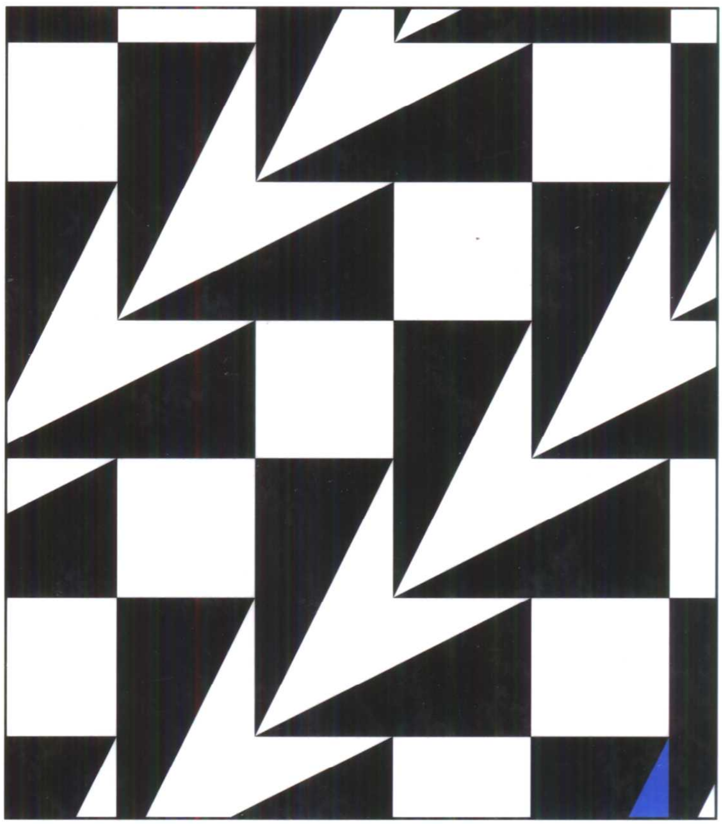
# Recent Trends in Combinatorics

THE LEGACY OF PAUL ERDŐS



*Edited by* Ervin Győri & Vera T. Sós

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# RECENT TRENDS IN COMBINATORICS

## The legacy of Paul Erdős

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge CB2 1RP, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, United Kingdom
40 West 20th Street, New York, NY 10011–4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org



First published 2001

Printed in the United Kingdom at the University Press, Cambridge

Typeset in 10/13pt Monotype Times

*A catalogue record for this book is available from the British Library*

ISBN 0 521 80170 2 hardback

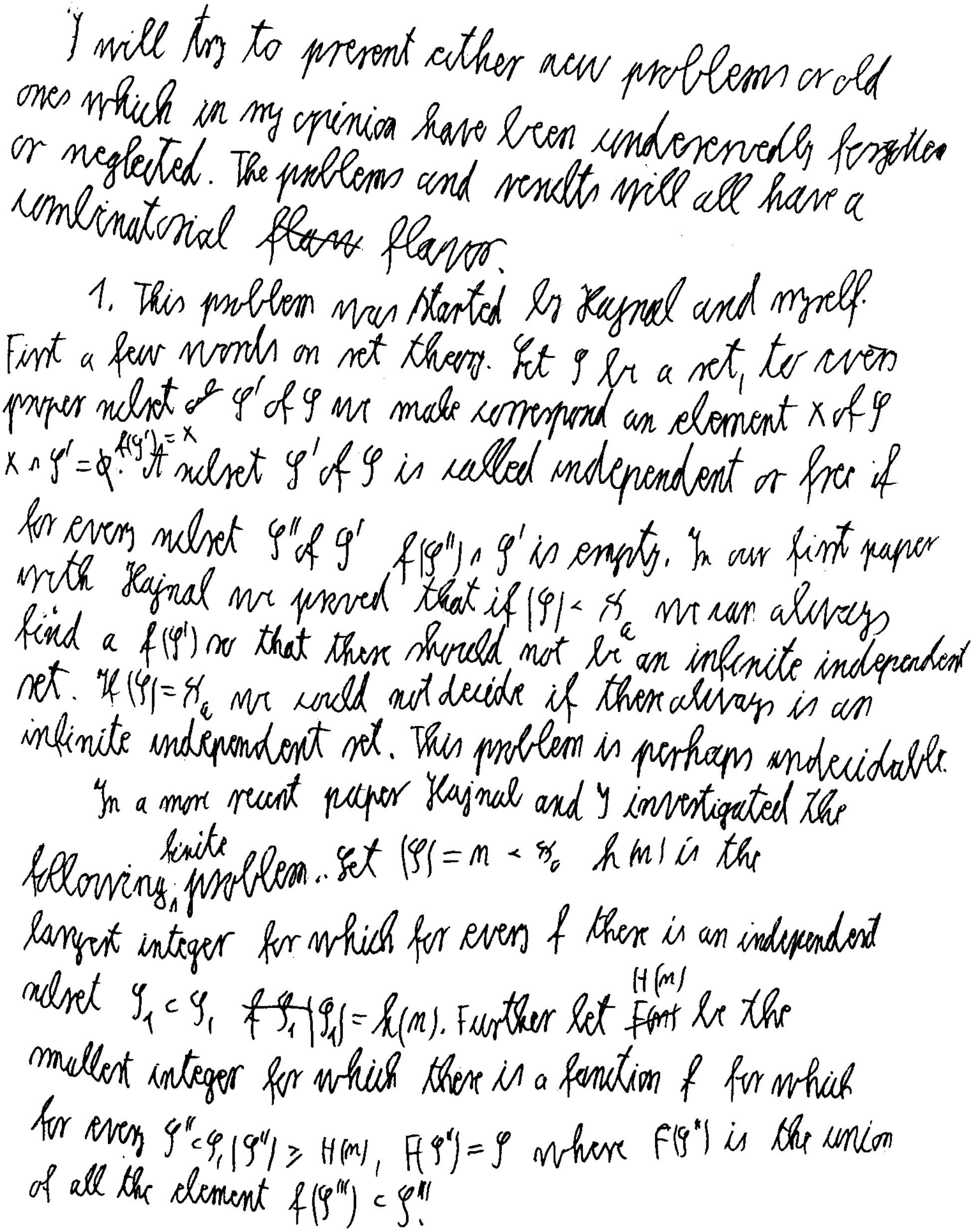
# Preface

The combinatorial workshop 'Some Trends in Discrete Mathematics' was held in Mátraháza, Hungary, from 22 to 28 October 1995. The aim of the workshop was to expose connections between distant parts of combinatorial mathematics, such as pure combinatorics, graph theory, combinatorial number theory and random graphs, by bringing together researchers from diverse fields. To emphasize the workshop character of this meeting, we invited many distinguished mathematicians but asked only ten of them to give lectures. (Unfortunately, illness prevented Claude Berge from attending the meeting.) There were no contributed talks, but the lectures were followed by long discussions involving all the participants: these sessions played a crucial role in the success of the workshop. A tangible result of these evening discussions is the Cameron–Erdős paper in this volume.

A highlight of the volume is the paper Paul Erdős was writing on the eve of his sudden death in Warsaw on 20 September, 1996. This paper had no title and, except for light editing, this very special manuscript is published as he left it. The other eight papers of this issue are surveys and research papers written by the invited speakers and their collaborators.

We want to thank all the participants of the workshop for their contribution to its success. We are also grateful to DIMANET and its main coordinator, Professor Walter Deuber, for providing the financial support that made the workshop possible. We wish to express our sincere thanks to Béla Bollobás who made it possible to publish these papers in a special issue of *Combinatorics, Probability and Computing*.

Ervin Győri and Vera T. Sós



I will try to present either new problems or old ones which in my opinion have been undeservedly forgotten or neglected. The problems and results will all have a combinatorial ~~flavor~~ flavor.

1. This problem was started by Hajnal and myself. First a few words on set theory. Let $S$ be a set, to every proper subset ~~of~~ $S'$ of $S$ we make correspond an element $x$ of $S$ $x \cap S' = \emptyset$. $f(S') = x$ A subset $S'$ of $S$ is called independent or free if for every subset $S''$ of $S'$ $f(S'') \cap S'$ is empty. In our first paper with Hajnal we proved that if $|S| < \aleph_\omega$ we can always find a $f(S')$ so that there should not be an infinite independent set. If $|S| = \aleph_\omega$ we could not decide if there always is an infinite independent set. This problem is perhaps undecidable.

In a more recent paper Hajnal and I investigated the following finite problem. Let $|S| = n < \aleph_0$ $h(n)$ is the largest integer for which for every $f$ there is an independent subset $S_1 \subset S$, ~~$f S_1$~~ $|S_1| = h(n)$. Further let $H(n)$ ~~$F(n)$~~ be the smallest integer for which there is a function $f$ for which for every $S'' \subset S$, $|S''| \geq H(n)$, $F(S'') = S$ where $F(S'')$ is the union of all the element $f(S''') \subset S''$.

A facsimile page of the first draft of the paper of Professor Erdős in this volume.

# Foreword
# Paul Erdős: The Man and the Mathematician (1913–1996)

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## 1. Preface

Paul Erdős was one of the greatest mathematical figures of the twentieth century. An enormous number of obituaries have appeared since his death, which is very unusual in the world of science.[1]

Paul Erdős had already become a legend in his own lifetime. With a strong character, a clear moral compass and an incredible love of mathematics he created a new mould for a life style. (Will anyone else ever fit it?) This did not require him to pay attention to some of the details of living that most of us deal with. Thus it is not surprising that he developed the eccentricities which are often related by people who knew him and in articles about him – sometimes accurate and sometimes exaggerated. Some of these stories he did not care for, but others he liked to remember, and would retell himself, contributing to the canonization of these anecdotes. As with other passionate geniuses, stories about his eccentricities are a way for writers to show how unusual he was. However, to those who knew him closely, these stories, although amusing, do not in themselves capture the essence of this person, who was so very connected to the world.

Here we shall try to depict Erdős as *we* saw him.

### 1.1. Mátraháza, 1995

This volume is the Proceedings of the Mátraháza Workshop, one of those workshops which he liked so much, where he felt at home, where he was surrounded by old friends and young mathematicians, all eager to speak with him, to ask him questions, to tell him their results.

There was something special about Mátraháza, a small village 100 km from Budapest. It is a tourist resort on the Mátra mountain, which is the highest mountain in Hungary,

[1] Among other places, one could read about him in the *New York Times*, the *Washington Post*, the (London) *Times*, and even in the latest edition of the *Encyclopaedia Britannica*.

although only about 1000 metres high. The Hungarian Academy of Sciences has a nice holiday home there for researchers and their families.

Erdős liked to come here. In the late 1960s and early 70s, Erdős often spent a week or two here, with his mother and his friends, like the Kalmárs, the Rényis and the Turáns,[2] or with others, mathematicians and non-mathematicians, working, walking and discussing politics, history, economics, often with some of the best experts in Hungary. This was also the place where he played much table tennis (in his funny but very effective way), chess and go. (For a long time he was one of the best Hungarian go players.) Some of our favourite papers were born here, e.g. [36], [37].

Erdős had his own favourite trails here, one of them leading to a small tower which he liked to climb. Yes, Erdős liked this place and, above all, he liked to be here with his friends.

In 1995 we organized a one week workshop at Mátraháza for a small group of people in combinatorics and number theory. We spent a week there "proving and conjecturing". We remember Erdős working hard with Peter Cameron on some problems in combinatorial number theory. The theorems proved there in the evenings or late nights, while a few metres away other people were discussing completely different mathematical problems, can be read in this volume. Their paper was partly born in front of us. Erdős was surrounded here with love, reverence and care, and there was a warm and friendly atmosphere.

### 1.2. Warsaw, one year later

Erdős died in Warsaw, on 20 September, 1996. The circumstances of his death were profoundly moving. There was a five week long Mini Semester in Combinatorics at the Banach Center, in Warsaw. He participated in this series of workshops for two weeks. He enjoyed the mathematical atmosphere, but at the same time he often complained about the cold weather. He gave two lectures, the second one on Wednesday, 18 September, receiving a big round of applause. Early on Saturday morning he was taken to hospital from his hotel room, following a heart attack. It is very likely that he soon lost his ability to communicate and then he lost consciousness. Around 3 pm he had a second, much more serious, heart attack and this one killed him.

What we all feel was expressed by Vojta Rödl:

"Things won't be the same without Uncle Paul".

* * *

Wherever he travelled in the world, during almost every moment of the day, he was surrounded by colleagues and friends who cared about him. This was something which meant more and more to him as he grew older. Unfortunately, his last hours were spent in an unfamiliar hospital, alone.

[2] László Kalmár helped Erdős to write his first paper, a new, simplified proof of Chebyshev's theorem on prime numbers.

Erdős often explained that the most beautiful way to "finish life" is "... to give a lecture, finish a proof, put down the chalk and die... ". In some sense he passed away as he wished. His brain was engaged in mathematics until the very last day. This is what he wished so much all the time.

## 2. The travelling mathematician; the co-author mathematician

In 1939, for some strange reason, Erdős's fellowship at the Institute for Advanced Studies in Princeton was not renewed. It was at that point that he started his "nomadic" life, travelling from place to place. Nobody can tell what kind of life he would have had if his fellowship at the Institute had been extended.[3]

The important part of his life style was not just that he visited many places. It was much more important that he was able to make "human" and "mathematical" friendships, very easily, wherever he went. Arriving at a new place he would meet several mathematicians, ask them about their research, start thinking about their questions, and quite often would give striking solutions to their mathematical problems. Needless to say, he was incredibly fast.

It often happens that someone so powerful and deep mathematically, and so fast-thinking, can be intimidating to work with. It can be rather frustrating if one thinks about a problem for months and somebody just comes around, asks a few questions, and then solves it. To work with Erdős was completely different. Whenever you spoke with him, even though you knew that a mathematical giant was sitting there, opposite you, he could still create a very special atmosphere, where you felt equal. He was always asking questions about problems near to your mathematical interests, or just questions where he got stuck. Everybody was happy to work on his questions, and felt proud if he succeeded in solving an "Erdős problem" and so had a chance to explain the solution to him. But he would not stop there: he went on by asking further related questions, and never stopped pursuing new directions connected to the original innocent-looking problem.

We all have stories about this. Let us illustrate with a story of András Hajnal. (Hajnal describes this whole story in more detail in [42].) Hajnal was a PhD student of László Kalmár, in Szeged, working in set theory. Erdős met Hajnal while visiting Kalmár. Erdős asked Hajnal what was he doing mathematically, and Hajnal explained his topic. Erdős listened politely to Hajnal (in spite of the fact that he was not interested much in that particular part of mathematics) and then, suddenly, switched to some other topic, by asking: "... and are you interested in normal set theory as well?" With that, they started discussing mathematics which was interesting for both of them. Later Erdős and Hajnal climbed the tower of the Szeged Memorial Church. (As mentioned above, Erdős very much liked climbing hills, mountains, towers.) Hajnal writes:

> "... I had by then lived for two years in Szeged, and I had never had the slightest difficulty in resisting any pressure to visit the tower. However, to my surprise, I could

[3] L. Babai [2] writes in his very detailed and informative article about Erdős: "To his [Erdős's] dismay, his IAS fellowship is not extended and he is left penniless. 'Leopold Infeld wasn't extended either,' he comments."

> *not resist this invitation. Climbing those stairs more results and conjectures were* formulated by him [Erdős], while, at the same time, he was complaining that he felt a little dizzy.
>
> That day ended with dinner at Kalmár's house where the conversation continued mainly about set mappings but was sometimes interrupted with some of his comments on "Sam and Joe" [the USA and the USSR]. When we parted, it was almost as from an old friend – there was a joint-paper half ready which could be completed by correspondence."

That day completely changed Hajnal's life.

* * *

Erdős helped people in various ways: mathematically, or writing letters of recommendation, or lending money. He kept visiting his old friends just to "keep them company". For emotional reasons, he stopped using his apartment in Budapest after his mother's death (instead, he lived at the guest house of the academy), and for many years his friends in need lived there: some Hungarians, while changing from one apartment to another, and many of his friends from abroad, while visiting Hungary. Erdős cared for people. He was a sympathetic and compassionate person. It really made him feel depressed if one of his friends got into a difficult situation.

He often was invited to his friends' houses to stay with them. The following poetic lines, written by Paul to the second author in 1976, could have been written from many places.

> "It is six in the morning. The house still sleeps. I am listening to lovely music; I write and conjecture."[4]

* * *

Friendship was very important to Paul. When Erdős became a university student, he met Tibor Gallai, Géza Grünwald, Eszter Klein, Pál Turán, Endre Vázsonyi and some others from the Pázmány Péter University, Budapest,[5] and also György Szekeres, a student in chemical engineering at the Technical University, who was shuttling between these two universities to attend classes in both. This was the start of several life-long collaborations and friendships, some tragically cut short by the coming Fascism, increasing antisemitism and then by the Second World War. Life became more and more difficult in Hungary. Erdős was among those who soon realized that Hungary was no longer safe for Jews. So he left.

Remarkably, these people, with the exception of Géza Grünwald, survived the holocaust, but many of their friends and relatives were victims of Fascism.

Let us return to the university years. According to their reminiscences, their regular meetings, which were day-long excursions around Budapest or in the City Park, had

[4] In Hungarian: "Reggel hat van s a ház még alszik – szép zenét hallgatok, s közben irok és sejtek."
[5] Now called Eötvös Loránd University.

a great impact on their future lives in all respects. It was a kind of open, peripatetic university, brought about by the socio-political discrimination of the era which affected most of them.

This is what Szekeres wrote about these years: "We had a very close circle of young mathematicians, foremost among them Erdős, Turán and Gallai; friendships were forged which became the most lasting that I have ever known and which outlived the upheavals of the thirties, a vicious world war and our scattering to the four corners of the world. Our discussions centered around mathematics, personal gossip and politics." (Later Szekeres and his wife also left Hungary.)

The relationship of Erdős to his mother was legendary. Because of the war and the political situation he could not see his mother for ten years. After 1955 he visited Hungary regularly and his relationship with her changed: they spent more and more time together and this became one of the dominating features of his life. His mother's death shocked and changed him completely.

### 3. *Ars mathematica*: To conjecture and prove ...

The best way to learn about Erdős' mathematics is to read some of his original papers. Some of these, primarily in combinatorics, but also in number theory and geometry, are collected in *The Art of Counting, Selected Writings of Paul Erdős* [12].

Several survey papers on his work appeared on the occasion of his 80th birthday or since his death, and several further ones are planned for publication. Here we mention only a few: Bollobás [4], [5], Hajnal [42], and Simonovits [45]. Further, we warmly recommend the paper of Turán [52], a paper of Erdős himself [27], and also the long paper of Babai [2] and the paper of Sós [47]. We should also mention the book of Chung and Graham on problems of Erdős in graph theory [7].

Finally, we can learn a lot about Erdős' mathematics and personality by reading *his* papers about his friends: Turán, [14], [15], [16], [17], Gallai [20], Kalmár, [13], Gödel [24], Gabriel Dirac, [18], Ernst Straus [21], Ulam, [22], Richard Rado [23], to name only a few.

Erdős was a mathematician primarily interested in "concrete problems". Many mathematicians prefer to build theories, and when they get stuck they look at special cases to try to clarify the details.

Erdős used the opposite method. He set out from subcases, attacked them, proved more and more results around these particular problems, and thus built up first a small kernel of what later developed into a whole theory. For example, trying to answer some problems of Sidon he started using random methods in number theory. At the same time, he observed that some number theoretical functions behave very much like sums of independent random variables: with Kac [30], [31] and Wintner [40], [41] he started a whole new branch of number theory. This work was continued by many others, for example Kubilius, and is now described in several monographs, including the book of Elliott [8].

Erdős and Szekeres started from a "strange", seemingly tiny problem of Eszter Klein:

Given $n$ points in general position in the plane, for how large a $k$ can a convex $k$-gon always be selected from the set? One can easily see that out of five points one can always choose four in a convex position. But to prove that out of any nine points one can select five points in a convex position is much harder. Erdős and Szekeres proved a general theorem. To prove this theorem, in fact, they rediscovered Ramsey's theorem [39], which had been proved by Ramsey somewhat earlier.[6]

Answering a Ramsey question of Turán, Erdős started using random methods in graph theory.[7] Later, using more and more sophisticated versions of this method, he proved the existence of graphs with high girth and high chromatic number [10], [11]. Soon Rényi joined the investigation and they wrote a whole series of papers [32], [35], [34], [33] trying to answer the question:

What is a random graph and what does it typically look like?

Today the theory of random graphs is an important branch of combinatorics.[8] The significance and impact of Erdős's results cannot be separated from his ars mathematica. The concept of uniform distribution and the related discrepancy theory play a fundamental role in several branches of mathematics (diophantine approximation, measure theory, ergodic theory, geometry, discrete geometry, numerical analysis, theoretical computer science, etc).

* * *

His friend and co-author Ernst Straus (who was also a co-author of Albert Einstein) wrote about him: "The prince of problem solvers and the absolute monarch of problem posers". Erdős describes this as follows: "Problems have always been an essential part of my mathematical life. A well chosen problem can isolate an essential difficulty in a particular area, serving as a benchmark against which progress in this area can be measured. An innocent looking problem often gives no hint as to its true nature..." [28]. His problems typically have a very simple formulation, and often even the experts of a field realize only years later that Erdős posed the first generic question for a basic, general theory (which can be extremely difficult to answer): see e.g. [19].

The age when mathematicians could deeply understand and influence several distant fields in mathematics seems to have passed. But Erdős was definitely a giant of this kind. He did pioneering work in several unrelated fields, often jointly, and is helped create several branches of today's mathematics. On some occasions somebody else had written one or two pioneering papers and then Erdős began asking questions, conjecturing and

[6] For a detailed description of the story see e.g. Szekeres' paper [48] in *The Art of Counting: Selected Writings of Paul Erdős* [12].

[7] Setting out from Ramsey's theorem, which was reinvented by Erdős and Szekeres, Turán arrived at his famous extremal graph problem [51] and then turned back to Ramsey theory, by asking: How large a complete subgraph must occur in every graph or its complementary graph? Turán thought that $\sqrt{n}$ is the correct answer, and asked Erdős, who proved instead that $\log n$ is the correct order of magnitude. In some sense, this was the birth of random graph theory. For details see [53].

[8] The interested reader may try to read the original papers of Erdős and Rényi, or the beautiful books of Erdős and Spencer [38], Bollobás [3], Alon and Spencer [1], or the survey of Karoński and Ruciński [44].

proving theorems, and the whole area changed from a collection of a few isolated results into a blossoming theory. Often his colleagues did not immediately realize what was going on: whether Erdős' questions and answers would just be the last missing pieces of a small area or whether they would develop into a substantial theory.

When Erdős turned 50, Turán wrote a survey [52] of Paul's mathematical achievements. Turán's lines are even today among the best sources for understanding Erdős' mathematics.[9] In his "introduction" Turán wrote:

> "... his works published so far (!!!) relate roughly to the following topics: Number Theory, Probability Theory and Ergodic Theory, Graph Theory and Asymptotical Combinatorics, Constructive Theory of Functions, Set Theory and Set-theoretical Topology, Theory of Series, Theory of Analytic Functions, Geometry...
>
> ... But to write [about Erdős's work] has been made especially difficult by the fact that reviewing his work and influence is hard to separate from his mathematical personality. This is partially indicated by the fact that he has joint papers with more than 90 co-authors from four continents.... In a certain sense he is an occidental Ramanujan, with his strength and limits, a singular and unique phenomenon...."

On another occasion, when Erdős was 43, Turán compared Erdős's performance in mathematics to that of Mozart in music, and mentioned that Erdős was barely 20 when he was called the magician of Budapest (*der Zauberer von Budapest*) by I. Schur.

These lines were written more than 30 years ago. At that time Erdős had published roughly 400 papers; today his total publication list approaches 1600 papers. In his later years he moved away from analytic fields towards the more combinatorial parts of mathematics. It is interesting to observe the resulting changes in the number of papers and co-authors.

## 4. Connecting distant fields

One basic feature of Erdős's style was to ask surprising questions connecting distant mathematical areas. He did it so many times that we got used to it and often forgot to be surprised.

In the first case he forged links between number theory, combinatorics and graph theory and geometry. More than fifty years ago Paul asked the following question [9]:

> "At most how many unit distances can occur among $n$ points in the plane?"

The question may seem to concern geometry but, obviously, it bears no similarity to the usual questions in geometry.

The conjecture is that there are at most $O(n^{1+o(1)})$ unit distances among $n$ points, and the motivation for this conjecture is that one can scale a grid so that the number of unit distances in a disk of radius $c\sqrt{n}$ is slightly more than linear. The conjecture says that this is optimal.

There is a clear connection with the following classical question in number theory:

[9] It is not that surprising that many of us, writing of Paul, still use Turán's lines. Perhaps it is more interesting that Erdős, while writing one of his last papers [27], also liked going back to this source.

given an integer $m > 0$, in how many ways can it be represented in the form $x^2 + y^2$, where $x, y$ are also integers?

This problem of Erdős is extremely simple to state, but it is still unsolved. He immediately noticed that a simple extremal graph theorem concerning forbidding a two by three complete bipartite subgraph provides the estimate $O(n^{3/2})$. A much more involved argument of Józsa and Szemerédi [43] gave an upper bound $o(n^{3/2})$. Many years later Spencer, Szemerédi and Trotter [46] proved that the number of unit distances can be estimated from above by $O(n^{4/3})$, and Brass [6] has shown that this may be sharp in some Minkowski geometries. The importance of this last result is that it shows that – if the conjecture is true – one has to use the Euclidean structure of the plane in a non-trivial way.

In the last 50 years the distribution of distances (in general metric spaces as well) has become a difficult and extensively investigated topic. Rather surprisingly, these kinds of problems have also become relevant in computer science.

A similar case is connected with the celebrated theorem of Van der Waerden about arithmetic progressions. In the early forties Erdős and Turán asked: What is the maximum length of a sequence of integers in $[1, n]$ not containing a $k$-term arithmetic progression? They conjectured that this maximum is $o(n)$. K.F. Roth proved this in 1954 for $k = 3$, and in 1973 E. Szemerédi proved it for the general case [49] (for which he collected a \$1000 prize from Erdős). H. Fürstenberg and I. Katznelson proved many generalizations of this result using tools from ergodic theory. In the last few decades a whole new area has developed around this innocent looking question.

## 5. Papers, lectures, letters

Papers, lectures, correspondence and personal conversations were all important means by which Erdős could communicate about mathematics. He developed a specific genre in his papers and lectures. He wrote about 200 problem-papers that formulate dozens of problems clustered around a particular topic, always providing the background and some remarks in connection with the partial results of the moment.

In recent decades his lectures, whether plenary addresses at international congresses or educational lectures for teachers and students, took on the same characteristic form: results, partial results and conjectures interspersed with some stories that became known as "Erdősisms". It was a pleasure for him to engage in a mathematical conversation, should his partner be a high-school student or a famous mathematician.

He had a legendary memory. He could recall results of his own and others with the place and date of their publications, conversations with hundreds of mathematicians several decades ago, as one recalls yesterday's events.

His correspondence, an integral part of his daily routine, should also be mentioned. Some mathematicians may have kept several hundreds, others perhaps a dozen of his letters, which he wrote in his special characteristic style, switching abruptly from old and new mathematical problems to politics, friends and colleagues, and back.

He formulated thousands of problems and described related results and passed them on in his letters, between countries, through geographical and political borders, stimulating close friends and casual acquaintances to work together. Many of his results were obtained entirely through correspondence.

## 6. Shifting the emphasis to combinatorics?

The first two decades of Erdős's mathematics were dominated by number theory and analysis. Nowadays we think of the majority of his work as combinatorial mathematics: graph theory, combinatorial number theory, combinatorial geometry, combinatorial set theory. As a matter of fact, his earlier work had a strong combinatorial flavour as well.

Combinatorics, in close interaction with computer science, has undergone explosive development in the past 2 or 3 decades. Though Erdős himself was never directly involved in computer science, the indirect influence of computer science can be found in his mathematics. Yet the influence of Erdős's mathematics in computer science is much stronger. For example, his "random method" became one of the most powerful tools in Theoretical Computer Science; among other things, it is used to give lower bounds on the running time of algorithms.

We asked him to write a paper for the volume which was published in honour of his 80th birthday with the title "On my favourite theorems" [27], instead of the usual "On my favourite problems". This helps us to select some of his most important results by giving us his own opinion.

Erdős found elementary proofs for many classical theorems in prime number theory. (Here, "elementary" means that it does not use complex function theory.) In 1948 Erdős and Selberg found a long-awaited elementary proof of the prime number theorem.

Erdős and Rényi founded the theory of random graphs in the early 1960s. The significance of this field keeps growing. The systematic application of random methods in number theory, graph theory, and in many other fields is one of Paul's most important contributions.

In addition to the above mentioned fields, Erdős contributed to many others from his early youth until the end. An example of such an important area is interpolation theory. It is not the aim of this paper to cover these topics. They all deserve one or more separate surveys, which will surely be written by various authors in the future.

## 7. Did Erdős care only for numbers?

Paul's lifestyle centred around three values – independence, search for truth, and caring humanitarianism.

To secure his personal independence (independence from the political powers, doctrines and conventions of the surrounding world), he gave up possessions, family, and a safe job. This was not possible without sacrifices, as he himself often remarked. He paid the price for it all his life. The search for truth in science, politics, and everyday life provided the motivation for his round-the-clock activity and productivity, ignoring pains, sickness and aging. He continued his nomadic life style until the very end, writing 20–30 papers and giving several dozens of lectures yearly.

Despite the opinion of many people, he was not ascetic, even though he did not value possessions. On the contrary, he enjoyed life very much. He liked nature. He liked reading books on various topics. Sometimes, upon entering one's room, he picked up a book, read a few pages from it, asked if he might borrow the book, and then sent it back by mail from some other part of the world. He was astonishingly well versed in literature, biology, history and politics. He liked music, particularly Bach, Händel and Mozart. While working on mathematical problems, he often asked us to put on records or listened to his radio.

He liked going to restaurants and had a gourmet's taste (without a gourmand's appetite). He liked to invite people to restaurants and he liked to be invited to a good party or to a good dinner. He liked to live in good hotels.

However, in India he refused to eat decent food and would not go to good restaurants. It was not for shortage of money. He thought that in a country where hundreds of millions go hungry he should not eat like a gourmet. When he delivered a lecture about child prodigies at the Tata Institute in Bombay, he remarked to his mostly well-to-do audience that he did not understand how one could accept prosperity in the midst of such omnipresent poverty.

He did not value possessions but, as a matter of fact, he did not need them either. When he wanted to leave a place, the airplane ticket was there and he just left.

* * *

Of course, there were different periods in Erdős's life. Whatever we write about him will be more characteristic of some of these and less typical of others. Yet this is not the place to go into a detailed discussion of these differences.

In spite of countless stories, many of which have been further embellished after his death, he will be remembered clearly as a brilliant, unique mathematician of pure character, and as a warm, compassionate and charitable person. He was exceptional as a mathematician, as an intellectual, and as a human being with deep feelings who carved out a remarkable life for himself. We hope that this is the picture of Paul Erdős that will emerge in time and will endure.

## Acknowledgement

We would like to thank John Goldwasser for his very valuable help in improving the style of this paper.

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