Dynamical and Physical METEOROLOGY

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PREFACE

The rapid advance in the fields of dynamical and physical meteorology during the last decade has created a need for texts incorporating the results of recent research. These fields have become so broad that to confine a text of this kind to a reasonable size it is necessary to make a selection of topics. In general only those phases of physical meteorology that are more closely linked with dynamic meteorology are included. The authors endeavor to develop most topics from first principles and to bring the subject to a point near its present stage. However, the detailed development of any topic is carried only as far as the limited mathematics required of the reader permits. Beyond this point, a qualitative discussion of further advances is frequently made. Probably no two instructors would agree exactly on a list of subjects to be covered in a text of limited size; nevertheless the authors feel, on the basis of their experience in teaching meteorology, that the selection of topics here should afford a sufficiently broad basis from which more detailed discussions may proceed.

The scope of the text is somewhat restricted by the fact that little mathematics beyond differential and integral calculus is assumed. Some vector calculus is employed, but the object here is mainly to simplify the mathematical equations and the physical interpretations. Since the vector operations used in this text are relatively few in number, they are reviewed in Chapter 1, so that the student may gain adequate facility with them. The experience of the authors indicates that the student, even though unfamiliar with the vector notation at first, soon becomes accustomed to it and ultimately benefits greatly through the simplification achieved. It should also be mentioned that brief reference to the terminology of statistics is made occasionally. However, these references occur so infrequently that the meaning, in general, is evident even if the student has no previous knowledge of statistics.

No attempt has been made *to give the reference* for original authors in every case, especially in connection with classical material. However, appropriate credit is generally given to the contributors of more recent research. The authors are indebted to the American Meteorological Society for permission to reproduce Figs. 6-2, 7-7, 8-10, 9-1, 14-5, 14-6, 19-7, 21-2, 21-3, 22-5, 23-1, 23-2, 23-4, 23-5, 23-6, 23-7*a*, *b*, 23-9, 23-10, 23-11, 23-12,

and Plate II; to the Royal Meteorological Society for Figs. 14-11 and 22-4; to the editor of *Tellus* for Figs. 23-8, 23-16, 23-17, 23-21; to Dr. B. Haurwitz for Figs. 7-5 and 8-1; to Dr. P. J. Kiefer for Plate I; to the Smithsonian Institution for Fig. 7-3; and to Harvard University Press for Fig. 15-3 and Plate II.

It should be mentioned that, in every sense, this text represents a joint contribution by the authors. We wish to thank our respective families for their patience while the work proceeded. We would like to express our thanks to Professor W. D. Duthie, Chairman, Department of Aerology, U.S. Naval Postgraduate School, for his cooperation during the preparation of the text. Our thanks also go to Mrs. Maurine McDonald for her assistance in typing the manuscript.

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CONTENTS

CHAPTER 1

VECTOR OPERATIONS

1-1. Dimensions, Units, and Time. In meteorology there are four fundamental physical quantities, length, mass, time, and temperature. In general the centimeter-gram-second (cgs) system of units will be employed; however, other units of measure will be used when considered more suitable.

The concept of time is related to the rotation of the earth and its revolution about the sun. There are two units of time based on rotation. The first is the *sidereal day*, which is the length of time required for one rotation of the earth with respect to a fixed star (first point of Aries). Since the earth's angular velocity is very nearly constant, the sidereal day is constant. The *solar day* is the time required for one rotation of the earth with respect to the sun. This period varies slightly with the position of the earth in its orbit about the sun, hence it is more convenient to use the *mean solar day* as the unit of time.

The other basic unit of time is the *year*, which is the period required to complete one revolution about the sun (between vernal equinoxes). The year is practically constant and is found to be 366.25 sidereal days. Because of the relative motion of the earth and sun, there are only 365.25 mean solar days during this time. Mean solar time is the unit of time normally used in physical measurements. In this system, the mean solar second is defined by the relationship 86,400 mean solar seconds = 1 mean solar day.

1-2. Vector Notation. It is advantageous to use vector notation because it greatly simplifies the mathematical treatment, and moreover, it generally provides for a simpler physical interpretation of the mathematical results. However, a complete course in vector analysis is not required; and a brief discussion of most of those vector operations which will be used in subsequent chapters will now be presented.

In physics, quantities are encountered that have only magnitude, such as temperature, pressure, etc. These are called scalars. On the other hand, other quantities, known as vectors, have both magnitude and direction. Examples of the latter are velocity, acceleration, force, etc. Vectors, which will be indicated by boldface type, may be represented by directed line segments.

1-3. Addition and Subtraction of Vectors. The sum of two vectors **A** and **B** is illustrated by Fig. 1-1. The vector $-\mathbf{B}$ is a vector of equal magnitude but of opposite direction to **B**. The vector difference $\mathbf{A} - \mathbf{B}$ is accordingly $\mathbf{A} + (-\mathbf{B})$.

Now let **i**, **j**, and **k** be vectors of unit magnitude (length) in the x, y, and z directions of a right-handed cartesian coordinate system. It follows from the definition of a sum of vectors that an arbitrary vector **A** may be represented as the sum

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

The scalars A_x, A_y, and A_z are the lengths of the projections of **A** on the x, y, and z axes, respectively, as shown in Fig. 1-2. It is also implied

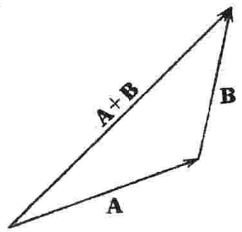

FIG. 1-1. Addition of vectors.

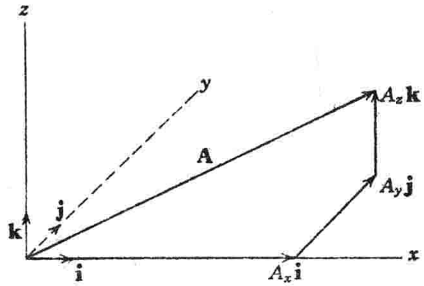

FIG. 1-2. Resolution of a vector into cartesian components.

that the product of a scalar multiplied by a vector, such as $b\mathbf{A}$, is a vector in the same direction as **A** but with magnitude b times the magnitude of **A**. The magnitude $|\mathbf{A}|$ of the vector expressed in the **i**,**j**,**k** system is simply

$$|\mathbf{A}| = A = (A_x^2 + A_y^2 + A_z^2)^{1/2}$$

Two vectors are equal if they have the same magnitude and direction, which is equivalent to the following:

$$\mathbf{A} = \mathbf{B} \qquad \text{if } A_x = B_x,\ A_y = B_y,\ A_z = B_z$$

It also follows from the preceding definitions that the sum and difference of two vectors **A** and **B** are given by

$$\mathbf{A} \pm \mathbf{B} = (A_x \pm B_x)\mathbf{i} + (A_y \pm B_y)\mathbf{j} + (A_z \pm B_z)\mathbf{k}$$

1-4. Dot Product. There are several ways of multiplying vectors, two of which will be described and used here. The *dot product* of two vectors, denoted by the symbol $\mathbf{A} \cdot \mathbf{B}$, is a scalar defined as follows:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta = AB\cos\theta$$

where θ is the angle ($\leq 180°$) between the vectors **A** and **B**. Applying this rule to the unit vectors gives

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1\ (\theta = 0);\ \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{i}\cdot\mathbf{k} = 0\ (\theta = 90°)$$

It follows that

$$\mathbf{A} \cdot \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$
$$= A_xB_x + A_yB_y + A_zB_z = \mathbf{B} \cdot \mathbf{A}$$

1-5. Cross Product. The cross product of two vectors $\mathbf{A} \times \mathbf{B}$ is a vector whose direction is that of a right-hand screw when turned in the direction from **A** to **B** (through $\theta \leq 180°$) and whose magnitude is $AB \sin \theta$. In Fig. 1-3 the product $\mathbf{A} \times \mathbf{B}$ is a vector perpendicular to the plane of the page, pointing out of the page toward the reader. It follows from the definition that $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$ ($\sin \theta = 0$); and

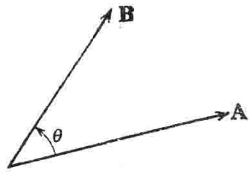

FIG. 1-3. The angle between two vectors.

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}$$

$\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}$, $\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}$ ($\theta = 90°$). Using these relationships $\mathbf{A} \times \mathbf{B}$ becomes

$$\mathbf{A} \times \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$
$$= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

This result may be conveniently expressed in determinant form as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad (1\text{-}1)$$

Example: As an example of the use of the cross product, assume a particle is rotating about the z axis with angular velocity $\mathbf{\Omega}$ (a vector). The direction of the vector $\mathbf{\Omega}$ is such that when the fingers of the right hand curl about the axis of rotation in the same sense as the motion, the thumb points in the direction of $\mathbf{\Omega}$. In Fig. 1-4, **r** is the position vector of the moving particle as measured from the origin of the coordinate system. The vector $\mathbf{\Omega} \times \mathbf{r}$ has magnitude $\Omega r \sin \theta$, which is just the linear speed of the particle. Moreover, the direction of $\mathbf{\Omega} \times \mathbf{r}$ is that of the instantaneous motion of the particle.

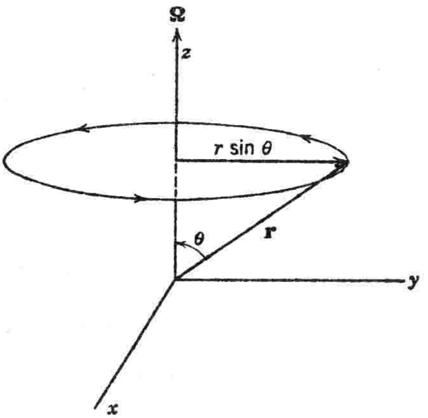

FIG. 1-4. Vector representation of angular velocity.

Hence the instantaneous particle velocity is simply

$$\mathbf{V} = \mathbf{\Omega} \times \mathbf{r} \qquad (1\text{-}2)$$

The dot and cross products described above may be applied successively as often as desired in multiple products of vectors. Several products of three vectors occur with sufficient frequency to warrant specific mention:

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \text{ (a scalar)} \tag{1-3}$$

Two triple products which result in vectors are

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \\ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \end{aligned} \tag{1-4}$$

1-6. Differentiation of Vectors. Consider a vector

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

in which the components are functions of time. Then, if the unit vectors are constant,

$$\frac{d\mathbf{A}}{dt} = \mathbf{i}\frac{dA_x}{dt} + \mathbf{j}\frac{dA_y}{dt} + \mathbf{k}\frac{dA_z}{dt}$$

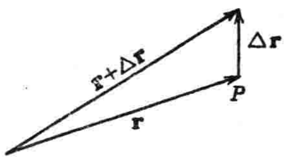

FIG. 1-5. Differentiation of vectors.

where the scalar derivatives are defined in the usual way. As a specific example of vector differentiation, the expressions for velocity and acceleration will be given. Let $\mathbf{r}$ represent the position vector of a moving particle P as shown in Fig. 1-5. If $\mathbf{r} + \Delta\mathbf{r}$ represents the position a short time, Δt, later, then the velocity of P is defined to be

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{V} \tag{1-5}$$

Similarly, a second differentiation with respect to time yields the acceleration $\mathbf{a}$:

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{V}}{dt} = \mathbf{a} \tag{1-6}$$

To illustrate differentiation further, consider a fixed cartesian coordinate system x,y,z and a second system x',y',z' which is rotating at angular velocity $\mathbf{\Omega}$ with respect to the fixed system (Fig. 1-6). Any vector $\mathbf{A}$ may be expressed in terms of both coordinate systems as follows:

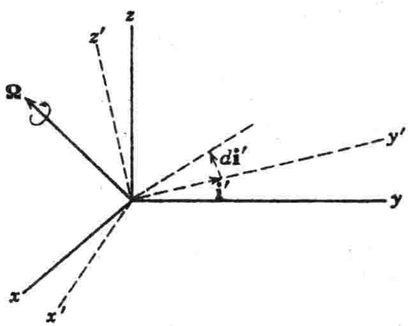

FIG. 1-6. Rotating coordinate system.

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} = A'_x\mathbf{i}' + A'_y\mathbf{j}' + A'_z\mathbf{k}'$$

Differentiation with respect to t yields

$$\frac{d\mathbf{A}}{dt} = \frac{dA_x}{dt}\mathbf{i} + \frac{dA_y}{dt}\mathbf{j} + \frac{dA_z}{dt}\mathbf{k} = \frac{dA'_x}{dt}\mathbf{i}' + \frac{dA'_y}{dt}\mathbf{j}' + \frac{dA'_z}{dt}\mathbf{k}' + A'_x\frac{d\mathbf{i}'}{dt} + A'_y\frac{d\mathbf{j}'}{dt} + A'_z\frac{d\mathbf{k}'}{dt}$$

In accordance with Eqs. (1-2) and (1-5),

$$\frac{d\mathbf{i}'}{dt} = \mathbf{\Omega} \times \mathbf{i}', \quad \frac{d\mathbf{j}'}{dt} = \mathbf{\Omega} \times \mathbf{j}', \quad \frac{d\mathbf{k}'}{dt} = \mathbf{\Omega} \times \mathbf{k}'$$

Substituting these results into the preceding equation gives

$$\frac{d\mathbf{A}}{dt}\text{ (fixed system)} = \frac{d\mathbf{A}}{dt}\text{ (rotating system)} + \mathbf{\Omega} \times \mathbf{A} \tag{1-7}$$

Thus it has been shown that the rate of change of an arbitrary vector **A** with respect to the fixed coordinate system equals the rate of change observed relative to the rotating system plus the term $\mathbf{\Omega} \times \mathbf{A}$.

1-7. Del Operator. In general the same rules apply to the differentiation of a sum, difference, or product of vectors as for scalars. Certain combinations of partial derivatives occur frequently in physical applications; and it is useful to introduce a vector differential operator called the *del* operator, denoted by the symbol ∇.

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

This operator may be used in essentially three different ways as follows.

Gradient. Firstly, del may be used on a scalar, for example, pressure, to give a vector:

$$\nabla p = \mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y} + \mathbf{k}\frac{\partial p}{\partial z} \tag{1-8}$$

In order to appreciate the significance of this operation consider a small displacement represented by the vector $\delta\mathbf{r} = \mathbf{i}\,\delta x + \mathbf{j}\,\delta y + \mathbf{k}\,\delta z$. Then by the definition of the dot product

$$\nabla p \cdot \delta\mathbf{r} = \frac{\partial p}{\partial x}\delta x + \frac{\partial p}{\partial y}\delta y + \frac{\partial p}{\partial z}\delta z = \delta p$$

where the total differential δp represents the difference in pressure through the displacement $\delta\mathbf{r}$.

If $\delta\mathbf{r}$ is taken along an isobaric (constant-pressure) surface, $\delta p = 0$. Hence, since neither $\delta\mathbf{r}$ nor ∇p is zero in general, they must be perpendicular in this case. It follows that ∇p is perpendicular to the isobaric surfaces.

Now let $\delta\mathbf{r}$ be taken in an arbitrary direction, making an angle θ with the direction of ∇p. Then

$$\delta p = |\nabla p|\,|\delta\mathbf{r}| \cos\theta = |\nabla p|\,\delta n \tag{1-9}$$

where $\delta n = \delta r \cos\theta$ is the perpendicular distance between the isobaric surfaces (Fig. 1-7). It is apparent from Eq. (1-9) that for any given pressure field and $|\delta\mathbf{r}|$, δp will be a maximum positive value when θ is zero, i.e., when $\delta\mathbf{r}$ is taken in the direction of ∇p. It follows that ∇p is in the direction of the maximum rate of increase of pressure, which is perpendicular to the isobaric surfaces toward higher pressure. Furthermore, by (1-9), the magnitude of ∇p is that of $\delta p/\delta n$.

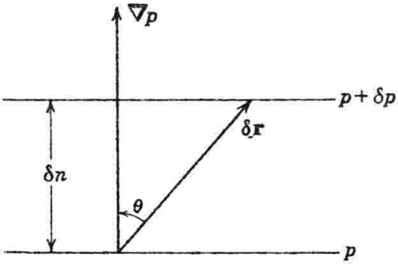

FIG. 1-7

In this text the term *pressure gradient* will be used to designate the quantity $-\nabla p$. This convention conforms with common meteorological usage and appears more logical in physical applications. However, mathematical texts normally define the gradient to be grad $p = \nabla p$; hence the reader should be careful in this regard. The term will also be used in connection with other scalars such as temperature, density, etc.

Divergence of a Vector. A second type of operation with ∇ yields a scalar. If $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is any vector, the divergence of the vector is defined to be

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{1-10}$$

The physical interpretation of this expression will be given in Sec. 19-5.

Curl of a Vector. Finally, the curl of a vector is defined to be

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k} \tag{1-11}$$

If $\nabla \times \mathbf{V} = 0$, $\mathbf{V}$ is said to be an irrotational vector. Moreover, it may be shown that any irrotational vector is always expressible as the gradient of a scalar function. Thus if $\nabla \times \mathbf{V} = 0$, there exists a function Φ such that $\mathbf{V} = -\nabla\Phi$.

1-8. Nondivergence and the Stream Function. Assume a horizontal wind field in which $\mathbf{V} = u\mathbf{i} + v\mathbf{j}$ is the velocity of an air particle. If

$\nabla \cdot \mathbf{V} \equiv 0$ (throughout space and time), the flow is said to be nondivergent; and it may be shown that there exists a scalar function Ψ such that

$$\mathbf{V} = \mathbf{k} \times \nabla\Psi = \mathbf{k} \times \left(\mathbf{i}\frac{\partial \Psi}{\partial x} + \mathbf{j}\frac{\partial \Psi}{\partial y}\right) = -\mathbf{i}\frac{\partial \Psi}{\partial y} + \mathbf{j}\frac{\partial \Psi}{\partial x} \tag{1-12}$$

Equating the coefficients of $\mathbf{i}$ and $\mathbf{j}$ gives

$$u = -\frac{\partial \Psi}{\partial y}, \; v = \frac{\partial \Psi}{\partial x} \tag{1-13}$$

From the expression $\mathbf{V} = \mathbf{k} \times \nabla\Psi$, it may be seen that $\mathbf{V}$ is perpendicular to $\nabla\Psi$ and is thus parallel to lines of constant Ψ, with low values of Ψ to the left of $\mathbf{V}$ (Fig. 1-8). Isolines of Ψ are called streamlines. Streamlines are defined as curves which, at any fixed time, are everywhere parallel to the wind. According to Eqs. (1-9) and (1-1), the wind speed is given by $V = \delta\Psi/\delta n$. Moreover, between any two adjacent streamlines, as shown in Fig. 1-8, $\delta\Psi$ is a constant. Hence the horizontal wind speed of nondivergent flow is inversely proportional to the spacing (δn) of the streamlines.

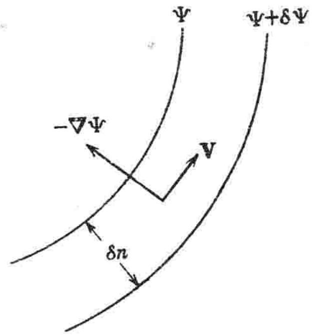

FIG. 1-8. Fluid velocity and the stream function.

1-9. Total Differential; Local Change. Consider any scalar or vector quantity $f(x,y,z,t)$, a function of position and time. Then by the calculus the total differential is

$$df = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \tag{1-14}$$

In vector notation this becomes

$$df = \frac{\partial f}{\partial t}\,dt + \nabla f \cdot d\mathbf{r}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of some particle or identifiable point, etc. When $\mathbf{r}$ is a function of time,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{c} \cdot \nabla f \tag{1-15}$$

Here $\partial f/\partial t$, which is called the local change, is the rate of change of f with respect to time at a fixed location; $d\mathbf{r}/dt = \mathbf{c}$; and df/dt is the rate of change of f with time following a point moving with velocity $\mathbf{c}$.

CHAPTER 2

THERMODYNAMICS OF DRY AIR

2-1. Physical Variables. In physics three distinct states of matter are recognized, solid, liquid, and gaseous. The state of a particular substance depends mainly on the temperature and external pressure. At sufficiently low temperatures all substances solidify; and, on the other hand, at sufficiently high temperatures all substances become gaseous.

In liquids and gases, the molecules are not so closely packed, and these forms possess essentially no rigidity. The form or shape of a liquid or gas is then determined by the container. Liquids are only slightly compressible and normally leave a free surface. A gas, on the other hand, occupies any volume in which it is contained.

Some of the properties of gases will now be described. *Density* ρ is defined as mass per unit volume:

$$\rho = \frac{M}{V} \tag{2-1}$$

and *specific volume* is the inverse of density, i.e.,

$$\alpha = \frac{1}{\rho} = \frac{V}{M} \tag{2-2}$$

Pressure p is defined as force per unit area. Experiment and theory show that the pressure at a point in an ideal fluid is the same in all directions. The cgs unit of pressure is the dyne per square centimeter; however, the unit more commonly used in meteorology is the millibar, which is equal to 10^3 dynes cm^{-2}. In addition to the millibar, the following expressions are also used:

$$1 \text{ cb} = 10 \text{ mb}, \ 1 \text{ in. Hg} = 33.86 \text{ mb}, \ 1 \text{ mm Hg} = 1.333 \text{ mb} \tag{2-3}$$

A rigorous definition of temperature may be found in standard texts in thermodynamics. Here it will suffice to accept the results of this theory as needed and, for the present, merely state the various temperature scales commonly used. The centigrade scale may be defined as follows: A substance in thermal equilibrium with a mixture of ice and pure water at a pressure of 1013.3 mb (1 atm) is said to be at a temperature of 0°C; and when in thermal equilibrium with steam over boiling water at 1 atm, the temperature is said to be 100°C. Now consider a gas at some fixed