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Scaling, self-similarity, and intermediate asymptotics

标度、自相似性和中间渐近

G. I. BARENBLATT

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Grigory Isaakovich Barenblatt
Emeritus G. I. Taylor Professor of Fluid Mechanics, University of Cambridge;
Fellow of Gonville and Caius College, Cambridge

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To the glowing memory of his beloved parents,

Dr. Nadezhda Veniaminovna Kagan, physician–virologist,
heroically lost for the sake of the healthy future of humanity,

and

Dr. Isaak Grigorievich Barenblat, physician–endocrinologist,

the author dedicates his work.

Preface

Scaling (power-law) relationships have wide application in science and engineering. Well-known examples of scaling relations are the following (we will discuss them later in detail):

G.I. Taylor's scaling law for the shock-wave radius r_f after a nuclear explosion,[†]

$$r_f = \left(\frac{Et^2}{\rho_0}\right)^{1/5};$$

the scaling law for the velocity distribution u near a wall in a turbulent shear flow,[‡]

$$u = Ay^n;$$

the scaling law for the breathing rate R of animals,[§]

$$R = AW^n;$$

and many others.

A very common view is that these scaling or power-law relations are nothing more than the simplest approximations to the available experimental data, having no special advantages over other approximations. It is not so. Scaling laws give evidence of a very deep property of the phenomena under consideration – their *self-similarity*: such phenomena reproduce themselves, so to speak, in time and space. Self-similar

† E explosion energy; t, time after explosion; ρ_0, air density.

‡ y distance from the wall; A, n, constants.

§ W body mass of an animal; A, n constants.

phenomena entered mathematical physics rather early, perhaps with the famous memoir of Fourier (1822) on the analytical theory of heat conduction. In this he arrived at a 'source-type' solution†

$$\theta(x,t) = \frac{A}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right), \quad f(\xi) = e^{-\xi^2/4}, \quad A = \text{Const}$$

to the heat conduction equation

$$\partial_t \theta = \partial^2_{xx} \theta .$$

Subsequently the phenomena under consideration, and the equations entering their mathematical models, became more and more complicated and very often nonlinear. Obtaining self-similar solutions was considered as a success, particularly in the pre-computer era. Indeed, the construction of such solutions always reduces to solving the boundary-value problems for ordinary, not partial, differential equations, which was considered as a substantial simplification. Moreover, in 'self-similar' coordinates $\theta\sqrt{t}$, $x/\sqrt{t}$ (and analogous coordinates in other problems), self-similar phenomena become time independent. This gives important evidence of a certain type of stabilization. Thus very often obtaining a self-similar solution was the only way to understand the qualitative features of the phenomena.

The exponents of the independent variables x, t in self-similar variables such as $\theta\sqrt{t}$, $x/\sqrt{t}$ in the heat conduction problem mentioned above were obtained at the outset in some simple way giving no special trouble to the researcher, often *dimensional analysis*. Dimensional analysis is merely a simple sequence of rules based on the fundamental *covariance principle* of physics: *all physical laws can be represented in a form equally valid for all observers*. Such classical self-similarities were discussed and summarized in a book by Sedov (1959) and in a monographic review by Germain (1973), in which a general approach to problems leading to such solutions was also discussed.

In fact, the situation changed drastically after the paper by Guderley (1942), in which a solution to the problem of a very intense implosion (a converging spherical or cylindrical shock wave) was obtained, and the papers by von Weizsäcker (1954) and Zeldovich (1956) treating the plane analogue of the implosion wave problem, the problem of an *impulsive*

† Here θ, x, t are the temperature, the spatial coordinate, and time. This solution is remarkable for two reasons. Firstly, the temperature θ, a function of two variables x, t, is represented via a function of one variable $x/\sqrt{t}$. Furthermore, according to this solution the temperature distributions at various times can be obtained one from another by a similarity transformation: the solution remains similar to itself.

loading. In these problems a delicate analytical procedure, qualitative investigation of the portrait in a phase plane, was needed to obtain the power n in which the time enters the self-similar variable x/t^n. These powers appeared generally speaking to be certain transcendental numbers rather than simple fractions as for classical self-similarities. In fact solutions with such 'anomalous' dimensions had appeared for different variables even earlier. I am referring to the fundamental papers by Kolmogorov, Petrovsky, and Piskunov (1937), and by Fisher (1937), devoted to the propagation of an advantageous gene, and by Zeldovich and Frank–Kamenetsky (1938a, b), dealing with flame propagation in gases. In these papers the wave-type solutions $\theta(x - \lambda t)$ of the nonlinear parabolic equation

$$\partial_t \theta = \partial^2_{xx} \theta + f(\theta)$$

were considered, and the wave phase speed λ has been calculated by a complicated analytical procedure: phase-plane portrait investigation. Transforming the variables $x = \ln \xi, t = \ln \tau$ one arrives at the same problem of determining the exponent of τ in the self-similar variable ξ/τ^{λ}.

An important question arose: what is the real nature of such a difference in self-similar solutions? To understand that, in the papers of Barenblatt and Sivashinsky (1969, 1970) two special problems were considered, containing a parameter that entered to the problem's formulation naturally. For a single value of this parameter a classical self-similar solution appeared, in which all powers were obtained from dimensional considerations. However, for all other values of the parameter anomalous dimensions appeared as continuous functions of the parameter; they are obtained from the solution to a nonlinear eigenvalue problem. These results allowed one to understand the fundamental nature of the difference between the two types of self-similar solution mentioned above. Indeed, self-similar solutions are always 'intermediate asymptotics' to the solutions of more general problems, valid for times, and distances from boundaries, large enough for the influence of the fine details of the initial and/or boundary conditions to disappear, but small enough that the system is far from the ultimate equilibrium state. So, the reason for the difference is the character of these intermediate asymptotics. If an asymptotics is represented by a function that tends to a finite limit when approaching the self-similar state, self-similarity of the first kind appears. If, however, a finite (different from zero) limit does not exist, but the asymptotics is a power-type (scaling) one, with the exponents depending on the fine details of the analytical properties of pre-self-similar behaviour, self-similarity of the second kind occurs. So, it became clear how anomalous, transcendental dimensions appear in self-similar solutions. It is also the case that only a power-type asymptotics preserves self-similarity.

Independently but later an activity started in theoretical physics, basically in quantum field theory and in the theory of phase transitions in statistical physics, related to the scaling and renormalization group. Anomalous dimensions entered the language of physicists. The names and works of Stückelberg and Peterman (1953), Gell-Mann and Low (1954), Bogolyubov and Shirkov (1955), Kadanoff (1966; see also Kadanoff *et al*, 1967), Patashinsky and Pokrovsky (1966), and Wilson (1971), as well as the books by Bogolubov and Shirkov (1959), Ma (1976), Amit (1989), and Goldenfeld (1992) should be mentioned. It is essential to emphasize, however, that in contrast with the researchers in applied mechanics mentioned above, researchers in theoretical physics considered problems where rigorous mathematical formulations such as initial and/or boundary value problems for partial differential equations were lacking.

Rather early it became clear that the concepts of intermediate asymptotics developed in applied mechanics and the concepts of scaling and renormalization group developed in theoretical physics are closely related. This relationship was emphasized in the author's first book concerning this subject (Barenblatt, 1979), the Foreword to which, by Acad. Ya.B. Zeldovich, follows this Preface; in that book, theoretical physicists were invited to look at how the approach of intermediate asymptotics can work in problems previously considered by the renormalization group approach.

In a remarkable series of works by N. Goldenfeld, Y. Oono, O. Martin, and their students (see the book by Goldenfeld, 1992) several problems in continuum mechanics (filtration, elasticity, turbulence, etc.) which had been solved previously by the method of intermediate asymptotics were solved by the traditional renormalization group method. Moreover, on the one hand using the singular expansion method widely applied in theoretical physics (ϵ-expansion) Goldenfeld, Oono and their colleagues were able to obtain some instructive and useful approximate solutions to these problems. On the other hand, they obtained by the method of intermediate asymptotics the solutions to several problems of statistical physics, solved previously by the renormalization group approach.

These important works helped to represent in final form the renormalization group approach from the viewpoint of intermediate asymptotics. In particular it appeared useful to give a proper definition of the renormalization group using the concept of intermediate asymptotics. Ultimately the works by Goldenfeld and his colleagues were among the basic stimuli for me to write this book. Of course, in this writing I have used essential materials from my previous books devoted to this subject (Barenblatt, 1979, 1987), so the continuity is completely preserved.

I want to express in conclusion my deep gratitude to the memory of my great mentors, A.N. Kolmogorov and Ya.B. Zeldovich whose approach in particular to self-similarities and intermediate asymptotics greatly influenced my views.

I want to thank Professor D.G. Crighton, FRS, for his kind offer to publish this book in the series under his editorship at Cambridge University Press. I am pleased to express my deep gratitude to him, to Professor G.K. Batchelor, FRS, and to Professor H.K. Moffatt, FRS, for the honour and pleasure of writing this book here at the Department of Applied Mathematics and Theoretical Physics in Cambridge. I am grateful to Professor M.D. van Dyke for his valuable advice. I thank Miss Sarah Kirkup for her help in preparing the manuscript.

Foreword

Professor Grigorii Isaakovich Barenblatt has written an outstanding book that contains an attempt to answer the very important question of how to *understand* complex physical processes and how to *interpret* results obtained by numerical calculations.

Progress in numerical calculation brings not only great good but also notoriously awkward questions about the role of the human mind. The human partner in the interaction of a man and a computer often turns out to be the weak spot in the relationship. The problem of formulating rules and extracting ideas from vast masses of computational or experimental results remains a matter for our brains, our minds.

This problem is closely connected with the recognition of patterns.It is not just a coincidence that in both the Russian and English languages the word 'obvious' has two meanings – not only something easily and clearly understood, but also something immediately evident to our eyes. The identification of forms and the search for invariant relations constitute the foundation of pattern recognition; thus, we identify the similarity of large and small triangles, and so on.

Let us assume now that we are studying a certain process, for example a chemical reaction in which heat is released and whose rate depends on temperature. For a wide range of parameters and initial conditions, a completely definite type of solution is obtained – flame propagation. The chemical reaction occurs in a relatively narrow region separating the cold combustible substance from the hot combustion products; this region moves relative to the combustible substance with a velocity that is independent of the initial conditions. (Of course, the very occurrence of combustion depends on the initial conditions.)

This result can be obtained by direct numerical integration of the partial differential equations that describe the heat transfer, diffusion, chemical reaction, and (in some cases) hydrodynamics. Such a computational approach is difficult; the result is obtained in the form of a listing of quantities such as temperature and concentration as functions of temporal and spatial coordinates. To make manifest the flame propagation, i.e., to extract from the mass of numerical material the regime of uniform temperature propagation, $T(x - ut)$, is a difficult problem! It is necessary to know the type of the solution in advance in order to find it; anyone who has made a practical attempt to apply mathematics to the study of nature knows this truth.

The term 'self-similarity' was coined and is by now widespread: a solution $T(x, t_1)$ at a certain moment t_1 is similar to the solution $T(x, t_0)$ at a certain earlier moment. In the case of uniform propagation considered above, similarity is replaced by simple translation. Similarity is connected with a change of scales:

$$T = \left(\frac{t_1}{t_0}\right)^n T\left(x\left(\frac{t_1}{t_0}\right)^m , t_0\right)$$

or

$$T = \varphi(t)\Psi(x/\xi(t)) .$$

In geometry, this type of transformation is called an affine transformation. The existence of a function Ψ that does not change with time allows us to find a similarity of the distributions at different moments.

Barenblatt's book contains many examples of analytic solutions of various problems. The list includes heat propagation from a source in the linear case (for constant thermal conductivity) and in the nonlinear case, and also in the presence of heat loss. The problem of the hydrodynamic propagation of energy from a localized explosion is also considered. In both cases, the problem in its ordinary formulation – without loss – was solved many years ago; in these problems the dimensions of the constants that characterize the medium (its density, equations of state, and thermal conductivity) and the dimensions of energy uniquely dictate the exponents of self-similar solutions.

However, with properly introduced losses the problems turn out to be essentially different. If $dE/dt = -\alpha E^{3/2}/R^{5/2}, dR/dt = \beta E^{1/2}/R^{3/2}$ (E being the total energy referred to the initial density of the gas, R the radius of the perturbed domain and t the time) so that $dE/dR =$

$-\gamma E/R$, then the conservation of energy does not hold:

$$E \sim R^{-\gamma}\,,\ E = E_0 R_0^{\gamma} R^{-\gamma} \neq \text{const};$$

however, self-similarity remains.

The dimensionless numbers α, β, and γ depend on the functions describing the solution, but the equations that determine these solutions contain indeterminate exponents. Mathematically we have to deal with the determination, from nonlinear ordinary differential equations and their boundary conditions, of certain numbers that can be called eigenvalues.

The new exponents in the problem are not necessarily integers or rational fractions; as a rule they are transcendental numbers that depend continuously on the parameters of the problem, including the parameters of energy loss. Thus arises a new type of self-similar solution, which we shall call the second type, reserving the title of first type for the case where naive dimensional analysis succeeds.

An important point arises here. The solution does not describe the point source asymptotically: if R_0 (the value of R at $t = 0$) is taken to be equal to zero, then it must necessarily be that $E_0 = \infty$ for $t = 0$, which is physically inconsistent. Hence the new solution is considered as an intermediate asymptotics. We assume that up to a certain finite time t_0 there is no loss. At this moment, when the radius of the perturbed domain reaches the finite value R_0, we switch on the loss. Or, to be more general, we can start with a finite energy E created by some other means, that has already spread out to the finite radius R_0. It is assumed that asymptotically, for sufficiently large time, the solution assumes a self-similar form corresponding to the given loss.

We want to emphasize the asymptotic character of the self-similar solution for $t \gg t_0$. In nonlinear problems, exact special solutions sometimes appear to be useless: since there is no principle of superposition, one cannot immediately find a solution of the problem for arbitrary initial conditions.

Here asymptotic behaviour is the key that partially plays the role of the lost principle of superposition. However, for arbitrarily given initial conditions this asymptotic behaviour must be proved. The problem is difficult, and in many cases numerical computations give only a substitute for rigorous analytic proof.

The preceding arguments may seem unusual in a Foreword: but I wanted, using the simplest examples, to introduce the reader as quickly as possible to the advantages and difficulties of the new world of solutions of the second kind.

There are also other types of solutions, among which convergent spherical shock waves are the most important. In this case there is no external loss, but the region in which self-similarity holds is contracting; it it therefore impossible to assume that the entire energy is always concentrated in the shrinking region, and this energy in fact decreases according to a power law, since part of the energy remains in the exterior regions of the gas. Again it is necessary to find the exponents as eigenvalues of a nonlinear operator.

The specific character of this class of equations is connected with the finiteness of the speed of sound; the point where the phase velocity of propagation of a self-similar variable is equal to the velocity of sound plays a decisive role in the construction of the solution.

Barenblatt also discusses in his book another problem of analogous type: the problem of a strong impulsive load in a half-space filled with gas. This problem abounds in paradoxes. In particular, why do the laws of conservation of energy and momentum not make it possible to determine the exponents? The answer to this question is contained in chapter 4, and it would be against the rules to give it here in the Foreword.

Problems involving the nonlinear propagation of waves on the surface of a heavy fluid, described by the Korteweg–de Vries equation, give a remarkable example. Here there are long-established and well-known solutions describing solitary waves (called 'solitons'), propagating with a velocity dependent on the amplitude. This example is remarkable in that there exist theorems proving the stability of solitons even after their collisions, and theorems determining the asymptotic behaviour of initial distributions of general type, which are transformed into a sequence of solitons. At first suggested by numerical computations, these properties are now rigorously proved by analytic methods of extraordinary beauty. In these solutions all the properties of ideal self-similar solutions of the second kind appear.

In some sense the problems of turbulence, considered at the end of the book, differ from those mentioned above. These are farther from my interests and I will not dwell on them here. A complete outline of all that is contained in the book can be found in the Table of Contents and should not be sought in the Foreword.

We shall now return to the nature of the book as a whole; we shall not hesitate to repeat for the general situation some considerations that have already been presented above in connection with simple examples.

The problems are chosen carefully. Each of them taken separately is a pearl, important and cleverly presented. In the solution of many of the problems the role of the author was essential, and this gives to the presentation the flavour of something lived. But I must emphasize that the importance of this book far exceeds its value as a collection of interesting special examples; from the special problems considered, very general ideas develop.

Most of the problems are nonlinear. What is the use of special solutions if there is no principle of superposition? The fact is that as a rule these special solutions represent the asymptotics of a wide class of other more general solutions that correspond to various initial conditions. Under these circumstances the value of exact special solutions increases immensely. This aspect of the question is reflected in the title of the book in the words 'intermediate asymptotics'. The value of solutions as asymptotics depends on their stability. The questions of the stability of a solution and of its behavior under small perturbations are also considered in this book; in particular, there is presented a rather general approach to the stability of invariant solutions developed in a paper by Barenblatt and myself.

The very idea of self-similarity is connected with the group of transformations of solutions. As a rule, these groups are already represented in the differential (or integro-differential) equations of the process. The groups of transformations of equations are determined by the dimensions of the variables appearing in them; the transformations of the units of time, length, mass, etc. are the simplest examples. This type of self-similarity is characterized by power laws with exponents that are simple fractions defined in an elementary way from dimensional considerations.

Such a course of argument has led to results of immense and permanent importance. It is sufficient to recall the theory of turbulence and the Reynolds number, of linear and nonlinear heat propagation from a point source, and of a point explosion. Nevertheless, we shall see that dimensional analysis determines only a part of the problem, the tip of the iceberg; we shall call the corresponding solutions *solutions of the first kind*, as mentioned above. We shall reserve the name *solutions of the second kind*, for the large and ever growing class of solutions for which the exponents are found in the process of solving the problem, analogously to the determination of eigenvalues for linear equations. For