Mathematics Monograph Series **12**

Nonlinear Complex Analysis and Its Applications

Guochun Wen   Dechang Chen   Zuoliang Xu

SCIENCE PRESS
Beijing

Responsible Editor: Zhang Yang

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

Printed in Beijing

ISBN 978-7-03-021296-2 (Beijing)

Preface

This book is a continuation and development of "boundary value problems for nonlinear elliptic equations and systems" and "elliptic, hyperbolic and mixed complex equations with parabolic degeneracy" (see [167]27),42)). A large portion of the work is devoted to boundary value problems for general elliptic complex equations of first, second and fourth order, initial-boundary value problems for nonlinear parabolic complex equations and systems of second order, and properties of solutions for hyperbolic complex equations of first and second order. Moreover, some results about first and second order complex equations of mixed (elliptic-hyperbolic) type are investigated. Applications of nonlinear complex analysis to continuum mechanics are also introduced.

In Chapters 1 and 2, various boundary value problems for general elliptic complex equations of first and second order under weaker conditions in multiply connected domains are discussed. These include the nonlinear Riemann-Hilbert problem and the Poincaré boundary value problem, where the lower terms of nonlinear elliptic complex equations contain an explicit nonlinear part, and domains may have non-smooth boundaries. In Chapter 3, we prove, in detail, the existence theorems of solutions of some boundary value problems for nonlinear elliptic systems of first, second and fourth order equations.

Chapter 4 addresses not only initial-boundary value problems for nonlinear nondivergent parabolic equations of second order with measurable coefficients, but also initial-boundary value problems for nonlinear nondivergent parabolic systems of second order equations with measurable coefficients. These materials are not available in any other published books.

In Chapter 5, the hyperbolic elements and hyperbolic complex functions are introduced, which are correspondents of complex functions in the theory of elliptic complex equations. On the basis of hyperbolic notations, the hyperbolic systems of first order equations and hyperbolic equations of second order are reduced to the complex forms. Boundary value problems for some hyperbolic complex equations of first and second order are then discussed. In Chapter 6, we consider boundary value problems for complex equations of mixed (elliptic-hyperbolic) type by using the complex analytic method. There are many open problems about complex equations of mixed type, which remain to be further investigated.

Applications of nonlinear complex analysis to continuum mechanics are considered, which can be seen in Chapter 7, where some free boundary problems in planar filtrations, gas dynamics and elastico-plastic mechanics are discussed.

Similarly to the book [168]1), the complex equations and boundary conditions studied in this book are rather general. However, two special features are presented in this book: one is that elliptic and parabolic complex equations are discussed in nonlinear cases and many boundary value problems are studied in multiply connected domains, and the other is that complex analytic methods are used to investigate various problems on elliptic, parabolic, hyperbolic equations and systems, as well as equations of mixed type.

The great majority of the contents in this book originates in studies of the authors and their cooperative colleagues, and a large number of results are published here for the first time. Many questions investigated in this book deserve further investigations. We sincerely hope the reader will enjoy reading the book.

Finally, the preparation of this book was supported by the National Natural Science Foundation of China (No.10671207), its support has provided a wonderful environment for us to obtain many results reported in this book. In the meantime the authors would like to acknowledge the editorial staff of Science Press for making the publication of this book possible.

Guochun Wen
Dechang Chen
and
Zuoliang Xu

Contents

Chapter 1

Nonlinear Elliptic Complex Equations of First Order

In this chapter, we mainly discuss continuous and discontinuous Riemann-Hilbert boundary value problems for some elliptic systems of first order equations including the degenerate elliptic systems of first order equations. Firstly we reduce the systems of first order equations with measurable coefficients to a class of complex equations, give the representations and a priori estimates of solutions of the boundary value problems for the class of elliptic complex equations, and then prove the existence and uniqueness of solutions for the boundary value problems.

1.1 Discontinuous Riemann-Hilbert Problem for Nonlinear Uniformly Elliptic Complex Equations of First Order

First of all, we reduce the general uniformly elliptic systems of first order equations with certain conditions to the complex equations, and then give a priori estimates of solutions of the discontinuous Riemann-Hilbert problem for the complex equations, finally we verify the solvability of the above boundary value problem.

1.1.1 Reduction of general uniformly elliptic systems of first order equations to the standard complex form

Let D be a bounded simply connected domain in $\mathbf{R}^2$ with the boundary ∂D. Without loss of generality, we consider that ∂D is a smooth closed curve $\partial D \in C^1_\mu$, where $\mu(0 < \mu < 1)$ is a positive number, because the requirement can be realized through a conformal mapping. We first consider the linear uniformly elliptic system of first order equations

$$\begin{cases} a_{11}u_x + a_{12}u_y + b_{11}v_x + b_{12}v_y = a_1u + b_1v + c_1, \\ a_{21}u_x + a_{22}u_y + b_{21}v_x + b_{22}v_y = a_2u + b_2v + c_2, \end{cases} \tag{1.1.1}$$

where the coefficients $a_{jk}, b_{jk}, a_j, b_j, c_j$ $(j, k = 1, 2)$ are known real bounded measurable functions of $(x, y) \in D$. The uniform ellipticity condition in D is as follows

$$\begin{aligned} J &= 4K_1K_4 - (K_2+K_3)^2 \\ &= 4K_5K_6 - (K_2-K_3)^2 \geqslant J_0 > 0, \quad K_1 > 0 \text{ in } D, \end{aligned} \tag{1.1.2}$$

in which J_0 is a positive constant and

$$K_1 = \begin{vmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{vmatrix}, \quad K_2 = \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix}, \quad K_3 = \begin{vmatrix} a_{12} & b_{11} \\ a_{22} & b_{21} \end{vmatrix},$$

$$K_4 = \begin{vmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{vmatrix}, \quad K_5 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad K_6 = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}.$$

From $J > 0$ it follows that

$$K_1K_6 > 0, \text{ or } K_1K_6 < 0, \quad \text{i.e. } K_1 > 0,\ K_6 \neq 0.$$

There is no harm in assuming that $K_6 > 0$. Hence from the elliptic system (1.1.1), we can solve v_x, v_y and obtain the system of equations

$$\begin{cases} v_y = au_x + bu_y + a_0u + b_0v + f_0, \\ -v_x = du_x + cu_y + c_0u + d_0v + g_0, \end{cases} \tag{1.1.3}$$

where $a = K_1/K_6$, $b = K_3/K_6$, $c = K_4/K_6$, $d = K_2/K_6$, and the uniform ellipticity condition (1.1.2) is transformed into the condition

$$\Delta = \frac{J}{4K_6^2} = ac - \frac{1}{4}(b+d)^2 \geqslant \Delta_0 > 0, \quad a > 0, \tag{1.1.4}$$

where Δ_0 is a positive constant and a, b, c, d are bounded for almost every point in D. Noting that

$$z = x + \mathrm{i}y, \quad w = u + \mathrm{i}v, \quad w_z = \frac{1}{2}(w_x - \mathrm{i}w_y), \quad w_{\bar z} = \frac{1}{2}(w_x + \mathrm{i}w_y),$$

$$u_x = \frac{1}{2}(w_z + \overline{w}_{\bar z} + w_{\bar z} + \overline{w}_z), \quad u_y = \frac{\mathrm{i}}{2}(w_z - \overline{w}_{\bar z} - w_{\bar z} + \overline{w}_z),$$

$$v_x = \frac{\mathrm{i}}{2}(-w_z + \overline{w}_{\bar z} - w_{\bar z} + \overline{w}_z), \quad v_y = \frac{1}{2}(w_z + \overline{w}_{\bar z} - w_{\bar z} - \overline{w}_z),$$

the system (1.1.3) can be written in the complex form

$$w_{\bar z} = Q_1(z)w_z + Q_2(z)\overline{w}_{\bar z} + A_1(z)w + A_2(z)\overline{w} + A_3(z), \tag{1.1.5}$$

where

$$Q_1(z) = \frac{-2q_2}{|q_1+1|^2 - |q_2|^2}, \quad Q_2(z) = \frac{|q_2|^2 - (q_1-1)(\overline{q_1}+1)}{|q_1+1|^2 - |q_2|^2},$$

$$q_1(z) = \frac{1}{2}[a + c + \mathrm{i}(d-b)], \quad q_2(z) = \frac{1}{2}[a - c + \mathrm{i}(d+b)].$$

On the basis of

$$|q_1+1|^2-|q_2|^2=\frac{1}{4}[(2+a+c)^2+(d-b)^2]-\frac{1}{4}[(a-c)^2+(d+b)^2]$$
$$=1+a+c+\left(\frac{d-b}{2}\right)^2+\Delta\geqslant 1+\Delta,$$

the uniform ellipticity condition (1.1.4) can be written in the complex form

$$|Q_1(z)|+|Q_2(z)|\leqslant q_0<1, \tag{1.1.6}$$

in which q_0 is a non-negative constant. If the coefficients $a_{jk}, b_{jk}\in W_p^1(D)$, $p>2$, $j,k=1,2$, then the following function $\eta(z)$ can be extended in $D_R=\{|z|\leqslant R\}$ $(\supset D,\ 0<R<\infty)$, such that $\eta(z)\in W_p^1(D_R)$, thus the Beltrami equation

$$\begin{cases}\zeta_{\bar z}-\eta(z)\zeta_z=0,\\ \eta(z)=\dfrac{2Q_1(z)}{1+|Q_1|^2-|Q_2|^2+\sqrt{[1+|Q_1|^2-|Q(z)|^2]^2-4|Q_1|^2}}\end{cases} \tag{1.1.7}$$

has a homeomorphic solution $\zeta(z)\,(\in W^2_{p_0}(D_R))$ with its inverse function $z(\zeta)\in W^2_{p_0}(G_R)$, herein $G_R=\zeta(D_R)$ and $p_0\,(2<p_0\leqslant p)$ is a positive constant. Setting $w=w[z(\zeta)]$, the complex equation (1.1.5) is reduced to the complex equation

$$w_{\bar\zeta}=Q(\zeta)\bar w_{\zeta}+B_1(\zeta)w+B_2(\zeta)\bar w+B_3(\zeta), \tag{1.1.8}$$

in which

$$Q(\zeta)=\frac{Q_2[z(\zeta)]}{1-\eta[z(\zeta)]\overline{Q_1[z(\zeta)]}},$$
$$B_1(\zeta)=\{A_1[z(\zeta)]+\overline{A_2[z(\zeta)]}Q(\zeta)\eta[z(\zeta)]\}\bar z_{\bar\zeta},$$
$$B_2(\zeta)=\{A_2[z(\zeta)]+\overline{A_1[z(\zeta)]}Q(\zeta)\eta[z(\zeta)]\}\bar z_{\bar\zeta},$$
$$B_3(\zeta)=\{A_3[z(\zeta)]+\overline{A_3[z(\zeta)]}Q(\zeta)\eta[z(\zeta)]\}\bar z_{\bar\zeta}.$$

Setting $W(\zeta)=w(\zeta)-Q(\zeta)\overline{w(\zeta)}$, the complex equation (1.1.8) can be transformed into the complex equation

$$W_{\bar\zeta}=C_1(\zeta)W+C_2(\zeta)\overline{W}+C_3(\zeta), \tag{1.1.9}$$

in which

$$C_1(\zeta)=\frac{B_1+(B_2-Q_{\bar\zeta})\overline{Q}}{1-|Q|^2},\quad C_2(\zeta)=\frac{B_1Q+B_2-Q_{\bar\zeta}}{1-|Q|^2},\quad C_3(\zeta)=B_3,$$

(see [167]9),[168]1)). This is a standard complex form of the uniformly elliptic system (1.1.1), which is called the nonhomogeneous generalized Cauchy-Riemann system, and the solution of homogeneous generalized Cauchy-Riemann system in D is called the generalized analytic function (see [159]1)).

For the nonlinear uniformly elliptic system of first order equations

$$F_j(x, y, u, v, u_x, v_x, u_y, v_y) = 0 \text{ in } D, \quad j = 1, 2 \tag{1.1.10}$$

under certain conditions, we can transform the system into the complex form

$$w_{\bar{z}} = F(z, w, w_z), \quad F = Q_1 w_z + Q_2 \overline{w}_{\bar{z}} + A_1 w + A_2 \bar{w} + A_3, \quad z \in D, \tag{1.1.11}$$

in which $Q_j = Q_j(z, w, w_z), j = 1, 2$, $A_j = A_j(z, w)$, $j = 1, 2, 3$ (see [167]9),[168]1)). We assume that equation (1.1.11) satisfy the following conditions.

Condition C

(1) $Q_j(z, w, U)\,(j = 1, 2)$, $A_j(z, w)\,(j = 1, 2, 3)$ are measurable in $z \in D$ for all continuous functions $w(z)$ in $D^* = \bar{D}\backslash Z$ and all measurable functions $U(z) \in L_{p_0}(D^*)$, and satisfy

$$L_p[A_j, \overline{D}] \leqslant k_0, \ j = 1, 2, \quad L_p[A_3, \overline{D}] \leqslant k_1, \tag{1.1.12}$$

where $Z = \{z_1, \cdots, z_m\}$, $z_1, \cdots, z_m$ are different points on the boundary ∂D arranged according to the positive direction successively, and $p_0, p\,(2 < p_0 \leqslant p)$, k_0, k_1 are non-negative constants.

(2) The above functions are continuous in $w \in \mathbf{C}$ for almost every point $z \in D$, $U \in \mathbf{C}$, and $Q_j = 0\,(j = 1, 2)$, $A_j = 0\,(j = 1, 2, 3)$ for $z \notin D$.

(3) The complex equation (1.1.11) satisfies the uniform ellipticity condition

$$|F(z, w, U_1) - F(z, w, U_2)| \leqslant q_0 |U_1 - U_2|, \tag{1.1.13}$$

for almost every point $z \in D$, in which $w, U_1, U_2 \in \mathbf{C}$ and $q_0\,(< 1)$ is a non-negative constant.

1.1.2 Representation of solutions of the discontinuous Riemann-Hilbert problem for elliptic complex equations

Let D be a bounded domain in $\mathbf{C}$ with the smooth boundary $\partial D = \Gamma$. Now we formulate the discontinuous Riemann-Hilbert problem for equation (1.1.11).

Problem A The discontinuous Riemann-Hilbert boundary value problem for (1.1.11) is to find a continuous solution $w(z)$ in D^* satisfying the boundary condition

$$\mathrm{Re}[\overline{\lambda(z)} w(z)] = r(z), \quad z \in \Gamma^* = \partial D \backslash Z, \tag{1.1.14}$$

where $\lambda(z), r(z)$ satisfy the conditions

$$C_\alpha[\lambda(z), \Gamma_j] \leqslant k_0, \quad C_\alpha[R_j(z)r(z), \Gamma_j] \leqslant k_2, \quad j = 1, \cdots, m, \tag{1.1.15}$$

in which $\lambda(z) = a(z) + ib(z), |\lambda(z)| = 1$ on ∂D, and $Z = \{z_1, \cdots, z_m\}$ are the first kind of discontinuous points of $\lambda(z)$ on ∂D, Γ_j is an arc from the point z_{j-1} to z_j on ∂D, and does not include the end points z_{j-1}, z_j $(j = 1, 2, \cdots, m)$, herein $z_0 = z_m$, $R_j(z) = |z - z_{j-1}|^{\beta_{j-1}}|z - z_j|^{\beta_j}$, $\alpha\,(1/2 < \alpha < 1), k_0, k_2, \beta = \min(\alpha, 1 - 2/p_0), \beta_j (0 < \beta_j < 1), \gamma_j$ are non-negative constants and satisfy the conditions

$$\beta_j + \gamma_j < \beta, \quad j = 1, \cdots, m, \tag{1.1.16}$$

where $\gamma_j (j = 1, \cdots, m)$ are as stated in (1.1.17) below. Problem A with $A_3(z) = 0$ in D, $r(z) = 0$ on Γ^* is called Problem A_0.

Denote by $\lambda(z_j - 0)$ and $\lambda(z_j + 0)$ the left limit and right limit of $\lambda(z)$ as $z \to z_j$ $(j = 1, 2, \cdots, m)$ on ∂D, and

$$\left\{\begin{array}{l} e^{i\phi_j} = \dfrac{\lambda(z_j - 0)}{\lambda(z_j + 0)}, \quad \gamma_j = \dfrac{1}{\pi i} \ln \dfrac{\lambda(z_j - 0)}{\lambda(z_j + 0)} = \dfrac{\phi_j}{\pi} - K_j, \\[2ex] K_j = \left[\dfrac{\phi_j}{\pi}\right] + J_j, \quad J_j = 0 \text{ or } 1, \quad j = 1, \cdots, m, \end{array}\right. \tag{1.1.17}$$

in which $0 \leqslant \gamma_j < 1$ when $J_j = 0$, and $-1 < \gamma_j < 0$ when $J_j = 1$, $j = 1, \cdots, m$. The index K of Problems A and A_0 is defined as follows

$$K = \frac{1}{2}(K_1 + \cdots + K_m) = \sum_{j=1}^{m} \left[\frac{\phi_j}{2\pi} - \frac{\gamma_j}{2}\right]. \tag{1.1.18}$$

If $\lambda(x)$ on Γ is continuous, then $K = \Delta_\Gamma \arg \lambda(x)/2\pi$ is a unique integer. Now the function $\lambda(x)$ on Γ is not continuous, we can choose $J_j = 0$ or 1, hence the index K is not unique. If we choose $K = -1/2$, then the solution of Problem A is unique.

In order to prove the solvability of Problem A for the complex equation (1.1.11), we need to give a representation theorem for Problem A.

Theorem 1.1.1 *Suppose that the complex equation* (1.1.11) *satisfies Condition* C, *and $w(z)$ is a solution of Problem* A *for* (1.1.11). *Then $w(z)$ is representable by*

$$w(z) = \Phi[\zeta(z)]e^{\phi(z)} + \psi(z), \tag{1.1.19}$$

where $\zeta(z)$ is a homeomorphism in $\bar{D}$, which quasiconformally maps D onto the unit disk $G = \{|\zeta| < 1\}$ with boundary $L = \{|\zeta| = 1\}$, such that three points on Γ

are mapped onto three points on L respectively, $\Phi(\zeta)$ *is an analytic function in G,* $\psi(z),\phi(z),\zeta(z)$ *and its inverse function* $z(\zeta)$ *satisfy the estimates*

$$C_\beta[\psi(z),\bar{D}]\leqslant k_3,\quad C_\beta[\phi(z),\bar{D}]\leqslant k_3,\quad C_\beta[\zeta(z),\bar{D}]\leqslant k_3,\quad C_\beta[z(\zeta),\bar{G}]\leqslant k_3, \tag{1.1.20}$$

$$L_{p_0}[|\psi_{\bar{z}}|+|\psi_z|,\bar{D}]\leqslant k_3,\quad L_{p_0}[|\phi_{\bar{z}}|+|\phi_z|,\bar{D}]\leqslant k_3, \tag{1.1.21}$$

$$C_\beta[z(\zeta),\bar{G}]\leqslant k_3,\quad L_{p_0}[|\chi_{\bar{z}}|+|\chi_z|,\bar{D}]\leqslant k_4, \tag{1.1.22}$$

in which $\chi(z)$ *is as stated in* (1.1.27) *below,* $\beta=\min(\alpha,1-2/p_0)$, $p_0\,(2<p_0\leqslant p)$, $k_j=k_j(q_0,p_0,\beta,k_0,k_1,D)\,(j=3,4)$ *are non-negative constants dependent on* q_0,p_0,β,k_0,k_1,D. *Moreover, if the coefficients* $Q_j(z)=0\,(j=1,2)$ *of the complex equation* (1.1.11) *in D, then the representation* (1.1.19) *becomes the form*

$$w(z)=\Phi(z)e^{\phi(z)}+\psi(z), \tag{1.1.23}$$

and when $K<0$, $\Phi(z)$ *satisfies the estimate*

$$C_\delta[X(z)\Phi(z),\bar{D}]\leqslant M_1=M_1(p_0,\delta,k,D)<\infty, \tag{1.1.24}$$

in which

$$X(z)=\prod_{j=1}^{m}|z-z_j|^{\eta_j},\quad \eta_j=\begin{cases}|\gamma_j|+\tau, & \gamma_j<0,\ \beta_j\leqslant|\gamma_j|,\\ |\beta_j|+\tau, & for\ other\ case.\end{cases} \tag{1.1.25}$$

Here $\gamma_j\,(j=1,\cdots,m)$ *are real constants as stated in* (1.1.17), $\tau,\delta\,(0<\delta<\min(\beta,\tau))$ *are sufficiently small positive constants,* $k=(k_0,k_1,k_2)$, *and* M_1 *is a non-negative constant dependent on* p_0,δ,k,D.

Proof We substitute the solution $w(z)$ of Problem A into the coefficients of equation (1.1.11) and consider the following system

$$\begin{cases}\psi_{\bar{z}}=Q\psi_z+A_1\psi+A_2\bar{\psi}+A_3,\quad Q=\begin{cases}Q_1+Q_2\overline{w_z}/w_z, & w_z\neq0,\\ 0, & w_z=0\ \text{or}\ z\notin D,\end{cases}\\ \phi_{\bar{z}}=Q\phi_z+A,\quad A=\begin{cases}A_1+A_2(\overline{w-\psi})/(w-\psi), & w(z)-\psi(z)\neq0,\\ 0, & w(z)-\psi(z)=0\ \text{or}\ z\notin D,\end{cases}\\ W_{\bar{z}}=QW_z,\quad W(z)=\Phi[\zeta(z)].\end{cases} \tag{1.1.26}$$

By using the continuity method and the principle of contracting mapping, we can find the solution

$$\begin{cases}\psi(z)=Tf=-\dfrac{1}{\pi}\displaystyle\iint_D\frac{f(\zeta)}{\zeta-z}d\sigma_\zeta,\\ \phi(z)=Tg,\quad \zeta(z)=\Psi[\chi(z)],\quad \chi(z)=z+Th\end{cases} \tag{1.1.27}$$

of (1.1.26), where $f(z), g(z), h(z) \in L_{p_0}(\overline{D})$, $2 < p_0 \leqslant p$, $\chi(z)$ is a homeomorphism in $\bar{D}$, $\Psi(\chi)$ is a univalent analytic function, which conformally maps $E = \chi(D)$ onto the unit disk G (see [159]1)), and $\Phi(\zeta)$ is an analytic function in G. We can verify that $\psi(z), \phi(z), \zeta(z)$ satisfy the estimates (1.1.20) and (1.1.21). It remains to prove that $z = z(\zeta)$ satisfies the estimate (1.1.22). In fact, we can find a homeomorphic solution of the last equation in (1.1.26) in the form $\chi(z) = z + Th$ such that $[\chi(z)]_z, [\chi(z)]_{\bar{z}} \in L_{p_0}(\bar{D})$ (see [168]1)). Next, we find a univalent analytic function $\zeta = \Psi(\chi)$, which maps $\chi(D)$ onto G, hence $\zeta = \zeta(z) = \Psi[\chi(z)]$. By the result on conformal mappings and the method of Lemma 2.1, Chapter II in [168]1), we can prove that (1.1.22) is true. When $Q_j(z) = 0$ in D, $j = 1, 2$, then we can choose $\chi(z) = z$ in (1.1.27). In this case $\Phi[\zeta(z)]$ can be replaced by the analytic function $\Phi(z)$, herein $\Psi(z), \zeta(z)$ are as stated in (1.1.27). It is clear that the representation (1.1.19) becomes the form (1.1.23). Thus the analytic function $\Phi(z)$ satisfies the boundary conditions

$$\mathrm{Re}[\overline{\lambda(z)}e^{\phi(z)}\Phi(z)] = r(z) - \mathrm{Re}[\overline{\lambda(z)}\psi(z)], \quad z \in \Gamma^*. \tag{1.1.28}$$

On the basis of the estimate (1.1.20), and by the methods in the proof of Theorem 1.1 or 1.8, Chapter IV in [168]1), we can prove that $\Phi(z)$ satisfies the estimate (1.1.24).

1.1.3 Existence of solutions of the discontinuous Riemann-Hilbert problem for nonlinear complex equations in the upper half-unit disk

We first consider a special domain, i.e. D is an upper half-unit disk with the boundary $\Gamma' = \Gamma \cup \gamma$, where $\Gamma = \{|z| = 1, \mathrm{Im} z > 0\}$ and $\gamma = \{-1 < x < 1, y = 0\}$.

Theorem 1.1.2 *Under the same conditions as in Theorem* 1.1.1 *for the above domain D, the following statements hold.*

(1) *If the index $K \geqslant 0$, then Problem* A *for* (1.1.11) *is solvable, and the general solution includes* $2K + 1$ *arbitrary real constants.*

(2) *If $K < 0$, then Problem* A *has* $-2K - 1$ *solvability conditions.*

Proof Let us introduce a closed, convex and bounded subset B_1 in the Banach space $B = L_{p_0}(\bar{D}) \times L_{p_0}(\bar{D}) \times L_{p_0}(\bar{D})$ $(2 < p_0 \leqslant p)$, whose elements are systems of functions $q = [Q(z), f(z), g(z)]$ with norms $\|q\| = L_{p_0}(Q, \bar{D}) + L_{p_0}(f, \bar{D}) + L_{p_0}(g, \bar{D})$ satisfying the conditions

$$|Q(z)| \leqslant q_0 < 1, \quad L_{p_0}[f(z), \bar{D}] \leqslant k_3, \quad L_{p_0}[g(z), \bar{D}] \leqslant k_3, \quad z \in D, \tag{1.1.29}$$

where q_0, k_3 are non-negative constants as stated in (1.1.13) and (1.1.21). Moreover we introduce a closed and bounded subset B_2 in B, the elements of which are systems of functions $\omega = [f(z), g(z), h(z)]$ satisfying the condition

$$L_{p_0}[f(z), \bar{D}] \leqslant k_4, \quad L_{p_0}[g(z), \bar{D}] \leqslant k_4, \quad |h(z)| \leqslant q_0|1 + \Pi h|, \tag{1.1.30}$$

where $\Pi h = -\frac{1}{\pi}\iint_D [h(\zeta)/(\zeta - z)^2]d\sigma_\zeta$.

Arbitrarily selecting $q = [Q(z), f(z), g(z)] \in B_1$, and using the principle of contracting mapping, we see that a unique solution $h(z) \in L_{p_0}(\bar{D})$ of the integral equation

$$h(z) = Q(z)[1 + \Pi h] \tag{1.1.31}$$

can be found, which satisfies the third inequality in (1.1.30). Moreover, $\chi(z) = z + Th$ is a homeomorphism in $\bar{D}$. Now, we find a univalent analytic function $\zeta = \Psi(\chi)$, which maps $\chi(D)$ onto the unit disk G as stated in Theorem 1.1.1. Moreover, we find an analytic function $\Phi(\zeta)$ in G satisfying the boundary condition in the form

$$\mathrm{Re}[\overline{\Lambda(\zeta)}\Phi(\zeta)] = R(\zeta), \quad \zeta \in L^* = \zeta(\Gamma^*), \tag{1.1.32}$$

in which $\zeta(z) = \Psi[\chi(z)]$ with $z(\zeta)$ as its inverse function, $\psi(z) = Tf$, $\phi(z) = Tg$, $\Lambda(\zeta) = \lambda[z(\zeta)]\exp[\overline{\phi(z(\zeta))}]$, $R(\zeta) = r[z(\zeta)] - \mathrm{Re}[\overline{\lambda(z(\zeta))}\psi(z(\zeta))]$, where $\Lambda(\zeta)$, $R(\zeta)$ on L^* satisfy conditions similar to those of $\lambda(z), r(z)$ in (1.1.15) and the index of $\Lambda(\zeta)$ on L is K. In the following, we first consider the case $K \geqslant 0$. By using Theorem 1.1.1, we can find the analytic function $\Phi(\zeta)$ in the form (1.73), Chapter I, [168]1), where $2K + 1$ arbitrary real constants can be chosen. Thus the function $w(z) = \Phi[\zeta(z)]e^{\phi(z)} + \psi(z)$ is determined. Afterwards, we find out the solution $[f^*(z), g^*(z), h^*(z), Q^*(z)]$ of the system of integral equations

$$f^*(z) = F(z, w, \Pi f^*) - F(z, w, 0) + A_1(z, w)Tf^* + A_2(z, w)\overline{Tf^*} + A_3(z, w), \tag{1.1.33}$$

$$Wg^*(z) = F(z, w, W\Pi g^* + \Pi f^*) - F(z, w, \Pi f^*) + A_1(z, w)W + A_2(z, w)\overline{W}, \tag{1.1.34}$$

$$\begin{aligned} S'(\chi)h^*(z)e^{\phi(z)} = {} & F(z, w, S'(\chi)(1 + \Pi h^*)e^{\phi(z)} + W\Pi g^* + \Pi f^*) \\ & - F(z, w, W\Pi g^* + \Pi f^*), \end{aligned} \tag{1.1.35}$$

$$Q^*(z) = \frac{h^*(z)}{[1 + \Pi h^*]}, \quad S'(\chi) = [\Phi(\Psi(\chi))]_\chi \tag{1.1.36}$$

and denote by $q^* = E(q)$ the mapping from $q = (Q, f, g)$ to $q^* = (Q^*, f^*, g^*)$. According to Lemma 5.5 from Chapter III in [168]1), we can prove that $q^* = E(q)$ continuously maps B_1 onto a compact subset in B_1. By means of the Schauder fixed-point theorem, there exists a system $q = (Q, f, g) \in B_1$, such that $q = E(q)$. Applying the above method, from $q = (Q, f, g)$, we can construct a function $w(z) =$

$\Phi[\zeta(z)]e^{\phi(z)} + \psi(z)$, which is just a solution of Problem A for (1.1.11). As for the case $K < 0$, it can be similarly discussed. But we first permit that the function $\Phi(\zeta)$ satisfying the boundary condition (1.1.32) has a pole of order $|[K + 1]|$ at $\zeta = 0$. If $-2K$ is an even integer, then we need to add a point condition: $\mathrm{Im}[\overline{\lambda(z_0')}w(z_0')] = b_0$, where z_0' is a fixed point on Γ^*, b_0 is a real constant, and then find the solution of the nonlinear complex equation (1.1.11) in this case. From the representation $w(z) = \Phi[\zeta(z)]e^{\phi(z)} + \psi(z)$, we can derive the $-2K - 1$ solvability conditions of Problem A for (1.1.11).

Besides, we can discuss the solvability of the discontinuous Riemann-Hilbert boundary value problem for the complex equation (1.1.11) in the upper half-plane and the zone domain. We mention that some problems in nonlinear mechanics as stated in [21]2) and [120]2) can be solved by the results in Theorem 1.1.2.

1.1.4 The discontinuous Riemann-Hilbert problem for nonlinear complex equations in general domains

In this subsection, let D' be a general simply connected domain with the boundary $\Gamma' = \Gamma_1' \cup \Gamma_2'$, herein $\Gamma_1', \Gamma_2' \in C_\mu^1\,(0 < \mu < 1)$ and their intersection points z', z'' with the inner angles $\alpha_1\pi, \alpha_2\pi\,(0 < \alpha_1, \alpha_2 < 1)$ respectively. We discuss the nonlinear uniformly elliptic complex equation

$$w_{\bar z} = F(z, w, w_z), \quad F = Q_1 w_z + Q_2 \overline{w}_{\bar z} + A_1 w + A_2 \bar w + A_3, \quad z \in D', \tag{1.1.37}$$

in which $F(z, w, U)$ satisfies Condition C in D'. There exist m points $Z = \{z_1 = z', \cdots, z_n = z'', \cdots, z_m = z_0\}$ on Γ' arranged according to the positive direction successively. Denoted by Γ_j the curve on Γ' from z_{j-1} to $z_j\,(j = 1, 2, \cdots, m)$, where Γ_j does not include the end points $z_{j-1}, z_j\,(j = 1, \cdots, m)$.

Problem A′ The discontinuous Riemann-Hilbert boundary value problem for (1.1.37) is to find a continuous solution $w(z)$ in $D^* = \overline{D'}\backslash Z$ satisfying the boundary condition

$$\begin{cases} \mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z), & x \in \Gamma^* = \Gamma'\backslash Z, \\ \mathrm{Im}[\overline{\lambda(z_j')}w(z_j')] = b_j, & j = 1, \cdots, 2K+1, \end{cases} \tag{1.1.38}$$

where $z_1', \cdots, z_{2K+1}'\,(\notin Z)$ are distinct points on Γ' and $b_j\,(j = 1, \cdots, 2K + 1)$ are real constants, and $\lambda(z)$, $r(z)$, $b_j\,(j = 1, \cdots, 2K + 1)$ are given functions satisfying

$$\begin{aligned} &C_\alpha[\lambda(z), \Gamma_j] \leqslant k_0, \quad C_\alpha[R_j(z) r(z), \Gamma_j] \leqslant k_2, \quad j = 1, \cdots, m, \\ &|b_j| \leqslant k_2, \quad j = 1, \cdots, 2K + 1. \end{aligned} \tag{1.1.39}$$

Herein $\alpha\,(1/2 < \alpha < 1)$, k_0, k_2 are non-negative constants, $R_j(z) = |z - z_{j-1}|^{\beta_{j-1}} \times |z - z_j|^{\beta_j}$, $\beta_j + \gamma_j < \beta = \alpha_0 \min(\alpha, 1 - 2/p_0)$, $\gamma_j, \beta_j (j = 1, \cdots, m)$ are similar to those in (1.1.16) and (1.1.17), and $\alpha_0 = \min(\alpha_1, \alpha_2)$. Problem A$'$ with $A_3(z) = 0$ in D', $r(z) = 0$ on Γ', and $b_j = 0\,(j = 1, \cdots, 2K+1)$ is called Problem A_0', in which K ($\geqslant -1/2$) is the index of $\lambda(z)$ on Γ' as defined in (1.1.18).

In order to give the uniqueness result of solutions of Problem A$'$ for equation (1.1.37), we need to add one condition: For any complex functions $w_j(z) \in C(D^*), U_j(z) \in L_{p_0}(D^*)\,(2 < p_0 \leqslant p, j = 1, 2)$, the following equality holds

$$F(z, w_1, U_1) - F(z, w_1, U_2) = Q(U_1 - U_2) + A(w_1 - w_2) \text{ in } D', \tag{1.1.40}$$

in which $|Q(z, w_1, w_2, U_1, U_2)| \leqslant q_0$, $A(z, w_1, w_2) \in L_{p_0}(D')$. Especially, if (1.1.37) is a linear equation, then the condition (1.1.40) obviously holds.

Applying a similar method as stated in the proof of Theorem 1.1.1, we can prove the following theorem.

Theorem 1.1.3 *If the complex equation* (1.1.37) *in* D' *satisfies Condition* C, *then Problem* A$'$ *for* (1.1.37) *is solvable. If Condition* C *and the condition* (1.1.40) *hold, then the solution of Problem* A$'$ *is unique. Moreover the solution* $w(z)$ *can be expressed as* (1.1.19) *satisfying the estimates* (1.1.20)–(1.1.22), *where* $\beta = \alpha_0 \min(\alpha, 1 - 2/p_0)$. *If* $Q_j(z) = 0\,(j = 1, 2), z \in D'$ *in* (1.1.37), *then the representation* (1.1.19) *becomes the form*

$$w(z) = \Phi(z)\mathrm{e}^{\phi(z)} + \psi(z), \tag{1.1.41}$$

and $w(z)$ *satisfies the estimate*

$$C_\delta[X(z)w(z), \bar{D}] \leqslant M_2 = M_2(p_0, \delta, k, D') < \infty, \tag{1.1.42}$$

in which

$$\left\{\begin{array}{l} X(z) = \displaystyle\prod_{j=1, j\neq 1, n}^{m} |z - z_j|^{\eta_j} |z - z_1|^{\eta_1/\alpha_1} |z - z_n|^{\eta_n/\alpha_2}, \\ \eta_j = \left\{\begin{array}{ll} |\gamma_j| + \tau, & \gamma_j < 0,\ \beta_j \leqslant |\gamma_j|, \\ |\beta_j| + \tau, & \gamma_j \geqslant 0 \text{ and } \gamma_j < 0, \beta_j > |\gamma_j|. \end{array}\right. \end{array}\right. \tag{1.1.43}$$

Here $\gamma_j (j = 1, \cdots, m)$ *are real constants as stated in* (1.1.17), $\tau, \delta\,(0 < \delta < \min(\beta, \tau))$ *are sufficiently small positive constants, and* $M_2 = M_2(p_0, \delta, k, D')$ *is a non-negative constant dependent on* p_0, δ, k, D' (see [167]42),[183]6)).

1.2 Boundary Value Problems for Elliptic Complex Equations with Nonsmooth Boundary

This section mainly deals with the Riemann-Hilbert problem for general nonlinear elliptic systems of first order equations in bounded domains with a non-smooth