Chaotic Dynamics of Nonlinear Systems

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S. NEIL RASBAND
Department of Physics and Astronomy
Brigham Young University
Provo, Utah

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PREFACE

In recent years the scientific community has witnessed the birth and initial development of a new paradigm for understanding complicated and seemingly unpredictable behavior. This new paradigm goes by the name of *Chaos*, referring to a scientific philosophy, an approach, and a set of methods to deal with manifestations of chaos in the physical sciences. To a large extent the enthusiasm that has developed for Chaos is the result of the breadth of its applications. These applications of Chaos for understanding complex and unpredictable behavior range across the spectrum of scientific disciplines. Indeed, Chaos has much the same flavor as classical thermodynamics in that the fundamental ideas and results seem applicable to a wide variety of different physical systems. There is probably no physical system exhibiting unpredictable behavior that is not presently being scrutinized through the lens of Chaos by someone.

Despite widespread interest and broad application, Chaos is a young science and as a consequence, the traditional examples are fewer and the standardization of methods is less well developed. With its roots in many areas of scientific inquiry, only in recent years have the examples and methods been welded into a new structure. In this textbook I have tried to introduce Chaos by presenting those topics and examples that seem to have risen to the top and become standard fare. However, for reasons of length choices must be made and the topics selected certainly reflect my own preferences. But I have tried to represent what most people in the community seem to feel are the important topics. Naturally, not all would agree on every point, and I would not expect anyone to agree with all of the selections or with the depth to which I have discussed them. Nevertheless, I believe that most of the "classical" topics in Chaos are represented.

To eliminate from the beginning any false expectations, I mention some topics that are not discussed: quantum chaos, noisy chaos, symbolic dynamics, and many, many examples of chaos occurring in specific physical systems. First and foremost, the book is intended to be useful as a textbook in a one-semester course taught in a physics department for seniors or first-year graduate students, and I have used this material for such a course. The audience, however, has not consisted only, or even primarily, of physics students. Certain topics and chapters require decidedly more background than others.

Chapter 8 on conservative dynamics expects the reader to be familiar with Hamiltonian dynamics. Sections 5.5 and 9.2 use some basic mathematical tools from differential geometry. However, the vast majority of the presentation depends only on some familiarity with differential equations and linear vector spaces. Even the reader with a limited knowledge of Hamiltonian dynamics or certain mathematical tools should be able to follow the presentation with only rarely a feeling of unfamiliarity.

The absolutely essential prerequiste the author expects the reader to bring to a study of this book is a willingness to do considerable numerical experimentation. The programming and numerical skills required for most of the examples are minimal, but a great deal of insight comes from personally performing numerical experiments on some of the classical problems in Chaos. Personal computers are adequate for doing everything in this book but, of course, may not suffice for tackling research problems in Chaos.

I wish to express my personal gratitude to colleagues who have fueled my interest by giving me copies of articles from diverse places and by generally encouraging me in this writing project. Particularly, I thank G. Mason, G. Hart, R. Shirts and E. Räuchle. I thank H. Stokes for frequent suggestions on TEX formatting and T. Knudsen for help in preparation of the manuscript. I especially thank my colleague and friend Ross Spencer for reading the manuscript and making literally hundreds of suggestions. The book is significantly better than it would otherwise be as a result of his help.

My deepest thanks go to my family since a considerable portion of the time necessary to complete this work has been taken from hours that rightfully belonged to them. The completion of this project would not have been possible without the love and support of my wife and children.

S. Neil Rasband

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July 1989

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CHAPTER ONE

INTRODUCTION

There are more things in heaven and earth, Horatio, than are dreamt of in your philosophy.
(W. Shakespeare, Hamlet, Act I, Scene 5)

Arguably the most broad based revolution in the worldview of science in the twentieth century will be associated with chaotic dynamics. Yes, I know about Quantum Mechanics and Relativity, and for physicists and philosophers these theories must rank above Chaos for their impact on the way we view the world. My assertion, however, refers to science in general, not just to physics. Leaving improved diagnostic instrumentation aside, it is not clear that Quantum Mechanics or Relativity have had any appreciable effect whatever on medicine, biology, or geology. Yet chaotic dynamics is having an important impact in all of these fields, as well as many others, including chemistry and physics.

Surely part of the reason for this broad application is that chaotic dynamics is not something that is part of a specific physical model, limited in its application to one small area of science. But rather chaotic dynamics is a consequence of mathematics itself and hence appears in a broad range of physical systems. Thus, although the mathematical representations of these physical systems can be very different, they often share common properties. In this introductory chapter we outline in a qualitative way some of the common features of chaos and introduce the reader to some chaotic phenomena. We further introduce some of the methods employed in the study of chaotic dynamics. Precision is left to discussions in subsequent chapters.

1.1 Chaos and Nonlinearity

The very use of the word "chaos" implies some observation of a system, perhaps through some measurement, and that these observations or measurements vary unpredictably. We often say observations are chaotic when there is no discernable regularity or order. We may refer to spatial patterns as chaotic if they appear to have less symmetry than other, more ordered states. In more technical terms we would say that the correlation in observations separated by either space or time appears to be limited. However, from the outset we must make clear that we are not speaking of the observation of random events, such as the flipping of a coin. Chaotic dynamics refers to *deterministic development* with chaotic outcome. Another way to say this is that from moment to moment the system is evolving in a deterministic way, i.e., the current state of a system depends on the previous state in a rigidly determined way. This is in contrast to a random system where the present observation has no causal connection to the previous one. The outcome of one coin toss does not depend in any way on the previous one. A system exhibiting chaotic dynamics evolves in a deterministic way, but measurements made on the system do not allow the prediction of the state of the system even moderately far into the future.

Whenever dynamical chaos is found, it is accompanied by *nonlinearity.* Nonlinearity in a system simply means that the measured values of the properties of a system in a later state depend in a complicated way on the measured values in an earlier state. By complicated we mean something other than just proportional to, differing by a constant, or some combination of these two. Although by these remarks, we do not mean to imply that somewhat complicated phenomena cannot be modeled by linear relations.

A simple, nonlinear, mathematical example would be for the observable x in the $(n+1)$th state to depend on the square of the observable x in the nth state, i.e., $x_{n+1} = x_n^2$. Such relations are termed *mappings*, and this is a simple example of a nonlinear map of the nth state to the $(n+1)$th state. A familiar physical example would be the temperature from one moment to the next as water is brought to a boil. At the end of this process the temperature in the $(n+1)$th state is just equal to the temperature in the nth state, but this is clearly not true as the water is being heated to its boiling temperature. Frequently the problem of modeling real-world systems with mathematical equations begins with a linear model. But when finer details or more accurate results are desired, additional nonlinear terms must be added.

Naturally, an uncountable variety of nonlinear relations is possible, depending perhaps on a multitude of parameters. These nonlinear relations are frequently encountered in the form of difference equations, mappings, differential equations, partial differential equations, integral equations, or even sometimes combinations of these. As we look deeper into specific causes of chaos, we shall see that chaos is not possible without nonlinearity. Nonlinear relations are not sufficient for chaos, but some form of nonlinearity is necessary for chaotic dynamics.

Having considered briefly nonlinear mappings, we now consider somewhat more closely systems modeled by differential equations. It is convenient when discussing the properties of differential equations to write them in a standard, first-order form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t). \tag{1.1}$$

If the $\mathbf{f}$ in (1.1) is independent of t, then the equation is said to be *autonomous*; otherwise it is *nonautonomous*. For such a system to be chaotic it must have more than one degree of freedom, or be nonautonomous. We illustrate this with the familiar example of a simple pendulum. The differential equation for a simple pendulum is often written in the form

$$\ddot{x} + \omega_0^2 \sin x = 0, \tag{1.2}$$

where x represents the angular displacement of the pendulum from the vertical position, two overdots denote two derivatives with respect to time in the usual way, and ω_0 denotes the natural frequency of the pendulum for small angular displacements. Even though this system is highly nonlinear, it does not exhibit chaotic dynamics. There is only the single degree of freedom associated with x and the right-hand side is the constant zero. If, instead, we replaced the zero in (1.2) with some function $f(x,t)$, then the system becomes nonautonomous and may exhibit chaotic dynamics, depending of course, on the exact nature of the function $f(x,t)$. In effect the time t has become an additional degree of freedom.

To put the differential equation (1.2) in the standard form (1.1) and to make explicit the notion that time is a degree of freedom, we define a new independent variable θ, and a new dependent variable $y = dx/d\theta$. Then with the driving term $f(x,t)$ on the right, (1.2) becomes the system

$$\frac{dx}{d\theta} = y, \quad \frac{dy}{d\theta} = -\omega_0^2 \sin x + f(x,t), \quad \frac{dt}{d\theta} = 1.$$

In this form the system consists of three, first-order differential equations and is nonautonomous. Frequently, such a system is said to have $1\frac{1}{2}$ degrees of freedom, since very often dynamical systems, particularly those resulting from Hamiltonian mechanics, have a pair of equations for every degree of freedom.

Although simple quadratic maps and forced, nonlinear oscillators like the preceeding examples may not appear to offer much promise for displaying a rich diversity of chaos, the opposite is true. We will see that indeed within these very simple nonlinearities lurk the seed of nearly all chaotic phenomena, and the bulk of this work is devoted to the study of such simple systems.

One of our major objectives is to classify and characterize deterministic systems exhibiting chaotic dynamics. Thus our characterization of nonlinearity as an essential ingredient for chaotic dynamics marks the beginning of this classification effort. We have further pointed out that for a system with one degree of freedom the differential equation must be nonautonomous. We now illustrate these points and the development of chaos with the familiar example of a simple harmonic oscillator.

1.2 The Kicked Harmonic Oscillator

To introduce many of the concepts and ideas that will be studied in subsequent chapters, we study the motion of a simple harmonic oscillator subject to

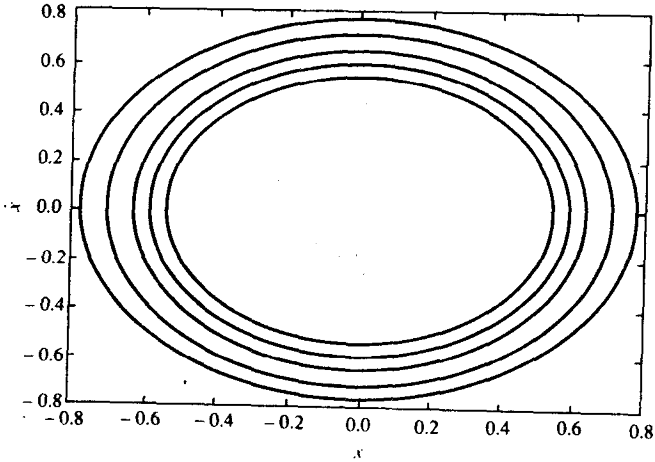

FIGURE 1.1 Sample phase-plane trajectories for the simple harmonic oscillator without kicks.

a periodic impulse. We refer to this system as the *kicked harmonic oscillator*. The equation of this system is given by

$$\ddot{x} + \omega_0^2 x = Af(x) \sum_{n=1}^{\infty} \delta(t - nT), \tag{1.3}$$

where ω_0 is the natural frequency of the oscillator, A is the amplitude of the kicks, and $f(x)$ is an arbitrary function of x, but not of t. Figure 1.1 shows the familiar phase-plane trajectories for the case where $A = 0$, i.e., the harmonic oscillator without kicks. Each ellipse corresponds to a fixed value of the energy of the oscillator. With $A \neq 0$, the right-hand side of (1.3) depends on time t; this differential equation is therefore nonautonomous.

In an interval between kicks the right-hand side of (1.3) is zero, and the solution is familiar:

$$x(t) = A_k \cos \omega_0 t + B_k \sin \omega_0 t, \qquad (k-1)T < t < kT, \tag{1.4}$$

and

$$\dot{x}(t) = -\omega_0 A_k \sin \omega_0 t + \omega_0 B_k \cos \omega_0 t, \tag{1.5}$$

where $k = 1, 2, \ldots$. For each k, at $t = kT$ we demand that the position of the one-dimensional oscillator be continuous but that the velocity (momentum) change discontinuously. This discontinuous change in the velocity is computed by integrating (1.3) from $(kT - \epsilon)$ to $(kT + \epsilon)$ and then taking the limit as $\epsilon \to 0$. We find easily the following relationship between the coefficients in the k interval and those in the $(k+1)$ interval.

$$A_{k+1} = A_k - \frac{A}{\omega_0} f(x_k) \sin \omega_0 kT, \tag{1.6}$$

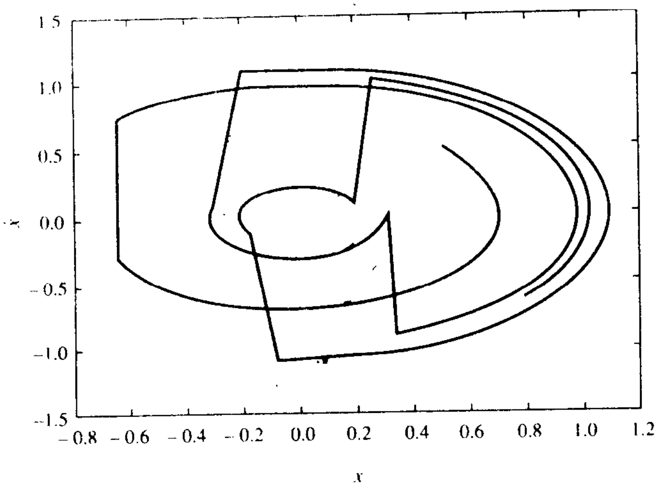

FIGURE 1.2 A section of a phase-space trajectory for a linear kicked harmonic oscillator. The discontinuous jumps in $\dot{x}$ are a result of the kicks.

$$B_{k+1} = B_k + \frac{A}{\omega_0} f(x_k) \cos \omega_0 kT, \tag{1.7}$$

where

$$\begin{aligned} x_k &= A_k \cos \omega_0 kT + B_k \sin \omega_0 kT, \\ \dot{x}_k &= -\omega_0 A_k \sin \omega_0 kT + \omega_0 B_k \cos \omega_0 kT. \end{aligned} \tag{1.8}$$

The subscript k on x and $\dot{x}$ refer to a time infinitesimally prior to the kick at kT. Using (1.8) with (1.6) and (1.7), plus a little algebra, yields the relation

$$\begin{pmatrix} x_{k+1} \\ \dot{x}_{k+1} \end{pmatrix} = \begin{pmatrix} \cos \omega_0 T & \omega_0^{-1} \sin \omega_0 T \\ -\omega_0 \sin \omega_0 T & \cos \omega_0 T \end{pmatrix} \begin{pmatrix} x_k \\ \dot{x}_k + A f(x_k) \end{pmatrix}, \tag{1.9}$$

which gives the position and velocity just before the time $(k+1)T$ in terms of the position and velocity just before the time kT.

The relationship between the coefficients in the k interval to those in the $(k+1)$ interval is an example of a two-dimensional mapping. Choosing the driving term in (1.3) to be a sum of delta functions is the feature that allows us to obtain the solution to the differential equation for the kicked harmonic oscillator in terms of the mapping represented by (1.9). The nonlinearities are introduced by the choice made for $f(x)$. With A and $f(x)$ not equal to zero, the system is nonautonomous and thus equivalent to more than one degree of freedom.

For $f(x) = 1$ or x, the mapping (1.9) is linear and invertible. In light of our previous remarks, no chaotic dynamics is to be expected. Such a case is, however, still nonautonomous — just not nonlinear. A plot of a segment of a phase-space trajectory for $f(x) = 1$ is given in Fig. 1.2. The trajectory crossings are a consequence of the time dependent driving term but can be eliminated by plotting the trajectory in extended phase space as in Fig. 1.3.

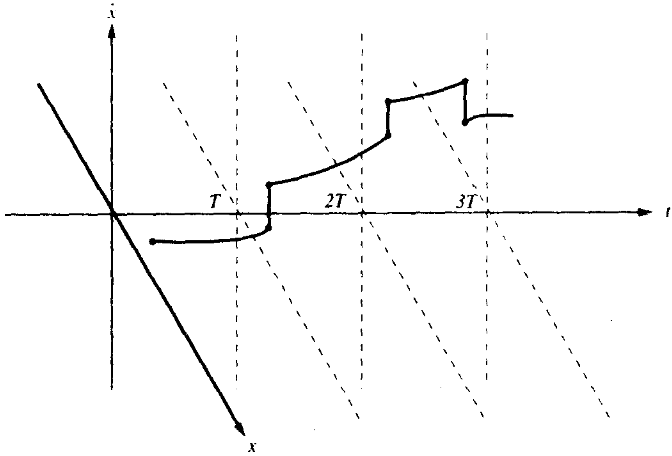

FIGURE 1.3 A sketch of a possible trajectory in extended phase space for the kicked harmonic oscillator. The kicks, and consequently discontinuous jumps in the velocity, occur at $t = T, 2T, \ldots$.

From (1.6) and (1.7) with $f = 1$ we obtain immediately

$$
\begin{aligned}
A_k &= A_1 - \frac{A}{\omega_0} \sum_{n=1}^{k-1} \sin(2\pi n \omega_0/\Omega), \\
B_k &= B_1 + \frac{A}{\omega_0} \sum_{n=1}^{k-1} \cos(2\pi n \omega_0/\Omega),
\end{aligned}
\tag{1.10}
$$

with $k = 2, 3, \ldots$ and $\Omega = 2\pi/T$. If $\omega_0 = \Omega$, i.e., if the kicks come at a frequency equal to the natural frequency of the oscillator, the coefficient $B_k \to \infty$ with k. The velocity and hence the energy of the oscillator become unbounded. This situation is called *resonance*. Resonance is a phenomenon occurring in a great many nonlinear systems leading to the destruction of the integrable behavior. The issue of resonance will reappear often in subsequent sections as we consider dynamics of nonlinear systems.

For $\omega_0 \neq \Omega$ the series in (1.10) can be summed to give

$$
\begin{aligned}
A_k &= A_1 + \frac{A}{\omega_0} \sin \pi k \left(\frac{\omega_0}{\Omega}\right) \left[\cos \pi k \left(\frac{\omega_0}{\Omega}\right) - \cot \pi \left(\frac{\omega_0}{\Omega}\right) \sin \pi k \left(\frac{\omega_0}{\Omega}\right)\right] \\
B_k &= B_1 + \frac{A}{\omega_0} \left[\sin \pi k \left(\frac{\omega_0}{\Omega}\right) \left[\sin \pi k \left(\frac{\omega_0}{\Omega}\right) + \cot \pi \left(\frac{\omega_0}{\Omega}\right) \cos \pi k \left(\frac{\omega_0}{\Omega}\right)\right] - 1\right]
\end{aligned}
\tag{1.11}
$$

If the ratio (ω_0/Ω) is a rational number, then there will always exist some k for which A_k and B_k return to their inital values, and the system is periodic.

As an alternative to a trajectory plot in extended phase space, which becomes impractical after a few periods, it is convenient to study the time

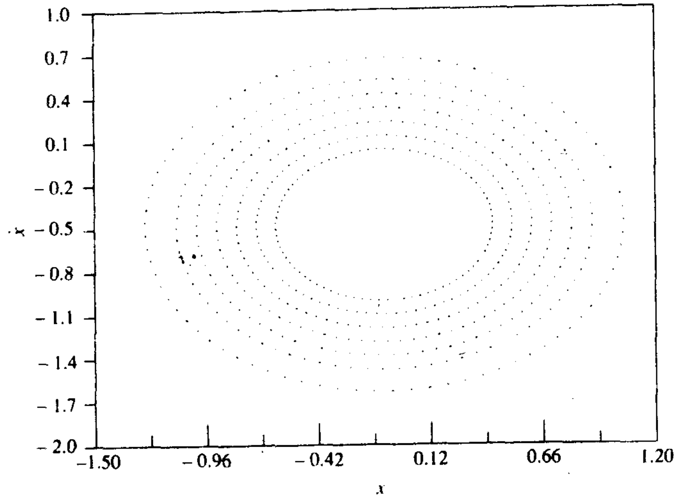

FIGURE 1.4 Poincaré Section plot for a kicked harmonic oscillator but with the driving term independent of x.

evolution of this system by making a point in the $(x, \dot{x})$ phase plane at the values of $t = T, 2T, \ldots$, i.e., at values of t corresponding to multiples of the period of the driving function. Such a plot for a dynamic system is called a ***Poincaré section***. Figure 1.4 is a Poincaré section plot for the system represented by (1.10), (1.11) and we see that the phase points always lie on ellipses, just as for the oscillator without kicks.

Comparing an orbit in Fig. 1.1 with the orbit in Fig. 1.2 dramatically demonstrates that a linear, time-dependent driving term alters the orbits in phase space. But this change in the nature of the phase-space orbits still does not go so far as to produce any chaotic dynamics. The relation between the (A_k, B_k) and (A_{k+1}, B_{k+1}) is still linear in (1.10) and (1.11). Nonlinearity is still absent in the system producing Fig. 1.4. Exercise 1.3 considers the same issues with $f(x) = x$.

We now change from $f(x) = 1$ to $f(x) = x^4$ and examine the Poincaré section plots for orbits with initial conditions similar to those producing the plots of Fig. 1.4. The Poincaré sections now produce Fig. 1.5, which is quite different from Fig. 1.4. We see two highly distorted elliptical orbits, an inner and an outer one, enclosing a seven-period island chain. Around the outer edge of this island chain there is a small, but finite, layer of chaotic orbits. The centers of the islands are called O points and the points between, joining the individual "islands," are called hyperbolic or X points. The insert in the center of Fig. 1.5 shows a magnified view of the intersection points of a *single* orbit in the neighborhood of the indicated X point. The reader should bear in mind that the insert only shows one of the seven X points, all of which are

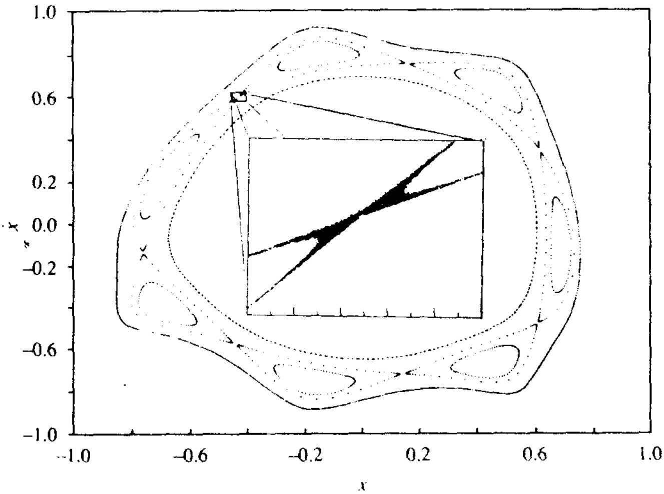

FIGURE 1.5 Poincaré section plot for a kicked harmonic oscillator with a dependence of the form x^4 in the drive. The intersection points of a single trajectory produce the points forming the island chain. The same is true for the outer closed curve, the inner closed curve, and the chaotic region magnified in the insert.

connected by a thin chaotic layer around the island structure. The chaotic region occupies a small but finite region in the phase plane. One of the most characteristic features of chaotic dynamics can be seen by considering two trajectories in the chaotic region that have nearly identical initial conditions. After a finite number of iterations of the map, the intersection point for one trajectory is completely unrelated to the intersection point for the second trajectory. This is our first example of chaotic behavior from deterministic dynamics. This feature is commonly referred to as ***sensitive dependence on initial conditions.*** Despite sensitive dependence on initial conditions and numerical roundoff, Hammel et al. (1988) have shown that the computation of chaotic orbits for a large number of periods as in Fig. 1.5 is still meaningful.

These few examples, and the kicked harmonic oscillator in particular, have illustrated the necessity for nonlinearity in producing chaotic dynamics. We further illustrated how Poincaré sections can be a useful tool in displaying chaotic consequences. For the kicked harmonic oscillator it was possible to obtain a mapping to advance the system in time, and it should be clear that this is much easier than the numerical integration of a system of differential equations. Partly because maps are easier to advance, and partly because of the importance of Poincaré section maps, considerable attention is devoted to mappings in subsequent chapters. This begins in the next chapter with a study of one-dimensional maps where we also develop additional methods to supplement Poincaré plots for studying and recognizing chaotic behavior.

1.3 Examples

The following is a selected list of some situtations where chaotic dynamics is manifest or appears to play a role.

1. Turbulence is believed to be the classic example of a system evolving deterministically, yet exhibiting chaotic behavior. Transistions to turbulence in Couette flow have been studied by Swinney and Gollub (1978), Swinney (1983, 1985), and Brandstater and Swinney (1987).
2. Thermal convection in fluids, called Rayleigh-Bernard convection, provides another example of transistion to turbulence. This has been one of the most fruitful applications experimentally and theoretically. It was in this system that chaotic dynamics was first appreciated theoretically with the work of Lorenz (1963). The Lorenz model is of such importance historically, and there has been so much work done on it, that the Lorenz equations have become one of the important examples for chaotic dynamics. Experimentally, it was careful measurements on liquid helium confined in a cell heated from below that led to stunning experimental confirmation of some of the predictions of chaotic dynamics by Libchaber and Maurer (1978). The Lorenz equations are considered in Chapter 6.
3. Supersonic panel flutter, important for supersonic aircraft and rockets, was studied by Kobayashi (1962).
4. Some chemical reactions, and in particular the Belousof-Zhabotinsky reaction, exhibit chaotic dynamics as discussed by Roux (1983) and Epstein (1983).
5. Optically bistable laser cavities have been studied by Ikeda et al. (1980) and Gibbs et al. (1981). Atmanspacher and Scheingraber (1986) have investigated several measures of chaos in a continuous-wave dye laser.
6. Cardiac dysrhythmias, or abnormal cardiac rhythms, have been discussed by Glass et al. (1983). In addition to the dynamics of the heart, its very structure has several manifestations of self-similar geometrical structures called fractals. Fractal structures are commonly the result of nonlinear dynamics, and, although the dynamics governing growth and development of the heart are unknown, fractal structures are detailed in the vascular network for the heart. Furthermore the cardiac impulse itself is transmitted to the ventricles via an irregular fractal network. Many such fractal structures in physiology are reviewed by West and Goldberger (1987).
7. There are many examples of nonlinear electrical circuits that exhibit chaotic dynamics. One famous example that has for many decades provided a model for nonlinear vibrations is the oscillator described by Van der Pol and Van der Mark (1927). Nonlinear circuits have provided analog devices for modeling many of the nonlinear equations discovered in one context or another.

8. Ecology and biological population dynamics provide a simple and instructive example of a dynamical system exhibiting chaotic dynamics. This example comes to us under the name of the "logistic equation." This equation may describe the variations in nonoverlapping biological populations from one year to the next. This equation and its importance were pointed out in an early review by May (1976). We study this classic example in the next two chapters.
9. Vibrations of buckled elastic systems have provided experimental examples of double-well potential systems. These systems have been studied experimentally and theoretically by Moon and Holmes (1979, 1980) and Holmes and Whitley (1983) as realizations of Duffing's equation, which is also one of the classical systems studied in nonlinear oscillations.
10. Chaotic dynamo models have been proposed for representing the geomagnetic field reversals and have been studied by Cook and Roberts (1970). A review has been given by Bullard (1978). We study this example in detail in Chapter 6.
11. Several types of standard chaotic behavior have been observed in simple plasma systems and reported in Cheung and Wong (1987) and Cheung et al. (1988).
12. A number of simple experiments suitable for classroom demonstration of chaos have been described by Briggs (1987).
13. Several researchers have claimed that EEG data suggests that chaotic neural activity plays a role in the processing of information by the brain. See Harth (1983), Nicolis (1984), and Skarda and Freeman (198).
14. By constructing a special computer for the single purpose of studying the stability of planetary orbits over long time scales Sussman and Wisdom (reported by Lewin, 1988) have found the orbit of Pluto to be chaotic on a time scale of about 20 million years.

This selected list of examples illustrates the broad range of scientific investigation that has been affected by studies in chaotic dynamics. I offer to the reader the personal challenge to find some previously unmentioned system in nature exhibiting chaotic behavior. Chaos can make life interesting in many ways.

Exercises

1.1 Consider a kicked rotor with its dynamics modeled by the equation

$$\ddot{\phi} + \gamma\dot{\phi} = Af(\phi)\sum_{n=1}^{\infty}\delta(t - nT),$$

where ϕ is the angle of the rotor, measured from some fiducial point, and γ is the damping constant (Schuster, 1984). If $\phi_n(t)$ is the solution for