FOURIER SERIES AND BOUNDARY VALUE PROBLEMS

CHURCHILL

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BY

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PREFACE

This is an introductory treatment of Fourier series and their application to the solution of boundary value problems in the partial differential equations of physics and engineering. It is designed for students who have had an introductory course in ordinary differential equations and one semester of advanced calculus, or an equivalent preparation. The concepts from the field of physics which are involved here are kept on an elementary level. They are explained in the early part of the book, so that no previous preparation in this direction need be assumed.

The first objective of this book is to introduce the reader to the concept of orthogonal sets of functions and to the basic ideas of the use of such functions in representing arbitrary functions. The most prominent special case, that of representing an arbitrary function by its Fourier series, is given special attention. The Fourier integral representation and the representation of functions by series of Bessel functions and Legendre polynomials are also treated individually, but somewhat less fully. The material covered is intended to prepare the reader for the usual applications arising in the physical sciences and to furnish a sound background for those who wish to pursue the subject further.

The second objective is a thorough acquaintance with the classical process of solving boundary value problems in partial differential equations, with the aid of those expansions in series of orthogonal functions. The boundary value problems treated here consist of a variety of problems in heat conduction, vibration, and potential. Emphasis is placed on the formal method of obtaining the solutions of such problems. But attention is also given to the matters of fully establishing the results as solutions and of investigating their uniqueness, for the process cannot be properly presented without some consideration of these matters.

The book is intended to be both elementary and mathematically sound. It has been the author's experience that careful attention to the mathematical development, in contrast to more formal procedures, contributes much to the student's interest as well as to his understanding of the subject, whether he is a student of pure or of applied mathematics. The few theorems that are stated here without proofs appear at the end of the discussion of the topics concerned, so they do not reflect upon the completeness of the earlier part of the development.

Illustrative examples are given whenever new processes are involved.

The problems form an essential part of such a book. A rather generous supply and wide variety will be found here. Answers are given to all but a few of the problems.

The chapters on Bessel functions and Legendre polynomials (Chaps. VIII and IX) are independent of each other, so that they can be taken up in either order. The continuity of the subject matter will not be interrupted by omitting the chapter on the uniqueness of solutions of boundary value problems (Chap. VII) or by omitting certain parts of other chapters.

This volume is a revision and extension of a planographed form developed by the author in a course given for many years to students of physics, engineering, and mathematics at the University of Michigan. It is to be followed soon by a more advanced book on further methods of solving boundary value problems.

The selection and presentation of the material for the present volume have been influenced by the works of a large number of authors, including Carslaw, Courant, Byerly, Bôcher, Riemann and Weber, Watson, Hobson, and several others.

To Dr. E. D. Rainville and Dr. R. C. F. Bartels the author wishes to express his gratitude for valuable suggestions and for their generous assistance with the reading of proof. In the preparation of the manuscript he has been faithfully assisted by his daughter, who did most of the typing, and by his wife and son.

RUEL V. CHURCHILL.

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CHAPTER I

INTRODUCTION

1. The Two Related Problems. We shall be concerned here with two general types of problems: (*a*) the expansion of an arbitrarily given function in an infinite series whose terms are certain prescribed functions and (*b*) boundary value problems in the partial differential equations of physics and engineering. These two problems are so closely related that there are many advantages, especially to those interested in applied mathematics, in an introductory treatment that deals with both of them together.

In fact an acquaintance with the expansion theory is necessary for the study of boundary value problems. The expansion problem can be treated independently. It is an interesting problem in pure mathematics, and its applications are not confined to boundary value problems. But it gains in unity and interest when presented as a problem arising in the solution of partial differential equations.

The series in the problem type (*a*) is a Fourier series when its terms are certain linear combinations of sines and cosines. Fourier encountered this expansion problem, and made the first extensive treatment of it, in his development of the mathematical theory of the conduction of heat in solids.* Before Fourier's work, however, the investigations of others, notably D. Bernoulli and Euler, on the vibrations of strings, columns of air, elastic rods, and membranes, and of Legendre and Laplace on the theory of gravitational potential, had led to expansion

* Fourier, "Théorie analytique de la chaleur," 1822. A translation of this book by Freeman appeared in 1878 under the title "The Analytical Theory of Heat."

problems of the kind treated by Fourier as well as the related problems of expanding functions in series of Bessel functions, Legendre polynomials, and spherical harmonic functions.

These physical problems which led the early investigators to the various expansions are all examples of boundary value problems in partial differential equations. Our plan of presentation here is in agreement with the historical development of the subject.

The expansion problem as presented here will stress the development of functions in Fourier series. But we shall also consider the related generalized Fourier development of an arbitrary function in series of orthogonal functions, including the important series of Bessel functions and Legendre polynomials.

2. Linear Differential Equations. An equation in a function of two or more variables and its partial derivatives is called a partial differential equation. The *order* of a partial differential equation, as in the case of an ordinary differential equation, is that of the highest ordered derivative appearing in it. Thus the equation

$$\frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial y} = 3xy \tag{1}$$

is one of the second order.

A partial differential equation is *linear* if it is of the first degree in the unknown function and its derivatives. The equation

$$\frac{\partial^2 u}{\partial x^2} + xy^2\frac{\partial u}{\partial y} = 3xy \tag{2}$$

is linear; equation (1) is nonlinear. If the equation contains only terms of the first degree in the function and its derivatives, it is called a *linear homogeneous* equation. Equation (2) is nonhomogeneous, but the equation

$$\frac{\partial^2 u}{\partial x^2} + xy^2\frac{\partial u}{\partial y} = 0$$

is linear and homogeneous.

Thus the general linear partial differential equation of the second order, in two independent variables x and y, is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

where the letters $A, B, \cdots, G$, represent functions of x and y. If F is identically zero, the equation is homogeneous.

The following theorem is sometimes referred to as the *principle of superposition* of solutions.

Theorem 1. *Any linear combination of two solutions of a linear homogeneous differential equation is again a solution.*

The proof for the ordinary equation

$$y'' + Py' + Qy = 0, \tag{3}$$

where P and Q may be functions of x, will show how the proof can be written for any linear homogeneous differential equation, ordinary or partial.

Let $y = y_1(x)$ and $y = y_2(x)$ be two solutions of equation (3). Then

$$y_1'' + Py_1' + Qy_1 = 0, \tag{4}$$

$$y_2'' + Py_2' + Qy_2 = 0. \tag{5}$$

It is to be shown that any linear combination of y_1 and y_2—namely, $Ay_1 + By_2$, where A and B are arbitrary constants—is a solution of equation (3). By multiplying equations (4) by A and (5) by B and adding, the equation

$$Ay_1'' + By_2'' + P(Ay_1' + By_2') + Q(Ay_1 + By_2) = 0$$

is obtained. This can be written

$$\frac{d^2}{dx^2}(Ay_1 + By_2) + P\frac{d}{dx}(Ay_1 + By_2) + Q(Ay_1 + By_2) = 0,$$

which is a statement that $Ay_1 + By_2$ is a solution of equation (3).

For an ordinary differential equation of order n, a solution containing n arbitrary constants is known as the general solution. But a partial differential equation of order n has in general a solution containing n arbitrary functions. These are functions of $k - 1$ variables, where k represents the number of independent variables in the equation. On those few occasions here where we consider such solutions, we shall refer to them as "general solutions" of the partial differential equations. But the collection of all possible solutions of a partial differential equation is not simple enough to be represented by just this "general solution" alone.*

* See, for instance, Courant and Hilbert, "Methoden der mathematischen Physik," Vol. 2, Chap. I; or Forsyth, "Theory of Differential Equations," Vols. 5 and 6.

Consider, for example, the simple partial differential equation in the function $u(x, y)$:

$$\frac{\partial u}{\partial x} = 0.$$

According to the definition of the partial derivative, the solution is

$$u = f(y),$$

where $f(y)$ is an arbitrary function. Similarly, when the equation

$$\frac{\partial^2 u}{\partial x^2} = 0$$

is written $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 0$, its general solution is seen to be

$$u = xf(y) + g(y),$$

where $f(y)$ and $g(y)$ are arbitrary functions.

PROBLEMS

1. Prove Theorem 1 for Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

2. Prove Theorem 1 for the heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

Note that k may be a function of x, y, z, and t here.

3. Show by means of examples that the statement in Theorem 1 is not always true when the differential equation is nonhomogeneous.

4. Show that $y = f(x + at)$ and $y = g(x - at)$ satisfy the simple wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2},$$

where a is a constant and f and g are arbitrary functions, and hence that a general solution of that equation is

$$y = f(x + at) + g(x - at).$$

5. Show that $e^{-n^2 t} \sin nx$ is a solution of the simple heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

If $A_1, A_2, \cdots, A_N$ are constants, show that the function

$$u = \sum_{n=1}^{N} A_n e^{-n^2 t} \sin nx$$

is a solution having the value zero at $x = 0$ and $x = \pi$, for all t.

3. Infinite Series of Solutions. Let u_n $(n = 1, 2, 3, \cdots)$ be an infinite set of functions of any number of variables such that the series

$$u_1 + u_2 + \cdots + u_n + \cdots$$

converges to a function u. If the series of derivatives of u_n, with respect to one of the variables, converges to the same derivative of u, then the first series is said to be *termwise differentiable* with respect to that variable.

Theorem 2. *If each of the functions $u_1, u_2, \cdots, u_n, \cdots$, is a solution of a linear homogeneous differential equation, the function*

$$u = \sum_{1}^{\infty} u_n$$

is also a solution provided this infinite series converges and is termwise differentiable as far as those derivatives which appear in the differential equation are concerned.

Consider the proof for the differential equation

$$\frac{\partial^2 u}{\partial x^2} + p\,\frac{\partial^2 u}{\partial x\,\partial t} + qu = 0, \tag{1}$$

where p and q may be functions of x and t. Let each of the functions $u_n(x, t)$ $(n = 1, 2, \cdots)$ satisfy equation (1). The series

$$\sum_{1}^{\infty} u_n(x, t)$$

is assumed to be convergent and termwise differentiable; hence if $u(x, t)$ represents its sum, then

$$\frac{\partial u}{\partial x} = \sum_{1}^{\infty} \frac{\partial u_n}{\partial x}, \qquad \frac{\partial^2 u}{\partial x^2} = \sum_{1}^{\infty} \frac{\partial^2 u_n}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial x\,\partial t} = \sum_{1}^{\infty} \frac{\partial^2 u_n}{\partial x\,\partial t}.$$

Substituting these, the left-hand member of equation (1) becomes

$$\sum_{1}^{\infty} \frac{\partial^2 u_n}{\partial x^2} + p \sum_{1}^{\infty} \frac{\partial^2 u_n}{\partial x\,\partial t} + q \sum_{1}^{\infty} u_n, \tag{2}$$

and if this quantity vanishes, the theorem is true. Now expression (2) can be written

$$\sum_{1}^{\infty}\left(\frac{\partial^2 u_n}{\partial x^2} + p\,\frac{\partial^2 u_n}{\partial x\,\partial t} + q u_n\right),$$

since the series obtained by adding three convergent series term by term converges to the sum of the three functions represented by those series. Since u_n is a solution of equation (1),

$$\frac{\partial^2 u_n}{\partial x^2} + p\,\frac{\partial^2 u_n}{\partial x\,\partial t} + q u_n = 0 \quad (n = 1, 2, \cdots),$$

and so expression (2) is equal to zero. Hence $u(x, t)$ satisfies equation (1).

This proof depends only upon the fact that the differential equation is linear and homogeneous. It can clearly be applied to any such equation regardless of its order or number of variables.

4. Boundary Value Problems. In applied problems in differential equations a solution which satisfies some specified conditions for given values of the independent variables is usually sought. These conditions are known as the boundary conditions. The differential equation together with these boundary conditions constitutes a boundary value problem. The student is familiar with such problems in ordinary differential equations. Consider, for example, the following problem.

A body moves along the x-axis under a force of attraction toward the origin proportional to its distance from the origin. If it is initially in the position $x = 0$ and its position one second later is $x = 1$, find its position $x(t)$ at every instant.

The displacement $x(t)$ must satisfy the conditions

$$\frac{d^2x}{dt^2} = -k^2 x, \tag{1}$$

$$x = 0 \text{ when } t = 0, \qquad x = 1 \text{ when } t = 1, \tag{2}$$

where k is a constant. The boundary value problem here consists of the equation (1) and the boundary conditions (2), which assign values to the function x at the extremities (or on the boundary) of the time interval from $t = 0$ to $t = 1$.

The general solution of equation (1) is

$$x = C_1 \cos kt + C_2 \sin kt.$$

According to the conditions (2), $C_1 = 0$ and $C_2 = 1/\sin k$, so the solution of the problem is

$$x = \frac{\sin kt}{\sin k}.$$

From this the initial velocity which makes $x = 1$ when $t = 1$ can be written

$$\frac{dx}{dt} = \frac{k}{\sin k} \qquad \text{when } t = 0.$$

This condition could have been used in place of either of the conditions (2) to form another boundary value problem with the same solution.

In general, the boundary conditions may contain conditions on the derivatives of the unknown function as well as on the function itself.

The method corresponding to the one just used can sometimes be applied in partial differential equations. Consider, for instance, the following boundary value problem in $u(x, y)$:

$$\frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

$$u(0, y) = y^2, \qquad u(1, y) = 1. \tag{4}$$

Here the values of u are prescribed on the boundary, consisting of the lines $x = 0$ and $x = 1$, of the infinite strip in the xy-plane between those lines.

The general solution of equation (3) is

$$u(x, y) = xf(y) + g(y),$$

where $f(y)$ and $g(y)$ are arbitrary functions. The conditions (4) require that

$$g(y) = y^2, \qquad f(y) + g(y) = 1, \tag{5}$$

so $f(y) = 1 - y^2$, and the solution of the problem is

$$u(x, y) = x(1 - y^2) + y^2.$$

But it is only in exceptional cases that problems in partial differential equations can be solved by the above method. The general solution of the partial differential equation usually cannot be found in any practical form. But even when a general solution is known, the functional equations, corresponding to equations (5), which are given by the boundary conditions are often too difficult to solve. A more powerful method will be developed in the following chapters—a method of combining particular solutions with the aid of Theorems 1 and 2. It is, of course, limited to problems possessing a certain linear character.

The number and character of the boundary conditions which completely determine a solution of a partial differential equation depend upon the character of the equation. In the physical applications, however, the interpretation of the problem will indicate what boundary conditions are needed. If, after a solution of the problem is established, it is shown that only one solution is possible, the problem will have been shown to be completely stated as well as solved.

PROBLEMS

1. Solve the boundary value problem

$$\frac{\partial^2 u}{\partial x\, \partial y} = 0; \qquad u(0, y) = y, \qquad u(x, 0) = \sin x.$$

Ans. $u = y + \sin x$.

2. Solve the boundary value problem

$$\frac{\partial^2 u}{\partial x\, \partial y} = 2x; \qquad u(0, y) = 0, \qquad u(x, 0) = x^2.$$

Ans. $u = x^2 y + x^2$.

3. Solve Prob. 2 when the second boundary condition is replaced by the condition

$$\frac{\partial u(x, 0)}{\partial x} = x^2.$$

Ans. $u = x^2 y + \frac{1}{3}x^3$.

4. By substituting the new independent variables

$$\lambda = x + at, \qquad \mu = x - at,$$

show that the wave equation $\partial^2 y/\partial t^2 = a^2(\partial^2 y/\partial x^2)$ becomes

$$\frac{\partial^2 y}{\partial \lambda\, \partial \mu} = 0,$$

and so derive the general solution of the wave equation (Prob. 4, Sec. 2).

5. Solve the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}; \qquad y(x, 0) = F(x), \qquad \frac{\partial y(x, 0)}{\partial t} = 0,$$

where $F(x)$ is a given function defined for all real x.

Ans. $y = \frac{1}{2}[F(x + at) + F(x - at)]$.

6. Solve Prob. 5 if the boundary conditions are replaced by

$$y(x, 0) = 0, \qquad \frac{\partial y(x, 0)}{\partial t} = G(x).$$

Also show that the solution under the more general conditions

$$y(x, 0) = F(x), \qquad \frac{\partial y(x, 0)}{\partial t} = G(x)$$

is obtained by adding the solution just found to the solution of Prob. 5.

Ans. $y = (1/2a) \int_{x-at}^{x+at} G(\xi)\, d\xi$.