Homological Algebra

By HENRI CARTAN
and SAMUEL EILENBERG

HOMOLOGICAL ALGEBRA

HENRI CARTAN

and

SAMUEL EILENBERG

1956

Preface

During the last decade the methods of algebraic topology have invaded extensively the domain of pure algebra, and initiated a number of internal revolutions. The purpose of this book is to present a unified account of these developments and to lay the foundations of a full-fledged theory.

The invasion of algebra has occurred on three fronts through the construction of cohomology theories for groups, Lie algebras, and associative algebras. The three subjects have been given independent but parallel developments. We present herein a single cohomology (and also a homology) theory which embodies all three; each is obtained from it by a suitable specialization.

This unification possesses all the usual advantages. One proof replaces three. In addition an interplay takes place among the three specializations; each enriches the other two.

The unified theory also enjoys a broader sweep. It applies to situations not covered by the specializations. An important example is Hilbert's theorem concerning chains of syzygies in a polynomial ring of n variables. We obtain his result (and various analogous new theorems) as a theorem of homology theory.

The initial impetus which, in part, led us to these investigations was provided by a problem of topology. Nearly thirty years ago, Künneth studied the relations of the homology groups of a product space to those of the two factors. He obtained results in the form of numerical relations among the Betti numbers and torsion coefficients. The problem was to strengthen these results by stating them in a group-invariant form. The first step is to convert this problem into a purely algebraic one concerning the homology groups of the tensor product of two (algebraic) complexes. The solution we shall give involves not only the tensor product of the homology groups of the two complexes, but also a second product called their *torsion* product. The torsion product is a new operation derived from the tensor product. The point of departure was the discovery that the process of deriving the torsion product from the tensor product could be generalized so as to apply to a wide class of functors. In particular, the process could be iterated and thus a sequence of functors could be obtained from a single functor. It was then observed that the resulting sequence possessed the formal properties usually encountered in homology theory.

In greater detail, let Λ be a ring, A a Λ-module with operators on the right (i.e. a right Λ-module) and C a left Λ-module. A basic operation is the formation of the tensor product $A \otimes_\Lambda C$. This is the group generated by pairs $a \otimes c$ with the relations consisting of the two distributive laws and the condition $a\lambda \otimes c = a \otimes \lambda c$. It is important to consider the behavior of this construction in relation to the usual concepts of algebra: homomorphisms, submodules, quotient modules, etc.

To facilitate the discussion of this behavior we adopt diagrammatic methods. A sequence of Λ-modules and Λ-homomorphisms

$$A_m \to A_{m+1} \to \cdots \to A_n \qquad m+1 < n$$

is said to be *exact* if, for each consecutive two homomorphisms, the image of the first is the kernel of the following one. In particular we shall consider exact sequences

$$(1) \qquad 0 \to A' \to A \to A'' \to 0.$$

In such an exact sequence A' may be regarded as a submodule of A with A'' as the quotient module.

If an exact sequence of right Λ-modules is tensored with a fixed left Λ-module C, the resulting sequence of groups and homomorphisms is, in general, no longer exact. However, some measure of exactness is preserved. In particular, if the sequence (1) is tensored with C, the following portion is always exact:

$$(2) \qquad A' \otimes_\Lambda C \to A \otimes_\Lambda C \to A'' \otimes_\Lambda C \to 0.$$

We describe this property by saying that the tensor product is a *right exact* functor.

The kernel K of the homomorphism on the left in the sequence (2) is in general not zero. In case A is a free module, it can be shown that (up to natural isomorphisms) K depends only on A'' and C. We define the *torsion product* $\mathrm{Tor}_1^\Lambda(A'',C)$ to be the kernel in this case. In the general case there is a natural homomorphism

$$\mathrm{Tor}_1^\Lambda(A'',C) \to A' \otimes_\Lambda C$$

with image K. Continuing in this way we obtain an infinite exact sequence

$$(3) \qquad \cdots \to \mathrm{Tor}_{n+1}^\Lambda(A'',C) \to \mathrm{Tor}_n^\Lambda(A',C)$$
$$\to \mathrm{Tor}_n^\Lambda(A,C) \to \mathrm{Tor}_n^\Lambda(A'',C) \to \cdots$$

which terminates on the right with the sequence (2) above, provided that we set

$$\operatorname{Tor}_0^{\Lambda}(A,C) = A \otimes_{\Lambda} C. \tag{4}$$

The homomorphisms in (3) which pass from index $n+1$ to n are called *connecting homomorphisms*.

The condition that A be free in the definition of Tor (A'',C) is unnecessarily restrictive. It suffices that A be *projective*, i.e. that every homomorphism of A into a quotient B/B' admit a factorization $A \to B \to B/B'$.

The inductive definition of the sequence (3) as described above is cumbersome, and does not exhibit clearly the connection with homology theory. This is remedied by a direct construction as follows. If A is a module, then an exact sequence

$$\cdots \to A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to A \to 0$$

is called a *projective resolution* of A if each A_i, $i = 0,1,2,\ldots$ is projective. Tensoring with C gives a sequence

$$\cdots \to A_n \otimes_{\Lambda} C \to \cdots \to A_0 \otimes_{\Lambda} C \tag{5}$$

which may not be exact but which is a complex (the composition of two consecutive homomorphisms is zero). The n-th homology group of the complex (5) is precisely $\operatorname{Tor}_n^{\Lambda}(A,C)$. Using the second definition of Tor, the sequence (3) is constructed in the usual manner as the homology sequence of an exact sequence of complexes

$$0 \to X' \otimes_{\Lambda} C \to X \otimes_{\Lambda} C \to X'' \otimes_{\Lambda} C \to 0$$

where X', X, X'' are appropriate projective resolutions of A', A, A''.

A basic property of Tor is

$$\operatorname{Tor}_n^{\Lambda}(A,C) = 0 \text{ if } n > 0 \text{ and } A \text{ is projective.} \tag{6}$$

In fact, this property, the exactness of (3), property (4) and the usual formal properties of functors suffice as an axiomatic description of the functors $\operatorname{Tor}_n^{\Lambda}$.

The description of $\operatorname{Tor}_n^{\Lambda}(A,C)$ given above favored the variable A and treated C as a constant. If the reversed procedure is adopted, the same functors $\operatorname{Tor}_n^{\Lambda}(A,C)$ are obtained. This "symmetry" of the two variables in $A \otimes_{\Lambda} C$ is emphasized by adopting a definition of Tor which uses simultaneously projective resolutions of both A and C. This symmetry should not be confused with the symmetry resulting from the natural isomorphism $A \otimes_{\Lambda} C \approx C \otimes_{\Lambda} A$ which is valid only when Λ is commutative.

Another functor of at least as great importance as the tensor product is given by the group $\mathrm{Hom}_\Lambda(A,C)$ of all Λ-homomorphisms of the left Λ-module A into the left Λ-module C. This functor is contravariant in the variable A, covariant in the variable C and is *left exact* in that when applied to an exact sequence (1), it yields an exact sequence

$$(2') \qquad 0 \to \mathrm{Hom}_\Lambda(A'',C) \to \mathrm{Hom}_\Lambda(A,C) \to \mathrm{Hom}_\Lambda(A',C).$$

A similar discussion to that above leads to an exact sequence

$$(3') \qquad \cdots \to \mathrm{Ext}^n_\Lambda(A'',C) \to \mathrm{Ext}^n_\Lambda(A,C) \to \mathrm{Ext}^n_\Lambda(A',C) \to \mathrm{Ext}^{n+1}_\Lambda(A'',C) \to \cdots$$

which is a continuation of (2′), provided that we set

$$(4') \qquad \mathrm{Ext}^0_\Lambda(A,C) = \mathrm{Hom}_\Lambda(A,C).$$

These properties together with the property

$$(6') \qquad \mathrm{Ext}^n(A,C) = 0 \text{ if } n > 0 \text{ and } A \text{ is projective}$$

and the usual formal properties of functors suffice as an axiomatic description of the functors $\mathrm{Ext}^n_\Lambda(A,C)$.

The description above favored A as a variable while keeping C constant. Again symmetry prevails, and identical results are obtained by treating A as a constant and varying C. In this case however, instead of projective modules and projective resolutions, we employ the dual notions of injective modules and injective resolutions. A module C is *injective* if every homomorphism $B' \to C$ admits an extension $B \to C$ for each module B containing B' as a submodule. An injective resolution of C is an exact sequence

$$0 \to C \to C^0 \to C^1 \to \cdots \to C^n \to C^{n+1} \to \cdots$$

with C^i injective for $i = 0,1,2,\ldots$.

With the functors Tor and Ext introduced we can now show how the cohomology theories of groups, Lie algebras and associative algebras fit into a uniform pattern.

Let Π be a multiplicative group and C an (additive) abelian group with Π as a group of left operators. The integral group ring $Z(\Pi)$ is defined and C may be regarded as a left $Z(\Pi)$-module. The group Z of rational integers also may be regarded as a $Z(\Pi)$-module with each element of Π acting as the identity on Z. The *cohomology groups of* Π *with coefficients in* C are then

$$H^q(\Pi,C) = \mathrm{Ext}^q_{Z(\Pi)}(Z,C).$$

These cohomology groups were first introduced by Eilenberg-MacLane (*Proc. Nat. Acad. Sci. U.S.A.* 29 (1943), 155–158) in connection with a topological application. Subsequently they found a number of topological and algebraic applications; some of these will be considered in Ch. XIV and XVI. Quite recently, the theory for finite groups has been greatly enriched by the efforts of Artin and Tate; Ch. XII deals with these new developments. This theory has had its most striking application in the subject of Galois theory and class field theory. As this is a large and quite separate topic we shall not attempt an exposition here, although we do prove nearly all the results of the cohomology theory of groups needed for this application.

Let $\mathfrak{g}$ be a Lie algebra over a commutative ring K and let C be a (left) representation space for $\mathfrak{g}$. The enveloping (associative) algebra $\mathfrak{g}^e$ is then defined and C is regarded as a left $\mathfrak{g}^e$-module. The ground ring K with the trivial representation of $\mathfrak{g}$ also is a left $\mathfrak{g}^e$-module. The *cohomology groups of* $\mathfrak{g}$ *with coefficients in* C are then

$$H^q(\mathfrak{g},C) = \mathrm{Ext}^q_{\mathfrak{g}^e}(K,C).$$

This theory, implicit in the work of Elie Cartan, was first explicitly formulated by Chevalley-Eilenberg (*Trans. Am. Math. Soc.* 63 (1948), 85–124). We shall give an account of this theory in Ch. XIII; however we do not enter into its main applications to semi-simple Lie algebras and compact Lie groups.

Let Λ be an associative algebra (with a unit element) over a commutative ring K, and let A be a two-sided Λ-module. We define the enveloping algebra $\Lambda^e = \Lambda \otimes_K \Lambda^*$ where Λ^* is the "opposite" algebra of Λ. A may now be regarded as a left Λ^e-module. The algebra Λ itself also is a two-sided Λ-module and thus a left Λ^e-module. The cohomology groups are

$$H^q(\Lambda,A) = \mathrm{Ext}^q_{\Lambda^e}(\Lambda,A).$$

This theory, closely patterned after the cohomology theory of groups, was initiated by Hochschild (*Ann. of Math.* 46 (1945), 58–67). A fairly complete account of existing results is given in Ch. IX.

In all three cases above, homology groups also are defined using the functors Tor.

So far we have mentioned only the functors $A \otimes_\Lambda C$ and $\mathrm{Hom}_\Lambda(A,C)$ and their derived functors Tor and Ext. It has been found useful to consider other functors besides these two; Ch. II–V develop such a theory for arbitrary *additive* functors. Both procedures that led to the definition of Tor are considered. The slow but elementary iterative procedure leads to the notion of *satellite functors* (Ch. III). The faster, homological method using resolutions leads to the *derived functors* (Ch. V). In most important cases (including the functors $\otimes$ and Hom) both procedures yield identical results.

Beginning with Ch. VI we abandon general functors and confine our attention to the special functors Tor and Ext and their composites. The main developments concerning homology theory are grouped in Ch. VIII–XIII.

The last three chapters (XV–XVII) are devoted to the method of spectral sequences, which has been a major tool in recent developments in algebraic topology. In Ch. XV we give the general theory of spectral sequences, while the subsequent two chapters give applications to questions studied earlier in the book.

There is an appendix by David A. Buchsbaum outlining a more abstract method of treating the subject of satellites and derived functors.

Each chapter is preceded by a short introduction and is followed by a list of exercises of varied difficulty. There is no general bibliography; references are made in the text, whenever needed. Crossreferences are made as follows: Theorem 2.1 (or Proposition 2.1 or Lemma 2.1) of Chapter X is referred to as 2.1 if the reference is in Chapter X, and as X,2.1 if the reference is outside of that chapter. Similarly VIII,3,(8) refers to formula (8) of § 3 of Chapter VIII.

We owe expressions of gratitude to the John Simon Guggenheim Memorial Foundation who made this work possible by a fellowship grant to one of the authors. We received help from several colleagues: D. A. Buchsbaum and R. L. Taylor read the manuscript carefully and contributed many useful suggestions; G. P. Hochschild and J. Tate helped with Chapter XII; J. P. Serre and N. E. Steenrod offered valuable criticism and suggestions. Special thanks are due to Miss Alice Krikorian for her patience shown in typing the manuscript.

H. CARTAN
S. EILENBERG

University of Paris
Columbia University
September, 1953

Contents

HOMOLOGICAL ALGEBRA

CHAPTER I

Rings and Modules

Introduction. After some preliminaries concerning rings, modules, homomorphisms, direct sums, direct products, and exact sequences, the notions of projective and injective modules are introduced. These notions are fundamental for this book. The basic results here are that each module may be represented as a quotient of a projective module and also as a submodule of an injective one.

In § 4–7 we consider special classes of rings, namely: semi-simple rings, hereditary rings, semi-hereditary rings, and Noetherian rings. It will be seen later (Ch. VII) that for integral domains the hereditary (semi-hereditary) rings are precisely the Dedekind (Prüfer) rings.

1. PRELIMINARIES

Let Λ be a ring with a unit element $1 \neq 0$. We shall consider (left) *modules* over Λ, i.e. abelian groups A with an operation $\lambda a \in A$, for $\lambda \in \Lambda$, $a \in A$ such that

$$\lambda(a_1 + a_2) = \lambda a_1 + \lambda a_2, \qquad (\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a,$$
$$(\lambda_1\lambda_2)(a) = \lambda_1(\lambda_2 a), \qquad 1a = a.$$

We shall denote by 0 the module containing the zero element alone.

In the special case $\Lambda = Z$ is the ring of rational integers, the modules over Z are simply abelian groups. If Λ is a (commutative) field, they are the vector spaces over Λ.

Given two modules A and B (over the same ring Λ), a *homomorphism* (or linear transformation, or mapping) of A into B is a function f defined on A with values in B, such that $f(x + y) = fx + fy$; $f(\lambda x) = \lambda(fx)$; $x,y \in A$, $\lambda \in \Lambda$. We then write $f\colon A \to B$, or $A \to B$ if there is no ambiguity as to the definition of f. The *kernel* of f is the submodule of A consisting of all $x \in A$ such that $fx = 0$; it will be denoted by Ker (f) or Ker ($A \to B$). The *image* of f is the submodule of B consisting of all elements of the form fx, $x \in A$; it will be denoted by Im (f) or Im ($A \to B$).

We also define the *coimage* and *cokernel* of f as follows:

$$\text{Coim}\ (f) = A/\text{Ker}\ (f),$$
$$\text{Coker}\ (f) = B/\text{Im}\ (f).$$

Of course, f induces an isomorphism Coim $(f) \approx$ Im (f) and because of this isomorphism the coimage is very seldom employed.

A homomorphism $f\colon A \to B$ as is called a *monomorphism* if Ker $f = 0$; f is called an *epimorphism* if Coker $f = 0$ or equivalently if Im $f = B$. If f is both an epimorphism and a monomorphism then f is an *isomorphism* (notation: $f\colon A \approx B$).

Let A be a module and $\{A_\alpha\}$ a (finite or infinite) family of modules (all over the same ring Λ) with homomorphisms

$$A_\alpha \xrightarrow{i_\alpha} A \xrightarrow{p_\alpha} A_\alpha$$

such that $p_\alpha i_\alpha =$ identity, $p_\beta i_\alpha = 0$ if $\beta \neq \alpha$. We shall say that $\{i_\alpha, p_\alpha\}$ is a *direct family of homomorphisms*.

If we assume that each $x \in A$ can be written as a finite sum $x = \sum i_\alpha x_\alpha$, $x_\alpha \in A_\alpha$, it follows readily that A is isomorphic with the direct sum $\sum A_\alpha$. We therefore say that the family $\{i_\alpha, p_\alpha\}$ yields a *representation of A as a direct sum* of the modules A_α. In this case the mappings $\{p_\alpha\}$ can be defined using the $\{i_\alpha\}$ alone.

If we assume that for each family $\{x_\alpha\}$, $x_\alpha \in A_\alpha$, there is a unique $x \in A$ with $p_\alpha x = x_\alpha$, it follows readily that A is isomorphic with the direct product $\prod A_\alpha$. We therefore say that the family $\{i_\alpha, p_\alpha\}$ yields a *representation of A as a direct product* of the modules A_α. In this case the homomorphisms $\{i_\alpha\}$ can be defined using the $\{p_\alpha\}$ alone.

If the family $\{A_\alpha\}$ is finite, the notions of direct sum and direct product coincide. A finite direct family yields a direct sum (or direct product) representation if and only if $\sum i_\alpha p_\alpha =$ identity.

A sequence of homomorphisms

$$A_m \to A_{m+1} \to \cdots \to A_n, \qquad m + 1 < n$$

is said to be *exact* if for each $m < q < n$ we have Im $(A_{q-1} \to A_q)$ $=$ Ker $(A_q \to A_{q+1})$. Thus $A \to B$ is a monomorphism if and only if $0 \to A \to B$ is exact and an epimorphism if and only if $A \to B \to 0$ is exact. We shall also allow sequences which extend to infinity to the left or to the right or in both directions.

In particular, we shall consider exact sequences

$$(*) \qquad 0 \to A' \to A \to A'' \to 0.$$

Since $A' \to A$ is a monomorphism we may regard A' as a submodule of A. Since $A \to A''$ is an epimorphism with A' as kernel, we may regard A' as the quotient module A/A'. Thus $(*)$ may be replaced by

$$0 \to A' \to A \to A/A' \to 0.$$