Soviet Scientific Reviews, Section C

# MATHEMATICAL PHYSICS REVIEWS

## Volume 2 (1981)

*Edited by*

S. P. Novikov

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# Preface to the Series

In the last few years many important developments have taken place in Soviet science which may have not received as much attention as deserved among the international community of scientists because of language problems and circulation problems.

In launching this new series of *Soviet Scientific Reviews* we are motivated by the desire to make accounts of recent scientific advances in the USSR more readily and rapidly accessible to scientists who do not read Russian. The articles in these volumes are meant to be in the nature of reviews of recent developments and are written by Soviet experts in the fields covered. Most of the manuscripts are translated from Russian. In the interest of speedy publication neither the authors nor the volume editors have an opportunity to see the translations or to read proofs. They are therefore absolved of any responsibility for inaccuracies in the English texts.

*Soviet Scientific Reviews* will appear annually, with the average of specific subject areas in each of the sciences varying from year to year. In 1979 we published volumes in Chemistry and Physics. In 1980 we expanded the series with the addition of annual volumes in Biology, Mathematical Physics and Astrophysics and Space Physics.

We are much indebted to the volume editors and individual authors for their splendid cooperation in getting these first volumes put together and sent to press under considerable time pressure.

The future success of this series depends, of course, on how well it meets the readers' needs and desires. We therefore earnestly solicit readers' comments and particularly suggestions for topics and authors for future volumes.

By taking this initiative we hope to contribute to the development of scientific cooperation and the better understanding among scientists.

# Preface

In this second volume of Soviet surveys in mathematical physics, we present papers on two themes, quantum fluctuations of classical solutions in field theory, and the problem of stochasticity (and other aspects of the qualitative behavior of solutions) in nontrivial multidimensional dynamical systems that arise in various physical problems.

The survey by Fateev, Frolov, Schwarz and Tyupkin contains a presentation of results recently obtained by the authors. This quasi-classical approach requires the calculation of classical extrema of the action in a Euclidean metric and of some related quantities in order to calculate the asymptotic classical Green's function. The problem is solved completely in a two-dimensional model. Some individual results show promise in helping to solve the analogous problem in the theory of the Yang-Mills SU(2) field in four-dimensional space.

The remaining four surveys are combined under the general title: "Stochasticity in Nonlinear Dynamical Systems."

The Bogoyavlenskii paper contains a description of certain geometric methods for qualitative investigation of multidimensional dynamical systems. Starting in 1971–1973, these methods originally were developed by Novikov, Bogoyavlenskii, and later also by Persetskii, for the qualitative investigation of the Einstein equations in spatially homogeneous models of the general theory of relativity at early stages in the evolution of the Universe. It proved to be possible to give a classification of all possible regimes of evolution, including the BLK (Belinskii-Lifshitz-Khalatnikov) stochastic regime discovered earlier by different methods, and some newer ones. It also helps to formulate and solve the problem of "typical" regimes (within the framework of homogeneous models) of expansion of the Universe during early stages of its evolution. In later work Bogoyavlenskii greatly improved and successfully applied these methods to problems of gas dynamics, the study of perturbations of the Toda lattice, etc.

The survey by Ya. G. Sinai and Ya. B. Pesin describes the present state of the well-known class of dynamical systems of hyperbolic type. In the early 1960's Smale, Anosov, Sinai, Arnold and others discovered this kind of system and found that it possessed remarkable topological (qualitative) and stochastic (ergodic) properties. Sinai and

Pesin consider many important examples that have been studied in recent years. They give a summary of results on the so-called "hyperbolic attractors" (i.e., attracting sets, in which the dynamics has the hyperbolic property). They discuss strange attractors in connection with the well-known Lorentz system.

The paper by Pikovskii and Rabinovich is concerned with the problem of stochasticity in dissipative systems. The survey contains a description and interpretation of various experiments from different branches of physics in which stochastic behavior is observed.

The survey by Chirikov, Izrailev and Shepelyanskii concerns the investigation of stochasticity in classical and quantum dynamical systems. A detailed discussion is given of the criteria for overlapping of nonlinear resonances which present the condition for the appearance of global diffusion in the phase space of a classical system.

They discuss the seeming contradiction between the stochasticity of the classical system and the behavior of the corresponding quantum system. They give a simple classical model of quantum stochasticity.

S. P. Novikov, Editor

# Contents: Volume 2 of Mathematical Physics Reviews

# Part 1: Quantum Fluctuations of Instantons

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Although a great deal of attention has been devoted in recent years to the role of instantons in gauge theories, it cannot be said that the question of the contribution of instantons has been explained in sufficient detail. In most papers it is assumed that the only important instantons with topological number $q$ are those that can be represented as a superposition of $q$ instantons with topological number 1 (the approximation of a dilute instanton gas [1]). This approximation leads to infrared divergences, which we now know are associated with the fact that this approximation is by no means always valid.

To get closer to an understanding of the role of instantons in gauge theories it is wise to consider two-dimensional models. These are the two-dimensional nonlinear $\sigma$-model, which is in many aspects similar to four-dimensional gauge theory with the gauge group SU(2), and the two-dimensional CP($n-1$) model, which is analogous to gauge theory with the group SU($n$). The first part of this paper is devoted to the study of these models. The most complete results are obtained for the nonlinear $\sigma$-model. We show that in this model it is convenient to associate with an instanton solution with topological charge $q$ a system of $q$ positively charged and $q$ negatively charged Coulomb particles ("instanton quarks"). The study of the instanton contribution reduces to an investigation of the neutral Coulomb gas of "instanton quarks." In particular this investigation shows that the infrared divergences that are characteristic of the dilute instanton gas approximation vanish when we sum over all instanton contributions (as the result of Debye screening in the Coulomb gas).

One might suppose that the qualitative features of the instanton contribution in the two-dimensional $\sigma$-model are preserved in four-dimensional gauge theories; heuristic arguments in support of this hypothesis are presented in [2]. In any case, the results obtained in the two-dimensional models give strong arguments in favor of investigating the instanton contribution in gauge theories beyond the framework of the dilute instanton gas approximation. The second part of the paper is devoted to this investigation. The results of this part are much less complete than in the two-dimensional case; a partial reason is that there is at present no sufficiently explicit description of all instantons in gauge theories.

From the mathematical point of view the calculation of quantum fluctuations of instantons reduces to finding determinants of certain elliptic operators. A short appendix contains the main facts about regularized determinants of elliptic operators. The operators considered in this paper can be regarded as elliptic operators acting in sections of vector bundles. We describe this interpretation of our operators, but we do not make essential use of the language of vector bundles in order not to make the reading of the paper difficult for physicists. (Places where we mention bundle concepts can be skipped.) We mention that many of the proofs in this paper could be based on the results of [3] and [4]. These papers contain the general theory of the partition function for a degenerate Lagrangian, which finds application not only in quantum field theory but also in topology. We shall not present this general theory here; however, our proofs use the methods developed in [3] and [4].

The basic results on calculating the contribution of instantons to the Euclidean Green's function in the two-dimensional $\sigma$-model and CP($n-1$) model were obtained in [5–7]. A somewhat later paper [8] contains equivalent results. Some of the results in Secs. 3 and 5 are new. Some of them are published in [9].

The results of calculations of the instanton contribution in four-dimensional gauge theory, presented in Secs. 6–11 were published in [2–4], [10–11]. Similar results were obtained in [12–15].

## §1. *Instantons in Models with Nonlinear Fields*

We consider fields defined on an $n$-dimensional Riemannian manifold $E$ and taking values in an $m$-dimensional Riemannian manifold $M$ (or, if we use mathematical terminology, mappings of the manifold $E$ into the manifold $M$). We shall assume that a coordinate system has been introduced in the manifold $M$ $(\varphi^1, \ldots, \varphi^m)$, and that the Riemannian metric is given in this coordinate system by the formula

$$ds^2 = h_{ij}(\varphi)\, d\varphi^i\, d\varphi^j.$$

We denote coordinates in $E$ by $x^1, \ldots, x^n$, and write the metric in the form

$$ds^2 = g_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta.$$

We associate with the field $\varphi(x) = (\varphi^1(x), \ldots, \varphi^m(x))$ the number

$$S(\varphi) = \frac{1}{2f}\int h_{ij}(\varphi)\frac{\partial\varphi^i}{\partial x^\alpha}\frac{\partial\varphi^j}{\partial x^\beta}\, g^{\alpha\beta}(x)\, dV \qquad (1.1)$$

(as always, $g^{\alpha\beta}$ is the reciprocal of the metric tensor, $dV = \sqrt{g}\, dx^1 \cdots dx^n$ is the volume element in $E$). The functional (1.1) can also be written in the invariant form

$$S(\varphi) = \frac{1}{2f} \int_E \mathrm{Tr}\, \mathcal{D}^*(x) \mathcal{D}(x)\, dV \tag{1.2}$$

where $\mathcal{D}(x)$ is the differential of the mapping $\varphi$ at the point $x$, considered as a linear operator acting from the tangent space $E_x$ to the manifold $E$ at the point $x$ into the tangent space $\mathfrak{M}_{\varphi(x)}$ to the manifold $M$ at the point $\varphi(x)$ (the adjoint operator $\mathcal{D}^*(x)$ is calculated using the scalar products in $E_x$ and $\mathfrak{M}_{\varphi(x)}$ defined by the Riemann metrics). Functionals of the form (1.1) occur in physics in various situations. For example if $E$ is Euclidean space while $M$ is the two-dimensional sphere with the usual metric, then the functional (1.1) has the meaning of the energy of an isotropic classical Heisenberg ferromagnet; if $M$ is an ellipsoid, then (1.1) represents the energy of an anisotropic ferromagnet. If $E$ is Euclidean space and $M$ is the sphere $S^n$ with the usual metric, then (1.1) can be interpreted as the Euclidean action of an $O(n + 1)$-symmetric $\sigma$-model. (If we introduce the Minkowski metric in $E$, then (1.1) represents the usual action.)

Let us consider in more detail the case where $E$ is the two-dimensional Euclidean space $E^2$, and $M$ is the two-dimensional sphere $S^2$. It will be convenient to parametrize the points of the sphere $S^2$ by complex numbers $w$ by using stereographic projection:

$$w = \frac{n^2 + in^2}{1 + n^3}$$

where $\mathbf{n} = (n^1, n^2, n^3)$, $|\mathbf{n}| = 1$. Then the element of length on the sphere is given by the formula

$$ds^2 = \frac{4\, dw\, d\bar{w}}{(1 + |w|^2)^2}\,.$$

The points of the plane $E^2$ are parametrized by complex numbers $z = x^1 + ix^2$. In our coordinates the functional (1.1) takes the form

$$S(w) = \frac{4}{f} \int (1 + |w|^2)^{-2} \left( \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} + \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right) dx^1\, dx^2, \tag{1.3}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We shall consider the functional $S$ only on fields that have a limit as

$|z| \to \infty$. In other words we shall assume that the field $w(z, \bar{z})$, considered as a mapping of the plane $E^2$ into the sphere $S^2$ can be extended to a continuous mapping of the sphere obtained from $E^2$ by adjoining the point at infinity. (This restriction is not important, since heuristic arguments show that the action can be finite only for the fields that we are considering.)

It is known that one can associate with the mapping of a sphere into a sphere an integer—the degree of the mapping. The degree of a mapping can be expressed analytically as the integral of the Jacobian. In our case the degree of the mapping (the topological charge of the field) can be represented in the form

$$q(w) = \frac{1}{\pi} \int (1 + |w|^2)^{-2} \left( \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}} - \frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z} \right) dx^1 dx^2. \quad (1.4)$$

From (1.3) and (1.4) we find

$$S(w) = \frac{4\pi q}{f}(w) + \frac{8}{f} \int (1 + |w|^2)^{-2} \left| \frac{\partial w}{\partial \bar{z}} \right|^2 dx^1 dx^2, \quad (1.5)$$

$$S(w) = -\frac{4\pi q}{f}(w) + \frac{8}{f} \int (1 + |w|^2)^{-2} \left| \frac{\partial w}{\partial z} \right|^2 dx^1 dx^2. \quad (1.6)$$

Relations (1.5) and (1.6) show that

$$S(w) \geqslant \frac{4\pi}{f} |q(w)|.$$

The minimum of the functional $S$ on the space of fields having topological charge $q$, with $q \geqslant 0$ is attained on fields satisfying the equation $\partial w / \partial \bar{z} = 0$ (i.e., on analytic mappings). But if $q < 0$, then the minimum of this functional is attained on fields satisfying the condition $\partial w / \partial z = 0$ (anti-analytic mappings). We see that analytic mappings are extrema of the functional $S(w)$; they are called instantons. The anti-analytic mappings are also extrema, and are called anti-instantons. It should be noted that the point $w = \infty$ is in no way singular (it corresponds to the north pole of the sphere). Thus the pole of the function $w$ must not be considered to be a singularity. Thus an analytic mapping is given by functions $w(z)$ having a finite number of poles i.e., by rational functions:

$$w(z) = \frac{P(z)}{Q(z)}, \quad (1.7)$$

where $P(z)$ and $Q(z)$ are polynomials. It is easy to verify that the topological charge of the field is equal to the larger degree of the polynomials $P(z)$ and $Q(z)$. It then follows that the analytic mappings having topological charge $q$ form a $(4q+2)$-dimensional manifold. We see that, on the manifold of fields with topological charge $q$, the functional $S$ has a $(4|q|+2)$-dimensional manifold $N_q$ of stationary points (analytic mappings if $q \geqslant 0$, and anti-analytic mappings if $q < 0$). On all the fields of the manifold $N_q$ the functional $S$ takes the value $4\pi f^{-1}|q|$; the functional $S$ has a larger value on all other fields with topological charge $q$ [16].

Similar statements can be proved in much more general situations. Let us consider, in particular, a case where the sphere $S^2$ has a metric different from the usual one. Since all metrics on the sphere $S^2$ are conformally equivalent, we can write an arbitrary metric on the sphere $S^2$ in the form

$$ds^2 = h(w,\bar{w})\frac{dw\,d\bar{w}}{(1+|w|^2)^2} \tag{1.8}$$

where $h(w,\bar{w})$ is a positive function that tends to a nonzero constant as $|w| \to +\infty$. The functional (1.1) is then expressed in the form

$$S(w) = \frac{4}{f}\int \frac{h(w,\bar{w})}{(1+|w^2|)^2}\left(\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} + \frac{\partial w}{\partial \bar{z}}\frac{\partial \bar{w}}{\partial z}\right)dx^1\,dx^2, \tag{1.9}$$

and the topological charge of the field $w(z,\bar{z})$ has the form

$$q(w) = L\int \frac{h(w,\bar{w})}{(1+|w|^2)^2}\left(\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}} - \frac{\partial \bar{w}}{\partial z}\frac{\partial w}{\partial \bar{z}}\right)dx^1\,dx^2 \tag{1.10}$$

where

$$L^{-1} = \int \frac{h(w,\bar{w})}{(1+|w|^2)^2}\,dw^1\,dw^2.$$

From the relation

$$fS = 4L^{-1}q + 8\int \frac{h(w,\bar{w})}{(1+|w|^2)^2}\frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}}\,dx^1\,dx^2 \tag{1.11}$$

it follows that in this case analytic mappings (rational functions) also provide the minimal value of the functional $S(w)$ on fields with topological charge $q$, and this value is $4L^{-1}f^{-1}q$.

A further generalization can be obtained if we take for the manifold $M$ an arbitrary Kähler manifold [17]. We recall that a complex manifold $M$ with hermitian metric

$$ds^2 = h_{ab}(w,\bar{w})\, dw^a \wedge d\overline{w^b} \tag{1.12}$$

is called a Kähler manifold if the 2-form

$$\Omega = \frac{i}{2} h_{ab}(w,\bar{w})\, dw^a \wedge d\overline{w^b} \tag{1.13}$$

is closed. The hermitian metric generates a Riemannian metric in $M$, so that the functional (1.1) is defined on fields given on the plane $E^2$ and taking values in $M$. Representing a point in the plane $E^2$ by a complex number $z = x^1 + ix^2$, we can write this functional in the form

$$S = \frac{4}{f}\int h_{ab}(w,\bar{w})\left(\frac{\partial w^a}{\partial z}\frac{\partial \overline{w^b}}{\partial \bar{z}} + \frac{\partial w^a}{\partial \bar{z}}\cdot\frac{\partial \overline{w^b}}{\partial z}\right)dx^1\,dx^2. \tag{1.14}$$

The topological charge of the field $w(z,\bar{z})$ can be determined from the formula

$$q = L\int w^*\Omega = L\int \frac{i}{2} h_{ab}(w,\bar{w})\left(\frac{\partial w^a}{\partial z}\cdot\frac{\partial \overline{w^b}}{\partial \bar{z}} - \frac{\partial w^a}{\partial \bar{z}}\cdot\frac{\partial \overline{w^b}}{\partial z}\right)dz\wedge d\bar{z}$$

$$= L\int h_{ab}(w,\bar{w})\left(\frac{\partial w^a}{\partial z}\cdot\frac{\partial \overline{w^b}}{\partial \bar{z}} - \frac{\partial w^a}{\partial \bar{z}}\cdot\frac{\partial \overline{w^b}}{\partial z}\right)dx^1\,dx^2, \tag{1.15}$$

where $w^*\Omega$ is the image of the form $\Omega$ under the mapping $w$ of the plane $E^2$ into $M$, and $L$ is a constant chosen so that the topological charge takes integral values. (As before, we assume that the field $w$ can be continued into a continuous mapping of the sphere obtained from $E^2$ by adjoining the point at infinity.) From the relations (1.14) and (1.15) we obtain

$$\begin{aligned} fS - 4L^{-1}q &= 8\int h_{ab}(w,\bar{w})\,\frac{\partial w^a}{\partial \bar{z}}\,\frac{\overline{\partial w^b}}{\partial \bar{z}}\,dx^1\,dx^2,\\ fS + 4L^{-1}q &= 8\int h_{ab}(w,\bar{w})\,\frac{\partial w^a}{\partial z}\,\frac{\overline{\partial w^b}}{\partial z}\,dx^1\,dx^2, \end{aligned} \tag{1.16}$$

from which we see that

$$S \geqslant 4L^{-1}|q|f^{-1}$$

where the equality $S = 4L^{-1}f^{-1}|q|$ is attained on analytic mappings when $q \geqslant 0$, and on anti-analytic mappings when $q < 0$.

The case when $E$ is the two-dimensional sphere $S^2$ while $M$ is a Kähler manifold is practically the same as the one just considered. In fact it is easy to see that the action (1.1) is unchanged by conformal transformations of the metric of the space $E$ (i.e., for changes of the metric $g_{\alpha\beta}(x)$ into $\rho(x)g_{\alpha\beta}(x)$). Noting that stereographic projection is a conformal mapping of the sphere $S^2$ onto $E^2$, we see that in stereographic coordinates the action (1.1) on the sphere takes the form (1.14). The topological charge of the mapping of $S^2$ into the Kähler manifold $M$, just like the mapping of $E^2$ into $M$, can be determined from formula (1.15).

## §2. *The Expression of the Instanton Contribution in Terms of Regularized Determinants*

We shall consider two-dimensional nonlinear models describing fields that take on values in a Kähler manifold. In the models we consider, the Euclidean Green's function can be represented as the ratio of two functional integrals

$$I(\Phi) = \int \Phi(w)\, e^{-S(w)}\, \mathcal{D}w \Big/ \int e^{-S(w)}\, \mathcal{D}w \qquad (2.1)$$

where $S(w)$ is the Euclidean action (1.14), the functional integral runs over fields $w(z)$ taking values in the manifold $M$, and $\Phi(w)$ is a functional of the field $w$. It is understood that to give meaning to the integrals in (2.1) one must first go over to finite-dimensional integrals (e.g., by means of a lattice cutoff) and then remove the cutoff, by assuming that the coupling constant depends on a cutoff parameter in momentum space.

We shall assume that an infrared cutoff has been fixed, for example, by assuming that the fields are defined in a sphere of radius $R$. The infrared cutoff can be removed in the final formulas.

In the case where the fields in the nonlinear two-dimensional model take on values in a symmetric space (e.g., on a sphere or in CP($n-1$)), one can show, in the framework of perturbation theory, that the dependence of the coupling constant on the cutoff parameter can be chosen so that the Euclidean Green's function has a finite limit (in other words, that the theory is renormalizable).

We shall try to apply the Laplace method to calculate the functional integrals appearing in (2.1). We recall that in the finite-dimensional case the asymptotic behavior of the integral

$$\int F(x) \exp\left(-\frac{S(x)}{f}\right) d^n x,$$

where $S(x)$ is a function that attains its minimum $\gamma$ on a $k$-

dimensional manifold $N$, has the form

$$(\pi f)^{(n-k)/2} \exp(-\gamma/f) \int_N F(x)(\det S''(x))^{-1/2} d\mu_0 \qquad (2.2)$$

where $d\mu_0$ is the measure on $N$ given by the Riemannian metric, while the determinant of the degenerate matrix $S''_{ij} = \partial^2 S/\partial x_i \, \partial x_j$ is taken to be the product of the nonzero eigenvalues of this matrix. (We are assuming that $N$ is a nondegenerate stationary manifold, i.e., at each point of the manifold the matrix $S''_{ij}(x)$ has the same number $k$ of zero eigenvalues.)

Formally applying the relation (2.2) for calculating the infinite-dimensional integrals appearing in (2.1), we arrive at an expression for the instanton contribution to the Euclidean Green's function

$$I(\Phi) = \frac{\sum_q (2^{-3}\pi f)^{(\dim M - \dim N_q)/2} \int_{N_q} \Phi(w)(\det S''(w))^{-1/2} d\mu_0(w) e^{-4q/Lf}}{\sum_q (2^{-3}\pi f)^{(\dim M - \dim N_q)/2} \int_{N_q} \Phi(\det S''(w))^{-1/2} d\mu_0(w) e^{-4q/Lf}} \qquad (2.3)$$

Here the symbol $S''(w)$ denotes the operator defined by the relation

$$S(w+\nu) \approx S(w) + \frac{8}{f}\langle \nu, S''(w)\nu \rangle + o(\|\nu\|^2). \qquad (2.4)$$

The explicit form of the operator $S''$ and the measure $d\mu_0$ will be described later. The scalar product of small deviations $\nu^{(1)}$, $\nu^{(2)}$ from the field $w$ is defined by

$$\langle \nu^{(1)}, \nu^{(2)} \rangle = \frac{1}{2} \int h_{ij} \left( \bar{\nu}_i^{(1)} \nu_j^{(2)} + \bar{\nu}_i^{(2)} \nu_j^{(1)} \right) dV. \qquad (2.5)$$

In order to give a meaning to the expressions appearing in (2.3) we must introduce a cutoff (e.g., by using a lattice). When the lattice is introduced, the integral in the definition of the scalar product (2.5) must be replaced by a sum; in the limit, as the lattice constant $a$ goes to zero, the lattice scalar product $\langle \nu^{(1)}, \nu^{(2)} \rangle_a$ is related to the scalar product (2.5) by the relation

$$\langle \nu^{(1)}, \nu^{(2)} \rangle_a = a^{-2} \langle \nu^{(1)}, \nu^{(2)} \rangle.$$

This means that the lattice analog $d\mu_0^a$ of the measure $d\mu_0$ as $a \to 0$ is related to the measure $d\mu_0$ by the relation

$$d\mu_0^a = a^{-\dim N_q} d\mu_0.$$

In investigating the determinant $S''$ it is convenient to introduce, in place of the lattice cutoff, a proper time cutoff (cf. the appendix), assuming that the proper time cutoff parameter is related to the lattice constant by the relation $\epsilon = \text{const}\, a^2$. From formula (A.6) it follows that the asymptotic form of $\log \det_\epsilon S''$ as $\epsilon \to 0$ is

$$\frac{\alpha_1}{\epsilon} + (\alpha_0 - p)\log \epsilon,$$

where $\alpha_1$ is independent of $w$, while $\alpha_0$ can be represented in the form $\alpha_0 = \gamma q + \beta$, where $\gamma$ and $\beta$ are independent of the field. The coupling constant $f$ should be regarded as depending on the cutoff parameter $\epsilon$ according to the formula

$$\frac{4}{fL} = \text{const} + \gamma \ln a$$

where $L$ is the constant appearing in (1.15). This formula is usually written in the form

$$f^{-1} = f_R^{-1} + \left(\frac{\gamma L}{4}\right) \ln a\nu \tag{2.6}$$

where $\nu$ is called the normalization point, and $f_R$ is the physical coupling constant corresponding to the normalization point $\nu$ (perturbation theory gives a result for the dependence of the coupling constant on $a$ that agrees with formula (2.6)). It is easily verified that with this choice of the dependence of $f$ on $a$ all the factors in (2.3) that diverge in the limit $a \to 0$ drop out.

As a result we obtain a finite expression for the instanton contribution to the Euclidean Green's function in terms of the regularized determinants

$$I_{\text{inst}}(\Phi) = \sum_q K^q \int_{N_q} \Phi(w)\, d\mu(w) \Big/ \sum_q K^q \int_{N_q} d\mu(w) \tag{2.7}$$

where

$$\begin{aligned} d\mu(w) &= (\det' S''(w))^{-1/2} d\mu_0(w) \Big/ \int_{N_0} \det' S''(w_0)\, d\mu_0(w_0); \\ K &= k f_R^{-\gamma} \nu^{\gamma} \exp(-4/Lf); \end{aligned} \tag{2.8}$$

$k$ is an unimportant constant that depends on the choice of cutoff, $\det' S''(w)$ is the proper-time regularized determinant of the operator $S''(w)$. (For convenience, we have normalized $d\mu(w)$ to the contribution of the zero-instanton solution in the denominator of (2.1).)

It is understood that one should also consider the contribution to the Green's function also from "almost stationary" fields, in particu-