Direct sum. Let V_1, V_2 be two vector subspaces of V (vector spaces over the same field whose vectors are all in V). We say that V is their direct sum and write

$$V = V_1 \oplus V_2$$