# THE PICTURE BOOK OF QUANTUM MECHANICS

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# Preface

Students of classical mechanics can rely on a wealth of experience from everyday life to help them understand and apply mechanical concepts. Even though a stone is not a mass point, the experience of throwing stones certainly helps them to understand and analyze the trajectory of a mass point in a gravitational field. Moreover, students can solve many mechanical problems on the basis of Newton's laws and, in doing so, gain additional experience. When studying wave optics, they find that their knowledge of water waves, as well as experiments in a ripple tank, are very helpful in forming an intuition about the typical wave phenomena of interference and diffraction.

In quantum mechanics, however, beginners are without any intuition. Because quantum-mechanical phenomena happen on an atomic or a subatomic scale, we have no experience of them in daily life. The experiments in atomic physics involve more or less complicated apparatus and are by no means simple to interpret. Even if students are able to take Schrödinger's equation for granted, as many students do Newton's laws, it is not easy for them to acquire experience in quantum mechanics through the solution of problems. Only very few problems can be treated without a computer. Moreover, when solutions in closed form are known, their complicated structure and the special mathematical functions, which students are usually encountering for the first time, constitute severe obstacles to developing a heuristic comprehension. The most difficult hurdle, however, is the formulation of a problem in quantum-mechanical language, for the concepts are completely different from those of classical mechanics. In fact, the concepts and equations of quantum mechanics in Schrödinger's formulation are much closer to those of optics than to those of mechanics. Moreover, the quantities that we are interested in—such as transition probabilities, cross sections, and so on—usually have nothing to do with mechanical concepts such as the position, momentum, or trajectory of a particle. Nevertheless, actual insight into a process is a prerequisite for understanding its quantum-mechanical description and interpreting basic properties in quantum mechanics like position, linear and angular momentum, as well as cross sections, lifetimes, and so on.

Actually, students must develop an intuition of how the concepts of classical mechanics are altered and supplemented by the arguments of optics in order to acquire a roughly correct picture of quantum mechanics. In particular, the time evolution of microscopic physical systems has to be studied to establish how it corresponds to classical mechanics. Here computers and computer graphics offer incredible help, for they produce a large number of examples which are very detailed and which can be looked at in any phase of their time development. For instance, the study of wave packets in motion, which is practically impossible without the help of a computer, reveals the limited validity of intuition drawn from classical mechanics and gives us insight into phenomena like the tunnel effect and resonances, which, because of the importance of interference, can be understood only through optical analogies. A variety of systems in different situations can be simulated on the computer and made accessible by different types of computer graphics.

Some of the topics covered are

- scattering of wave packets and stationary waves,
- the tunnel effect,
- decay of metastable states,
- bound states in various potentials,
- energy bands,
- distinguishable and indistinguishable particles,
- angular momentum,
- three-dimensional scattering,
- cross sections and scattering amplitudes,
- eigenstates in three-dimensional potentials, for example, in the hydrogen atom,
- partial waves and resonances.

The graphical aids range from

- time evolutions of wave functions for one-dimensional problems,
- parameter dependences for studying, for example, the scattering over a range of energies,
- three-dimensional surface plots for presenting two-particle wave functions,

to

- ripple tank pictures to illustrate three-dimensional scattering.

Whenever possible, how particles of a system would behave according to classical mechanics has been indicated by their positions or trajectories. In passing, the special functions typical for quantum mechanics, such as Legendre, Hermite, and Laguerre polynomials, spherical harmonics, and spherical Bessel functions, are also shown in sets of pictures.

The text presents the principal ideas of wave mechanics. The introductory Chapter 1 lays the groundwork by discussing the particle aspect of light, using the fundamental experimental findings of the photoelectric and Compton effects and the wave aspect of particles as it is demonstrated by the diffraction of electrons. The theoretical ideas abstracted from these experiments are introduced in Chapter 2 by studying the behavior of wave packets of light as they propagate through space and as they are reflected or refracted by glass plates. The photon is introduced as a wave packet of light containing a quantum of energy.

To indicate how material particles are analogous to the photon, Chapter 3 introduces them as wave packets of de Broglie waves. The ability of de Broglie waves to describe the mechanics of a particle is explained through a detailed discussion of group velocity, Heisenberg's uncertainty principle, and Born's probability interpretation. The Schrödinger equation is found to be the equation of motion.

Chapters 4 through 8 are devoted to the one-dimensional quantum-mechanical systems. Study of the scattering of a particle by a potential helps us understand how it moves under the influence of a force and how the probability interpretation operates to explain the simultaneous effects of transmission and reflection. We study the tunnel effect of a particle and the excitation and decay of a metastable state. A careful transition to a stationary bound state is carried out. Quasi-classical motion of wave packets confined to the potential range is also examined.

Chapters 7 and 8 cover two-particle systems. Coupled harmonic oscillators are used to illustrate the concept of indistinguishable particles. The striking differences between systems composed of different particles, systems of identical bosons, and systems of identical fermions obeying the Pauli principle are demonstrated.

Three-dimensional quantum mechanics is the subject of Chapters 9 through 13. We begin with a detailed study of angular momentum and discuss methods of solving the Schrödinger equation. The scattering of plane waves is investigated by introducing partial-wave decomposition and the concepts of differential cross sections, scattering amplitudes, and phase shifts. Resonance scattering, which is the subject of many fields of physics research, is studied in detail in Chapter 13. Bound states in three dimensions are dealt with in Chapter 12. The hydrogen atom and the motion of wave packets on elliptical orbits under a harmonic force are among the topics covered.

The last chapter is devoted to results obtained through experiments in atomic, molecular, solid-state, nuclear, and particle physics. They can be qualitatively understood with the help of the pictures and the discussion in the body of the book. Thus examples for

- typical scattering phenomena,
- spectra of bound states and their classifications with the help of models,
- resonance phenomena in total cross sections,
- phase shift analyses of scattering and Regge classification of resonances,
- radioactivity as decay of metastable states,

taken from the fields of atomic and subatomic physics, are presented. Comparing these experimental results with the computer-drawn pictures of the book and their interpretation gives the reader a glimpse of the vast fields of science that can be understood only on the basis of quantum mechanics.

There are more than a hundred problems at the ends of the chapters. Many are designed to help students extract the physics from the pictures. Others will give them practice in handling the theoretical concepts. On the endpapers of the book are a list of frequently used symbols, a short list of physical constants, and a brief table converting SI units to particle physics units. The constants and units will make numerical calculations easier.

All computer-drawn figures were produced with an interactive computer program developed especially for this book. Figure 9.5, the one exception, was made by Dr. Peter Janzen. The hand-drawn figures and the lettering of the others were done by Manfred Euteneuer. Rüdiger Schütz helped with some technical points of the computer graphs. Gertrud Kreuz carefully typed the manuscript. Professor Diethard H. Schiller, Professor Fritz W. Bopp, and Dr. Hans-Jürgen Meyer read the manuscript and offered helpful criticism. We are grateful to all of them for their kind cooperation.

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# Contents

# 1. Introduction

The basic fields of classical physics are mechanics and heat on the one hand and electromagnetism and optics on the other. Mechanical and heat phenomena involve the motion of particles as governed by Newton's equations. Electromagnetism and optics deal with fields and waves, which are described by Maxwell's equations. In the classical description of particle motion, the position of the particle is exactly determined at any given moment. Wave phenomena, in contrast, are characterized by interference patterns which extend over a certain region in space. The strict separation of particle and wave physics loses its meaning in atomic and subatomic processes.

Quantum mechanics goes back to Max Planck's discovery in 1900 that the energy of an oscillator of *frequency* $\nu$ is quantized. That is, the energy emitted or absorbed by an oscillator can take only the values $0, h\nu, 2h\nu, \ldots$ . Only multiples of *Planck's quantum of energy*

$$E = h\nu$$

are possible. *Planck's constant,*

$$h = 6.262 \cdot 10^{-34}\ \mathrm{W\,s}$$

is a fundamental constant of nature, the central one of quantum physics. Often it is preferable to use the *angular frequency* $\omega = 2\pi\nu$ of the oscillator and to write Planck's quantum of energy in the form

$$E = \hbar\omega$$

Here

$$\hbar = \frac{h}{2\pi}$$

is simply Planck's constant divided by $2\pi$. Planck's constant is a very small quantity. Therefore the quantization is not apparent in macroscopic systems. But in atomic and subatomic physics Planck's constant is of fundamental importance. In order to make this statement more precise, we shall look at experiments showing the following fundamental phenomena:

- the photoelectric effect,
- the Compton effect,
- the diffraction of electrons.

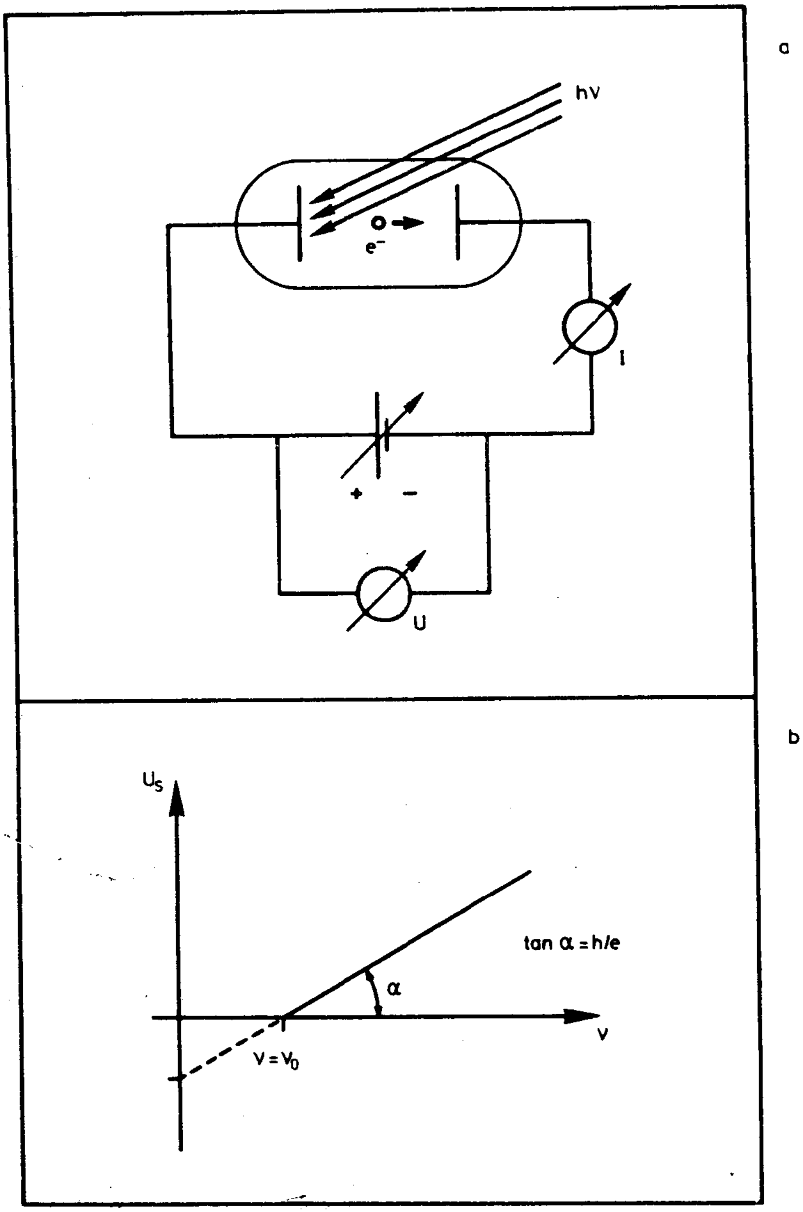


***Figure 1.1*** **Photoelectric effect. (a) The apparatus to measure the effect consists of a vacuum tube containing two electrodes. Monochromatic light of frequency $\nu$ shines on the cathode and liberates electrons which may reach the anode and create a current $I$ in the external circuit. The flow of electrons in the vacuum tube is hindered by the external voltage $U$. It stops once the voltage exceeds the value $U_s$. (b) There is a linear dependence between the frequency $\nu$ and the voltage $U_s$.**

The photoelectric effect was discovered by Heinrich Hertz in 1887. It was studied in more detail by Wilhelm Hallwachs in 1888 and Philipp Lenard in 1902. We discuss here the quantitative experiment, which was first carried out in 1916 by R. A. Millikan. His apparatus is shown schematically in Figure 1.1a. Monochromatic light of variable frequency falls onto a photocathode in a vacuum tube. Opposite the photocathode there is an anode—we assume cathode and anode to consist of the same metal—which is at a negative voltage $U$ with respect to the cathode. Thus the electric field exerts a repelling force on the electrons of charge $-e$ that leave the cathode. Here $e = 1.609 \cdot 10^{-19}$ coulomb is the elementary charge. If the electrons reach the anode, they flow back to the cathode through the external circuit, yielding a measurable current $I$. The kinetic energy of the electrons can therefore be determined by varying the voltage between anode and cathode. The experiment yields the following findings.

1. The electron current sets in, independent of the voltage $U$, at a frequency $\nu_0$ that is characteristic for the material of the cathode. There is a current only for $\nu > \nu_0$.
2. The voltage $U_s$ at which the current stops flowing depends linearly on the frequency of the light (Figure 1.1b). The kinetic energy $E_{\text{kin}}$ of the electrons leaving the cathode then is equal to the potential energy $eU_s$ of the electric field between cathode and anode:

$$E_{\text{kin}} = eU_s$$

If we call $h/e$ the tangent of the straight line representing the relation between the frequency of the light and the voltage,

$$U_s = \frac{h}{e}(\nu - \nu_0)$$

we find that light of frequency $\nu$ transfers the kinetic energy $eU_s$ to the electrons kicked out of the material of the cathode. When light has a frequency less than $\nu_0$, no electrons leave the material. If we call

$$h\nu_0 = eU_k$$

the ionization energy of the material that is needed to free the electrons, we must conclude that light of frequency $\nu$ has energy

$$E = h\nu = \hbar\omega$$

with

$$\omega = 2\pi\nu, \qquad \hbar = \frac{h}{2\pi}$$

3. The number of electrons set free is proportional to the intensity of the light incident on the photocathode.

In 1905 Albert Einstein explained the photoelectric effect by assuming that light consists of quanta of energy $h\nu$ which act in single elementary processes. The *light quanta* are also called *photons* or $\gamma$-quanta. The number of quanta in the light wave is proportional to its intensity.

If the light quanta of energy $E = h\nu = \hbar\omega$ are particles, they should also have momentum. The relativistic relation between the energy $E$ and momentum $p$ of a particle of rest mass $m$ is

$$p = \frac{1}{c}\sqrt{E^2 - m^2c^4}$$

where $c$ is the speed of light in vacuum. Quanta moving with the speed of light must have rest mass zero, so that we have

$$p = \frac{1}{c}\sqrt{\hbar^2\omega^2} = \hbar\frac{\omega}{c} = \hbar k$$

where $k = \omega/c$ is the wave number of the light. If the direction of the light is $\mathbf{k}/k$, we find the vectorial relation $\mathbf{p} = \hbar\mathbf{k}$. To check this idea one has to perform an experiment in which light is scattered on free electrons. The conservation of energy and momentum in the scattering process requires that the following relations be fulfilled,

$$E_\gamma + E_e = E'_\gamma + E'_e$$

$$\mathbf{p}_\gamma + \mathbf{p}_e = \mathbf{p}'_\gamma + \mathbf{p}'_e$$

where $E_\gamma$, $\mathbf{p}_\gamma$ and $E'_\gamma$, $\mathbf{p}'_\gamma$ are the energies and the momenta of the incident and the scattered photon, respectively. $E_e$, $\mathbf{p}_e$, $E'_e$, and $\mathbf{p}'_e$ are the corresponding quantities of the electron. The relation between electron energy $E_e$ and momentum $p_e$ is

$$E_e = c\sqrt{p_e^2 + m_e^2c^2}$$

where $m_e$ is the rest mass of the electron. If the electron is initially at rest, we have $\mathbf{p}_e = 0$, $E_e = m_ec^2$. Altogether, making use of these relations, we obtain

$$c\hbar k + m_ec^2 = c\hbar k' + c\sqrt{\mathbf{p}_e'^2 + m_e^2c^2}$$

$$\hbar\mathbf{k} = \hbar\mathbf{k}' + \mathbf{p}'_e$$

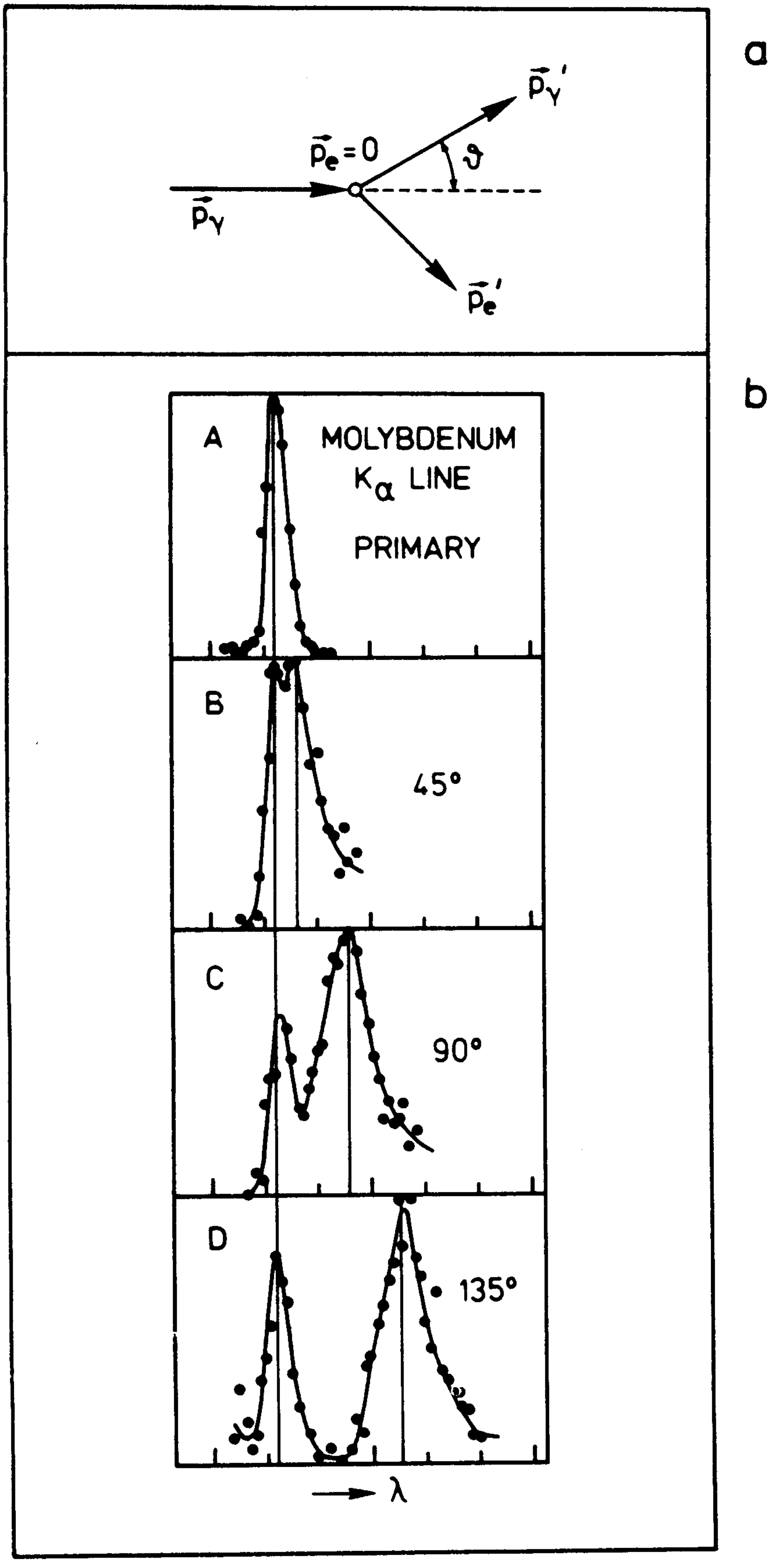


*Figure 1.2* The Compton effect. (a) Kinematics of the process. A photon of momentum $p_\gamma$ is scattered by a free electron at rest, one with momentum $p_e = 0$. After the scattering process the two particles have the momenta $p'_\gamma$ and $p'_e$, respectively. The direction of the scattered photon forms an angle $\vartheta$ with its original direction. From energy and momentum conservation in the collision, the absolute value $p'_\gamma$ of the momentum of the scattered photon and the corresponding wavelength $\lambda' = h / p'_\gamma$ can be computed.

(b) Compton's results. Compton used monochromatic X-rays from the $K_\alpha$-line of molybdenum to bombard a graphite target. The wavelength spectrum of the incident photons shows the rather sharp $K_\alpha$-line at the top. Observations of the photons scattered at three different angles $\vartheta$ (45°, 90°, 135°) yielded spectra showing that most of them had drifted to the longer wavelength $\lambda'$. There are also many photons at the original wavelength $\lambda$, photons which were not scattered by single electrons in the graphite. From A. H. Compton, *The Physical Review* 22 (1923) 409, copyright © 1923 by the American Physical Society, reprinted by permission.