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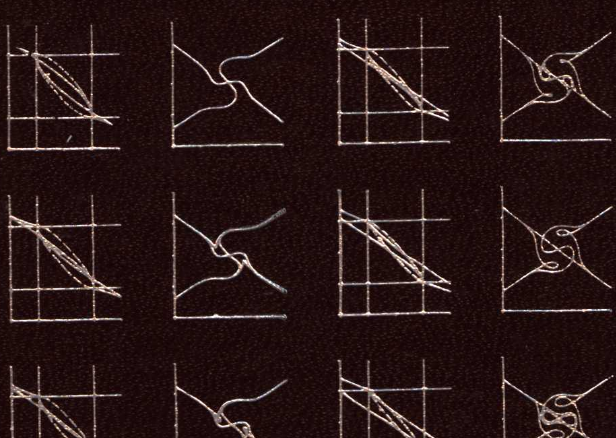
INTERMEDIATE CLASSICAL DYNAMICS WITH APPLICATIONS TO BEAM PHYSICS



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Intermediate Classical Dynamics with Applications to Beam Physics

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To my parents, Carlo and Anna Michelotti,
who together taught me all that was really necessary,
and my wife, Kathy, who kept our family going strong
while I took the time to write this.

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Preface

> This is not a work of scholarship. . . . I write for the unlearned about things in which I am unlearned myself. If an excuse is needed . . . for writing such a book, my excuse would be something like this. It often happens that two [students] can solve difficulties in their work for one another better than [a] master can. The fellow-pupil can help more than the master because he knows less. . . . I write as one amateur to another, talking about difficulties I have met . . . with the hope that this might at least interest, and sometimes even help, other inexpert readers.
>
> — C. S. Lewis
> *Reflections on the Psalms*

As there are available now a number of excellent modern books on dynamics, someone who produces yet another, perhaps not as excellent, had better be ready with an apology. Two superb examples of the more comprehensive and scholarly offerings currently in print are *Foundations of Mechanics*, by Abraham and Marsden [1] and *Analysis, Manifolds and Physics*, by Choquet-Bruhat and DeWitt-Morette [22]. If you were to purchase either of these two, say, instead of the one you are holding, and commit yourself to absorbing at least one page each day for a year or two, the (nonmonetary) rewards would be incalculable. Unfortunately, many of us who seriously need to know this material are either unfamiliar with or intimidated by advanced formalisms. The present volume is, in some sense, a mathematical halfway house between such works and more introductory material, such as Symon's *Mechanics* [87] or Goldstein's *Classical Mechanics* [41]. It may be considered a supplement to these excellent texts and is written at a level suitable for first-year graduate or senior undergraduate physics students.

Other books, such as Rasband's *Dynamics* [79], attempt this as well. One feature that may distinguish this book from it, and others, is the emphasis placed, via examples and problems, on beam physics, more particularly, on the orbits of particles in synchrotrons and storage rings. Partly, this is due to accidental, personal history. It is also true, however, that accelerator physics is one of the areas in which classical mechanics plays more than an academic role. To people designing accelerators and studying their behavior, it is the bread and butter of everyday existence. The historic successes of high energy physics has been due in no small measure to the ability of these people — from the designers and theorists to the operators in control rooms — to persuade large numbers of like-charged particles to travel billions of kilometers while staying within centimeters of each other, increasing their energy continually, and eventually either directing them to a target or colliding them at a focus narrow enough to make the entire enterprise worth the effort. Now, while several texts on dynamics contain numerous examples from fields like celestial mechanics or mechanical engineering, beam physics has been generally underrepresented. Given its impact on physics as a whole, it is fitting that at least one dynamics text draw from accelerator issues for its motivations and examples.

Much has been written recently about a revolution taking place in the asymptotic analysis of dynamical systems, usually illustrated with beautiful color photographs of objects like chaotic attractors, fractal basin boundaries, and other such Julia-Fatou sets. The first shots of this revolution were fired, however, not in the 1980s or 1970s but toward the close of the nineteenth century. By 1892, Henri Poincaré already had published his landmark work *Les Méthods Nouvelles de la Mécanique Céleste* in which he advanced the theses that differential equations should be viewed as *geometric* objects, in particular, as vector fields on manifolds, and that questions concerning the long-term stability of a dynamical system might be attacked by studying the topological properties of these objects as revealed by maps. In particular, he formulated the goal of finding the dynamically invariant regions of different dimensions and determining how they connect with each other to impose structure on dynamical systems. His work even led him to recognize the extraordinarily complicated behavior of orbits in the vicinity of a separatrix, what today we call "chaotic orbits" and identify as invariant regions of fractal dimension. Much like his predecessor Newton, Poincaré found that ideas and language which he needed did not yet exist and that he had to create entirely new mathematics in order to progress. In time the seeds that he planted grew into branches of modern topology, with all its trappings of tangent and cotangent bundles, differential forms, exterior algebra and calculus, homology and cohomology — all of which are frequently associated with advanced topics, such as general relativity or string theory, but are *almost never mentioned in connection with their primary source, good old classical mechanics.*

The benign neglect accorded to Poincaré in this country[1] was only partially shared by Sophus Lie, whose constructs are all too frequently not taught to physics students until the study of quantum angular momentum or, even worse, quark models. Because Lie's ideas are first presented in such contexts, their initial connection with dynamical systems tends to be forgotten, and it is easy for a student to think of Lie groups only in terms of finite representations and Dynkin diagrams. *The appropriate place to introduce Lie groups is in a course on classical mechanics.* We shall attempt to do that in as natural a manner as possible, following the track of his algebraic constructs

[1] A recent dismissal occurs in a very enjoyable article [19], in the beginning of which a great physicist, while paying homage to a late colleague, parenthetically laments, "I regret to say that he preferred Poincaré ... to Einstein." The irony is that you are likely to understand Einstein more easily — or, more precisely, a modern exposition, such as Misner, Thorne, and Wheeler's *Gravitation* — by paying some attention to Poincaré early in life.

from their setting in vector fields, through their use in Hamiltonian systems, and in the construction of normal forms.

In 1969 Deprit [28] discovered an algorithm, built on the Lie algebra of Poisson brackets, for performing perturbation theoretic calculations in celestial mechanics. In 1981 Dragt independently introduced another Lie algebraic method, based on discrete maps rather than continuous flows, into accelerator theory. (I was fortunate enough to be assigned the task of editing his lecture notes [2, 30], little suspecting that they would bifurcate my life.) To some these "new" formalisms at first appeared mysterious and somewhat contrived, computational tricks that seemed to have little physics motivation behind them. That this misperception arose may not be totally disconnected from the fact that the key ideas frequently do not appear in most core physics curricula, and practically never in connection with classical mechanics. (This neglect does not extend to generating function techniques, for example, which do possess a strong academic tradition.) We shall introduce the tools naturally in a classical setting so basic that it can be used even at the undergraduate level. We shall also attempt to demonstrate the relations between these "new" methods and the more "traditional" ones used in accelerator physics since the 1950's. What makes the "new" methods so exciting is their imbedding within an effective computing environment. Nonetheless, the underlying ideas go back more than a century. Those who earn their bread, sausage, and beer by using classical dynamics have a glorious heritage, one that is frequently disconnected from its roots.

It is ironic that as we approach the end of the twentieth century, classical mechanics is undergoing a renaissance; it remains a lively subject of active research, both in its own domain and in its connections with quantum theory. This is largely due to advances in computation, both symbolic and numeric. Advanced graphics has played a special role, giving us a new "lens" that has opened all our eyes to behavior that previously could barely be imagined only by the most brilliant. In contrast to the impression acquired by too many during their formal education that "classical mechanics" is a dull, closed subject with no mysteries left to explore, the work has just begun, and its prospects are exciting.

This volume grew from courses and lectures given at Northwestern University, at Harvard University, and at Accelerator Schools sponsored by the DoE and organized by Melvin Month. The Accelerator Schools especially have produced much needed documentation on the art and science of accelerators in the form of A.I.P. Conference Proceedings, and they have been a source of fine over-the-shoulder bags as well. My thanks to Melvin Month for giving me a forum for lecturing and for encouraging me to write this book. Additionally, I would like to acknowledge the good fortune of having worked closely with Don Edwards as my first supervisor at Fermilab; I continue to benefit from this early exposure to his physical insights, knowledge, and wisdom. Additionally, I owe a debt of gratitude to Shoroku Ohnuma, who would continually ask the most disturbing and tantalizing questions, only a few of which we ever got around to answering; some of the others are addressed here. As already implied, to Alex Dragt lies the credit for starting the whole business of using Lie operators to solve accelerator problems, and his influence has been pivotal. And I have benefitted from several recent interactions with Etienne Forest, whose insight into the perturbation theories of accelerator orbits is unmatched.

I want to thank one unknown reviewer who read the first rough draft of this book and made several suggestions, many of which I have tried to incorporate into the text. Most of all, however, I thank him for his generous, positive response, without which I probably would have dropped the project.

Regarding the technical errors, the manuscript suffers from not having had a horde of students critically reading the pages for several years, checking the proofs, repeating the calculations, and solving the problems. I regret that very much. However, unlike my colleagues in the universities, I do not have this marvelous resource of undervalued labor at my disposal. So, I must apologize for whatever errors remain — typographical, algebraic, and conceptual — and I urgently request that you communicate them to me as they are unearthed. If possible, I shall archive errata, and make them electronically available to interested reader(s).

A final word to the heroes who actually make accelerators work, most of whom already know it well, though it tends to get submerged: Your field deals with more than environmental impact reports, data bases, impedance measurements, or even the "kick turn turn kick turn . . ."[2] of tracking programs. As was written somewhat earlier, "[t]hose who earn their bread, sausage, and beer by using classical dynamics have a glorious [physics and mathematics] heritage" that should be enjoyed and, more importantly, used.

LEO MICHELOTTI

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[2]The beginning of a line taken from *The Producers*, by Mel Brooks, the end of which is also appropriate.

Contents

CHAPTER 1

Prologue: The Pendulum

If we want to know where Jupiter will be so as to plan properly the Jupiter shot, then we may proceed in one mathematical direction. If we are interested in whether the solar system is dynamically stable or unstable, we will have to proceed in another. In view of the inherent difficulties of the mathematics, the art of modelling is that of adopting the proper strategy.

— Philip J. Davis and Reuben Hersh
The Mathematical Experience

It is in high school that we are first taught to write Newton's Law as, "$F = ma$." This pedagogical blunder is usually compounded some time later by a teacher of Freshman Physics who one day draws a simple pendulum on a blackboard, more or less as in Figure 1.1, writes the equation of motion, from $F = ma$,

$$-mg\sin\theta = ml\ddot{\theta}, \tag{1.1}$$

and, barely pausing for breath, notes that $\sin\theta \approx \theta$, provided θ is "small," thus transforming this equation into that of a harmonic oscillator,

$$\ddot{\theta} \approx -(g/l)\theta,$$

with a basis of two linearly independent solutions,

$$\theta = \begin{cases}\sin\omega_o t, \\ \cos\omega_o t,\end{cases} \qquad \omega_o = \sqrt{g/l}.$$

All of which is correct, as far as it goes. Regrettably, the teacher of Freshman Physics probably stops the analysis at that point and never returns to consider the motion in

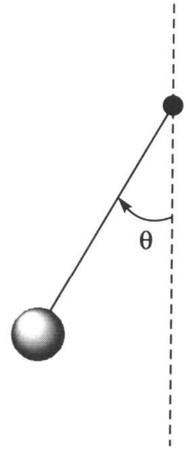


FIGURE 1.1 Simple pendulum: the archetypal nonlinear dynamical system.

more detail. The reason for abandoning the problem is obvious. If you ask the "orbit history" question—"Given initial position and velocity at time $t = 0$, what is the position at time $t \neq 0$?"—then the answer is complicated: in particular, it involves elliptic functions, and freshpersons are not expected to know about such things. This question was the dominant paradigm of physics during the eighteenth and most of the nineteenth centuries: "solving" a dynamical system meant determining its history.

However, there is another sort of question that is easier to answer and, in some ways, more interesting. For the purposes of this chapter, it can be motivated with the observation that the total energy of the pendulum,

$$\begin{aligned} E &= \tfrac{1}{2}mv^2 + V(\theta) \\ &= \tfrac{1}{2}m(l\dot{\theta})^2 - mgl\cos\theta \\ &= \frac{L^2}{2I} - I\omega_o^2\cos\theta, \end{aligned} \tag{1.2}$$

is conserved. In the final line, we have written E in terms of the moment of inertia, $I \equiv ml^2$, and angular momentum, $L \equiv I\dot{\theta}$. If we plot the level sets of E—that is, sets of phase space points for which the value of E is constant—

$$P_W \equiv \{(\theta, L) \mid E(\theta, L) = W\},$$

the resulting curves are paths constraining the system's orbits, as sketched in Figure 1.2. The direction of the orbits is obtained from the definition of L: $L\dot{\theta} > 0$. This picture of the orbit space, sometimes called a "flow diagram," does not contain enough "orbit history" information to answer the original question, but it does contain an enormous amount of qualitative information about the *topological classification of possible orbits* and the relative likelihood that an orbit will belong to one class or

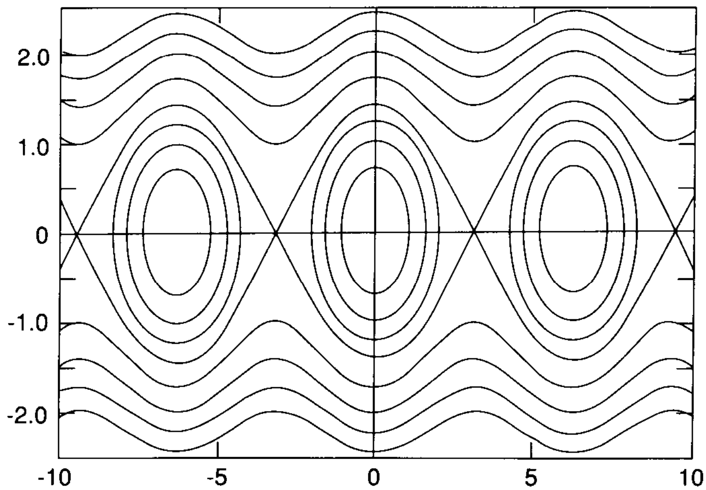


FIGURE 1.2 Level sets of E are paths for the pendulum.

another, the kind of information more appropriate for estimating long-term stability. What it shows is a view of *all possible* paths that the system can traverse, corresponding to all possible sets of initial conditions. Obtaining and interpreting such information form much of the subject matter of "qualitative dynamics."

The remainder of this opening chapter will be devoted to motivating some formal concepts and useful jargon as quickly and painlessly as possible. Before continuing, let me issue a small warning: those who are totally unfamiliar with the more abstract branches of mathematics should simply skim over parts that make them feel uncomfortable. On the other hand, those who are familiar with them to an unhealthy extent may be offended by the cavalier, heuristic manner in which objects such as "charts" or "bundles" are introduced. In either case, please read what follows in the spirit in which it is intended; try to get through it once quickly, getting a feel for the material without bogging down in details. There will be sufficient opportunities for that later.

Critical Orbits and Separatrix The principal feature that appears in Figure 1.2 is the existence of two fixed points — orbits for which $\dot{L} = \dot{\theta} = 0$ — positioned at $L = 0$ and $\theta = 0$ or π. Physically, these correspond to a pendulum at rest, either in the "down" or "up" position. The first fixed point, which is clearly the more stable one, is called an "elliptic" fixed point for the obvious reason that the paths of orbits in its neighborhood look like ellipses. For a similar reason, the second is called "hyperbolic." Loosely stated, the chief difference between them is this: orbits that pass close to an elliptic fixed point remain close to it; in contrast, almost all orbits that pass close to a hyperbolic fixed point do *not* remain close but pull away.

The disclaimer "almost all" is required in the previous sentence.[1] If we focus attention on the neighborhood of the hyperbolic fixed point, we find special orbits

[1]This phrase "almost all" almost always means "all but a set of measure zero," which in turn means "all but a set that has no volume."

that close in on it. All orbits that approach a hyperbolic fixed point as $t \rightarrow +\infty$ (resp., $t \rightarrow -\infty$) comprise its "stable (resp., unstable) manifold."[2] In this case, there is only one hyperbolic fixed point and two such orbits, one for $L > 0$, the other for $L < 0$, both of which belong to either kind of manifold. Therefore, in our example, the stable and unstable manifolds are, in fact, identical. That is not generally, or "generically," the case.

Taken together, the union of hyperbolic fixed points and their stable and unstable manifolds form an object called a "separatrix." Its name derives from the observation that this collection of orbits organizes the (θ, L) "phase space" by partitioning it into regions. To complete the terminology, the "center manifold"[3] of this system lies within the "island" that is bounded by the separatrix. The orbits in this island are characterized by bounded excursions in the angle coordinate, θ; speaking physically, their energies are too small to overcome the potential barrier associated with going over the top. Those "outside" the separatrix have unbounded angle excursions.

The pendulum is a very simple system, but this pattern, or its generalization,[4] appears again and again throughout dynamics. The search for separatrices and the study of their properties comprise much of "nonlinear dynamics." It is a fundamental problem, closely related to the search for the smallest invariant subspaces of different dimensions — a task akin to the group theoretical problem of finding irreducible representations.

Amplitude Dependent Frequencies An important feature, not apparent in the static diagram of Figure 1.2, is that the frequency of a pendulum's orbit depends on its amplitude. This is the first and simplest attribute of nonlinearity. The academic folklore of pendula, which overly emphasizes their connection with harmonic oscillators, contains a charming, apocryphal story about Galileo daydreaming in church, timing the swings of a chandelier with his pulse, and noting that their period remained the same as the motion wound down to rest.[5] In fact the period cannot be constant. By continuity, orbits must move slowly in the vicinity of the hyperbolic fixed point. Correspondingly, those orbits that pass closely to this fixed point will take longer to complete their cycle. Therefore if we plot the frequency of an orbit as a function of its energy, we shall get a curve, like that in Figure 1.3, which approaches ω_o for small E and vanishes at $E = I\omega_o^2$, the energy associated with the hyperbolic fixed point. Asymptotically, when the energy is large, the potential energy becomes negligible and we have $E \approx I\omega^2/2$. Therefore the frequency grows as $\sqrt{E}$ as $E \rightarrow \infty$.

As there is a one-to-one correspondence between the energy of a pendulum and the amplitude of its motion, for $E < I\omega_0^2$, suitably rescaling the abscissa of Figure 1.3 would change it into a plot of frequency versus amplitude.

[2] Simply calling it this does not establish that it *is* a manifold. Indeed, at this point we have not even discussed what constitutes a manifold.

[3] More precise definitions will be given in the next chapter.

[4] We shall see that separatrices possess more complicated structure in higher dimensions.

[5] If the story is true, I suspect that Galileo's pulse rate must have increased as he became more excited by his discovery.

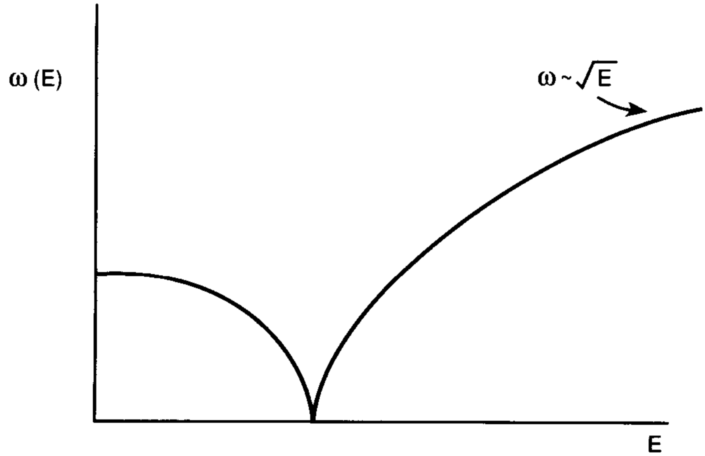


FIGURE 1.3 Rough sketch of frequency versus energy for a simple pendulum.

Phase Space Consider now the very space into which orbits are imbedded, the space that is, in fact, the union of all the paths followed by orbits. Figure 1.2 is drawn as though this were two-dimensional Cartesian space, $R^2 = R \times R$.[6] It is not: any two ordered pairs of the form (θ, L) and $(\theta + 2\pi n, L)$, where $n \in Z$ is an integer, represent the same physical state and therefore must be considered "equivalent." We express this formally by setting up an equivalence relation.[7]

$$(\theta, L) \cong (\theta', L') \quad \textbf{means} \quad L = L' \text{ and } \exists n \in Z : \theta' = \theta + 2\pi n.$$

The appropriate mathematical object that corresponds to a state of the pendulum is not a single ordered pair but a set of equivalent ordered pairs,

$$\lfloor \theta, L \rceil \equiv \{(\theta', L') \mid (\theta', L') \cong (\theta, L)\}, \tag{1.3}$$

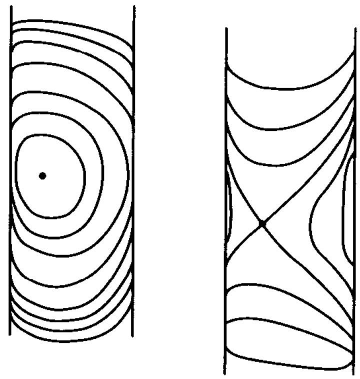
which is called the equivalence class of the ordered pair. The true phase space of the system, the space of physically inequivalent states, is technically a "quotient space," which means it is a set of equivalence classes.

$$\mathcal{P} \equiv \{\lfloor \theta, L \rceil \mid (\theta, L) \in R^2\}.$$

A moment's reflection is sufficient to convince one that $\mathcal{P}$ is a cylinder (see Figure 1.4), equivalent to the cross product of a circle with the reals, $\mathcal{P} = S^1 \times R$.

What this means is something more profound than is apparent at first: the stage on which a system *even as simple as a pendulum* plays out its history is not Cartesian space but a manifold, more particularly, a differentiable manifold. These objects are

[6]Notation for a few famous sets: R = reals; Z = integers; S^n = n-dimensional sphere; T^n = n-dimensional torus.
[7]Common set-theoretic notation: $\forall$ = "for all," $\exists$ = "there exists," and $\in$ = "is in the set."



(a) (b)

FIGURE 1.4 The phase space of a simple pendulum is a cylinder.

the subject matter of the most sophisticated, advanced mathematical disciplines ever devised. We shall not even pretend to attempt to begin to do them justice. Fortunately for our purposes, most physicists have an intuitive and generally sufficient understanding of "manifolds" as an extension of "surfaces," such as spheres, cylinders, or tori. Their important feature is that they are *locally Cartesian.* This means that each point on the manifold belongs to a local region on which we can introduce a system of Cartesian coordinates, formally called a chart. More correctly, localized, open sets of an n-dimensional manifold are coordinatized with open subsets of R^n, usually containing the origin. A collection of such charts which covers the entire manifold — with some overlap between charts so as to patch them together smoothly — is called, appropriately enough, an "atlas." (One need only think of real atlases used to map the Earth to get the idea.) Of course, we are glossing over innumerable technical details. For example, the business of patching charts together smoothly means that if two charts overlap, there must exist, as part of the atlas itself, a "smooth" transformation connecting the coordinates associated with those points that belong to both of them. If $\underline{z} \in R^n$ and $\underline{z}' \in R^n$ are the n-tuple representatives of the same point, $\mathbf{p} \in \mathcal{P}$, on two different charts, then the transformation is a one-to-one, highly (usually infinitely) differentiable function, $T : \underline{z} \longmapsto \underline{z}'$. Furthermore, if two atlases cover the same manifold, then there are smooth transformations connecting pieces of charts corresponding to the same open sets on the manifold, what physicists think of as "coordinate transformations." The key idea that these and other elaborate constructions are meant to express is that a manifold is an object which can be represented, or coordinatized, in a large number of ways but which we want to think of as an entity superceding all these representations.

A dynamical system — or a "dynamic" — is actually associated with a number of related manifolds. First, there is the configuration manifold, which labels its instantaneous "position" without information as to how that position is changing. For example, the configuration manifold of a harmonic oscillator marks its position along

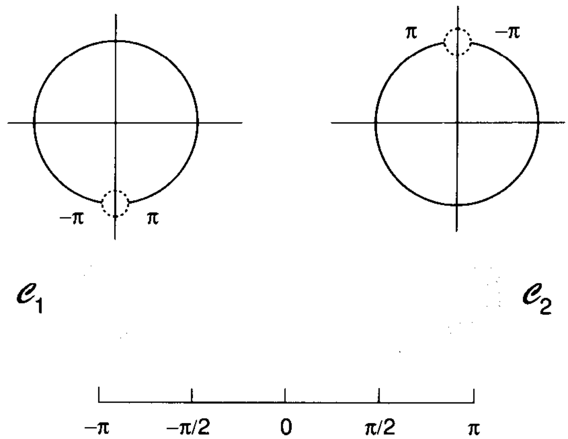


FIGURE 1.5 Charts for S^1, the configuration manifold of a pendulum.

a line and is (topologically equivalent to) R, the set of reals, while that of a pendulum would be (topologically equivalent to) a circle, S^1.[8]

$$S^1 = \{(x, y) \in R^2 \mid x^2 + y^2 = 1\}.$$

Now, although (x, y) Cartesian pairs are used here to define S^1, they cannot serve as coordinates: S^1 is a one-, not two-dimensional object. We have already used an angle coordinate, θ, to denote the pendulum's position, but in fact, $S^1 \neq \{\theta \mid \theta \in (-\pi, \pi)\}$, as this would neglect the upper position of the pendulum, (0, 1). Similarly, using a half-closed interval, $S^1 = \{\theta \mid \theta \in [-\pi, \pi)\}$, would not work, as it would produce a discontinuity in the coordinate.[9] An atlas containing at least two charts, say C_1 and C_2, is required to specify coordinates properly on S^1.One such atlas is sketched in Figure 1.5.

$$S^1 = S^1_1 \cup S^1_2,$$
$$S^1_1 = \{(x, y) \in R^2 \mid x^2 + y^2 = 1 \text{ and } y \neq -1\} = S^1 - \{(0, -1)\},$$
$$S^1_2 = \{(x, y) \in R^2 \mid x^2 + y^2 = 1 \text{ and } y \neq 1\} = S^1 - \{(0, 1)\},$$
$$C_1 : (-\pi, \pi) \to S^1_1,\ \theta_1 \mapsto (\sin\theta_1, \cos\theta_1),$$
$$C_2 : (-\pi, \pi) \to S^1_2,\ \theta_2 \mapsto (-\sin\theta_2, -\cos\theta_2).$$

We have made the overlap between these two charts rather large, almost all of S^1, in fact. That was not necessary; nothing would be lost by restricting the domains. It also

[8]We shall use the phrase "topologically equivalent to" only sparingly, as it tends to burden presentations without adding much information. It should be considered present but suppressed in sentences like, "This surface is a torus."

[9]Usually, charts are open sets; exceptions are made for manifolds with boundaries.