Fundamentals of ACOUSTICS

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Preface

The purpose of this book is to present, in as clear and concise a form as possible, the fundamental principles underlying the generation, transmission, and reception of acoustic waves. The extensive developments of the past few decades have so broadened this field that an exhaustive treatment of all its aspects could not be contained in any single volume, and it has therefore been necessary to restrict the number of topics considered and to limit the extent to which each is carried. In making this selection it has been the primary aim of the authors to familiarize the student with the fundamental concepts and terminology of the subject, and with the analytical methods that are available for attacking acoustical problems. The first nine chapters of the book provide an analysis of the various types of vibration of solid bodies, and of the propagation of sound waves through fluid media. These nine chapters will suffice for a one-semester course in the fundamentals of theoretical acoustics, and may also be used for the first semester of a full-year course in theoretical and applied acoustics. The remaining seven chapters are concerned with a limited number of applications of acoustics. No attempt has been made to cover all such applications, those discussed being selected either because of their outstanding importance or as concrete illustrations of the mathematical techniques developed in the earlier chapters. Since each of these last seven chapters is an independent, self-contained unit, an instructor presenting a two-semester course may omit any one or more of these chapters and substitute material from the more specialized textbooks of acoustics.

One factor that has been kept in mind in writing this book is the close association that exists between acoustics and communications engineering. Not only do nearly all modern devices used in the generation and reception of acoustic waves depend for their operation on a conversion of electrical into acoustical energy, or vice versa, but the mathematical formulation of many acoustical problems is also quite similar to that employed in corresponding problems involving the transmission of alternating currents through lines or networks. In addition, it has been found that the design and analysis of many acoustical devices is facilitated by converting their mechanical or acoustical properties, such as mass or pressure, into equivalent electrical analogues, and then carrying through either a theoretical or an experimental analysis of the resulting analogous electrical circuit. In view of these factors, the mechanical and acoustical notation employed has been chosen to emphasize the similarity between these fields and to facilitate the conversion of results from one to the other.

Although this book has been developed from notes used for several years in a course given to graduate students in Engineering Electronics, it is not assumed that the reader is proficient in the engineering aspects of electrical communications. The book may be studied with equal facility by advanced undergraduate or graduate students in either Physics or Engineering Electronics, the essential requirements being a knowledge of the fundamental principles of mechanics and electricity and an understanding of the methods of calculus, including partial derivatives. Since this book is intended primarily as a textbook for classroom use, rather than a reference work, no attempt has been made to include a complete bibliography, although occasional references are given where the treatment is necessarily incomplete. The authors have attempted to derive each important equation from the fundamental laws of physics and to show in some detail not only the mathematical steps but also the logical processes involved in these derivations. The derivations of a few of the less important equations have been intentionally omitted and are instead included as exercises for the student among the problems given at the end of each chapter. Considerable attention has been paid to the selection of a comprehensive set of problems, for the ultimate check on the student's understanding of the subject is his ability to apply his knowledge to new situations. Tables of physical constants and functions are given in the appendix. As far as possible, the proposed standards of acoustical terminology of the American Standards Association have been used throughout this book, and a glossary of symbols is incorporated in the appendix as a further aid in clarifying the confusion that might result from the multiplicity of physical quantities represented by certain of the more commonly used symbols.

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CHAPTER

I

Simple Harmonic Motion

1.1 Introduction. The whole study of acoustics is primarily concerned with the generation, transmission, and reception of energy in the form of *longitudinal* waves in matter, and is therefore a study of vibrations. As the atoms and molecules of a solid or fluid are displaced from their normal configuration an internal elastic force of stiffness arises. Typical examples of such a force include the tensile force produced when a spring is stretched, the pressure resulting from the compression of a fluid into a reduced volume, and the shearing force that appears when a shaft is twisted. It is the action of this restoring force of stiffness, coupled with the inertia of the system, that results in oscillatory vibrations.

Many different types of vibration occur in the generation and propagation of acoustic waves. In a narrow sense, their frequency is limited to the range from about 20 cycles/sec to 15,000 cycles/sec, which produce the auditory sensation of sound for the average person. However, in a broader sense they also include the *ultrasonic* frequencies above 15,000 cycles/sec, which although inaudible have important practical applications in numerous fields. The modes of vibration range from the simple sinusoidal sound waves produced in air by a mounted tuning fork vibrating at its fundamental frequency, through the complex pattern of periodic waves generated by a bowed violin string, to the nonperiodic waves associated with a noise or an explosion. In studying such vibrations it is advisable to begin with the simplest type, i.e., a sinusoidal vibration having only a single frequency component.

1.2 Simple Oscillator. If a mass m, fastened to some sort of spring and constrained to move back and forth in just one direction, is displaced from its central or rest position and is then released, the mass will be observed to vibrate. Measurement shows that the frequency of vibration is constant, and that the displacement of the mass from its rest position is a sinusoidal function of time. Sinusoidal vibrations of this type are called *simple harmonic vibrations.* It can be shown, both experimentally and theoretically, that the mass will vibrate with simple harmonic motion whenever the restoring force resulting from the stiffness of the spring is directly proportional to the displacement of the mass from its rest position. A very large number of vibrators used in acoustics are of this type, or are approximately equivalent to it. Loaded tuning forks, and loudspeaker diaphragms which are so constructed that at low frequencies their mass moves as a unit and may be considered to be concentrated near their center, are but two examples. Even more complex vibrating systems have many of the characteristics of the simple system and may be studied to a first approximation by being reduced to simple oscillators.

The only physical restriction placed upon the equations shortly to be developed for the motion of a simple oscillator is that the restoring force be directly proportional to the displacement. Whenever the amplitude of vibration is sufficiently small so that the *elastic limit* of the spring is not exceeded, the frequency of vibration is independent of amplitude and the motion is simple harmonic, but this is not true if this limit is exceeded. A similar restriction applies to more complex types of vibration, such as those corresponding to the transmission of an acoustic wave through a fluid medium. If the resulting acoustic pressures are so large that they are no longer proportional to the displacement of the particles of the fluid, it becomes necessary to modify the normal acoustic equations. With sounds of ordinary intensity this is not necessary, for even the noise generated by a large crowd at a football game rarely causes the amplitude of motion of the air molecules to exceed one-tenth of a millimeter, which is within the limit given above. The amplitude of the shock wave generated by a large explosion is, however, well above this limit, and hence the normal acoustic equations are not applicable.

Returning now to a consideration of the simple oscillator, such as that shown in Fig. 1.1, let us assume that the restoring force f can be expressed by the equation

$$f = -sx \tag{1.1}$$

where x is the displacement of the mass m from its rest position, s is the *stiffness constant* of the spring, and the minus sign indicates that the force is directed oppositely to the displacement. If forces are expressed in dynes and displacements in centimeters, then the constant s, which is assumed to be the same for tension as for compression, has the dimensions of dynes per centimeter.

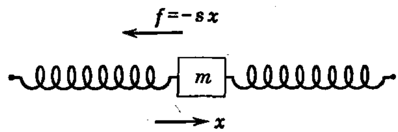

FIG. 1.1. Simple oscillator.

Substituting this expression for force into the general equation of linear motion

$$f = m\frac{d^2x}{dt^2} \tag{1.2}$$

and replacing the ratio of the two constants of the system, s/m, by a new single constant ω_0^2, we obtain

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \tag{1.3}$$

This equation is an important second-order linear differential equation whose solution is well known, and may be obtained by several methods.

One method is to assume a solution of the type

$$x = A_1 \cos \gamma t$$

Differentiation and substitution of this expression in equation 1.3 shows that it is a solution if we identify γ with ω_0. Furthermore, it may be similarly shown that

$$x = A_2 \sin \omega_0 t$$

is also a solution. The complete general solution is the sum of these two solutions, i.e.,

$$x = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t \tag{1.4}$$

where A_1 and A_2 are two arbitrary constants.

1.3 Initial Conditions. The constants A_1 and A_2 are determined by the manner in which the mass is started into motion, i.e., by the *initial conditions.* If at the time $t = 0$ the mass has an initial displacement x_0 and an initial velocity v_0, then the arbitrary constants A_1 and A_2 are fixed by these initial conditions, and the subsequent motion of the mass is completely determined. A direct substitution into equation 1.4 of $x = x_0$ at $t = 0$ will show that A_1 equals the initial displacement x_0. Differentiation of equation 1.4 and substitution of the initial velocity condition gives

$$v_0 = -\omega_0 A_1 \sin 0 + \omega_0 A_2 \cos 0$$

so that v_0 must equal $\omega_0 A_2$. Therefore $A_2 = v_0/\omega_0$, and equation 1.4 becomes

$$x = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t \tag{1.4a}$$

Another form of equation 1.4 may be obtained by letting $A_1 = A \cos \phi$ and $A_2 = -A \sin \phi$, where A and ϕ are two new arbitrary constants. Substitution and simplification then gives

$$x = A \cos (\omega_0 t + \phi) \tag{1.5}$$

where A is the *amplitude* of the motion and ϕ is the initial *phase angle* of the motion. Also one may show that A and ϕ have their values determined by the usual initial conditions and are

$$A = \left(x_0^2 + \frac{v_0^2}{\omega_0^2}\right)^{1/2}, \quad \text{and} \quad \phi = \tan^{-1} \frac{-v_0}{\omega_0 x_0} \tag{1.5a}$$

1.4 Frequency of Vibration. The frequency of vibration is determined by the value of the *angular frequency* constant ω_0. Since $\omega_0 = 2\pi f_0$, where f_0 is the frequency of vibration in cycles per second, then

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} \tag{1.6}$$

It is to be noted that either decreasing the value of the stiffness constant or increasing the mass of the oscillator results in a decreased frequency. This mathematical deduction is in agreement with what one would conclude from a logical consideration of the physics involved; i.e., increasing mass or decreasing stiffness would be expected to slow down the vibration. The *period*, T, of one complete vibration is given by the reciprocal of equation 1.6.

1.5 Series Method of Solution. An alternative method of solving the original differential equation (1.3) is to assume a power-series solution of the form

$$x = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$$

Upon differentiating this expression twice with respect to t and substituting in equation 1.3 we obtain

$$2a_2 + 6a_3 t + 12a_4 t^2 + \cdots + \omega_0^2 a_0 + \omega_0^2 a_1 t + \omega_0^2 a_2 t^2 + \cdots = 0$$

Since this equation must hold for every value of t, the sums of the coefficients of each power of t must be separately equal to zero. Applying this condition,

$$a_2 = -\omega_0^2 a_0/2 \qquad a_3 = -\omega_0^2 a_1/6$$

$$a_4 = -\omega_0^2 a_2/12 \qquad a_5 = -\omega_0^2 a_3/20$$

etc. Therefore the series that satisfies equation 1.3 is

$$x = a_0\left(1 - \frac{\omega_0^2 t^2}{2!} + \frac{\omega_0^4 t^4}{4!} - \cdots\right) + \frac{a_1}{\omega_0}\left(\omega_0 t - \frac{\omega_0^3 t^3}{3!} + \frac{\omega_0^5 t^5}{5!} - \cdots\right)$$

The series multiplying the constant a_0 is the well-known series expansion of $\cos \omega_0 t$, and similarly the series multiplying the constant a_1/ω_0 is the expansion of $\sin \omega_0 t$. The series solution is therefore equivalent to

$$x = a_0 \cos \omega_0 t + \frac{a_1}{\omega_0} \sin \omega_0 t$$

which is of the same general form as equation 1.4.

1.6 Complex Exponential Method of Solution. A third method of solving the original differential equation (1.3) is to assume a solution of the form

$$x = Ae^{\gamma t}$$

This expression will satisfy the equation if γ^2 is set equal to $-\omega_0^2$, which is equivalent to $\gamma = \pm j\omega_0$, where $j = \sqrt{-1}$. Letting γ equal both $+j\omega_0$ and $-j\omega_0$, the complete general solution may be written

$$x = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t} \tag{1.7}$$

where A_1 and A_2 are constants to be determined from the initial conditions of motion. Utilizing the well-known relations between exponential and trigonometric quantities,

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$$

and

$$e^{-j\omega_0 t} = \cos \omega_0 t - j \sin \omega_0 t$$

equation 1.7 may be reduced to

$$x = (A_1 + A_2) \cos \omega_0 t + j(A_1 - A_2) \sin \omega_0 t \tag{1.7a}$$

From physical considerations it is apparent that the displacement of the mass must be a *real* quantity (not involving j), and hence if A_1 and A_2 are chosen as real numbers this condition requires that $A_1 = A_2$. The solution then contains in effect only a single arbitrary constant, $A_1 + A_2 = 2A_1$, and is consequently incomplete. To obtain the complete solution we must assume that $\mathbf{A}_1$ and $\mathbf{A}_2$ are complex,[1] i.e.,

$$\mathbf{A}_1 = a_1 + jb_1 \quad \text{and} \quad \mathbf{A}_2 = a_2 + jb_2$$

where a_1, a_2, b_1, and b_2 are real numbers. Then equation 1.7 may be reduced to

$$\begin{aligned} x = (a_1 + a_2) \cos \omega_0 t - (b_1 - b_2) \sin \omega_0 t \\ + j[(b_1 + b_2) \cos \omega_0 t + (a_1 - a_2) \sin \omega_0 t] \end{aligned} \tag{1.7b}$$

and the displacement will be real at all values of t if the coefficients of the trigonometric expressions in the imaginary term are zero, i.e., if $b_1 + b_2 = 0$ and $a_1 - a_2 = 0$. Under these conditions $\mathbf{A}_1$ and $\mathbf{A}_2$ are complex conjugates, and equation 1.7b becomes

$$x = 2a_1 \cos \omega_0 t - 2b_1 \sin \omega_0 t \tag{1.7c}$$

which is identical in form with equation 1.4.

[1] In this book **boldface type** will be used to indicate **complex** quantities; *italic type* will represent *real* quantities.

In actual practice it is unnecessary to go through the mathematical steps required to make the imaginary part of the general solution vanish, for it is sufficient to adopt the convention that the *real part of the complex solution is by itself a complete general solution* of the physical problem indicated by the original differential equation. It is self-evident that the real part of the above complex solution (equation 1.7*b*) is a complete solution. Similarly, the real part of either $\mathbf{A}_1 e^{j\omega_0 t}$ or $\mathbf{A}_2 e^{-j\omega_0 t}$ is likewise a complete solution.

It will be the general practice in this textbook to analyze problems by the complex exponential method. The chief advantages of this procedure, as compared with the trigonometric method of solution, are its greater mathematical simplicity and the relative ease with which the phase relationships between the various mechanical and acoustic variables can be determined. In addition, many of the problems that arise in acoustics are similar to those encountered in alternating-current electrical theory, so that the results and techniques of electrical theory may be used in solving acoustic problems. Whenever possible, the notation used in this textbook is chosen to emphasize this similarity. The chief disadvantage of the complex exponential method is that the solutions obtained do not represent the *true* values of the various acoustic variables, and care must be taken to obtain the *real* part of the complex solution in order to arrive at the correct physical equation or numerical solution.

1.7 Physical Characteristics of Simple Harmonic Motion. Differentiation of equation 1.5 shows that the velocity is given by

$$v = \frac{dx}{dt} = -\omega_0 A \sin(\omega_0 t + \phi) \tag{1.8}$$

and the acceleration by

$$\text{Acceleration} = \frac{d^2x}{dt^2} = -\omega_0^2 A \cos(\omega_0 t + \phi) = -\omega_0^2 x \tag{1.8a}$$

From these equations it will be seen that the displacement lags 90° or $\pi/2$ radians behind the velocity and that the acceleration is out of phase with the displacement by 180° or π radians, as shown in Fig. 1.2.

Consideration of the complex form of the equation representing this type of motion leads to similar results. The expression $e^{j\omega_0 t}$ may be thought of as a vector of unit length, rotating in a counterclockwise direction in the complex plane with an angular velocity ω_0. Similarly, any complex quantity **A** having the components a and jb may be represented by $Ae^{j\phi}$, a vector of length $A = (a^2 + b^2)^{1/2}$ making a phase angle ϕ, whose tangent is b/a, with the axis of reals. It can readily be shown that the product of any two complex quantities is then represented by a

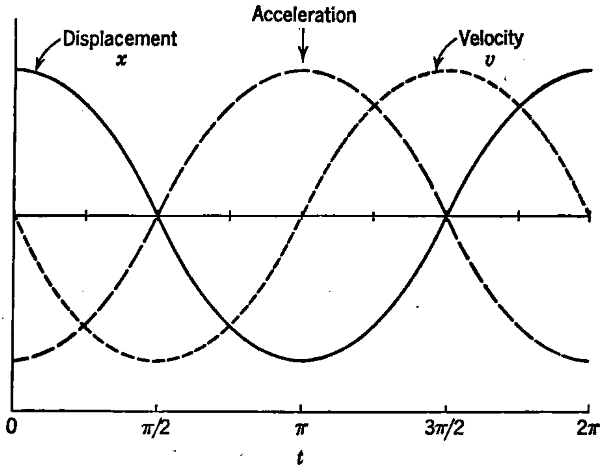

FIG. 1.2. The velocity v always leads the displacement x by a time interval corresponding to $\pi/2$ radians of phase-angle difference. Acceleration and displacement are always π radians out of phase with each other. Plotted curves correspond to $\phi = 0$ and $\omega_0 = 1$.

vector whose length is the product of the lengths of the individual vectors, and whose phase angle is the sum of their phase angles. The expression $\mathbf{A}e^{j\omega_0 t}$ consequently is equivalent to $Ae^{j(\omega_0 t+\phi)}$ and represents a vector of length A and initial phase angle ϕ, rotating counterclockwise in the complex plane with the angular velocity ω_0, Fig. 1.3. The real part of this rotating vector, i.e., its projection on the axis of reals, has the magnitude

$$A \cos (\omega_0 t + \phi)$$

and therefore varies with time in a simple harmonic manner. The reader may similarly show that the real part of $\mathbf{A}e^{-j\omega_0 t}$ also varies in a simple harmonic manner.

If the displacement x is represented by the complex equation

$$\mathbf{x} = \mathbf{A}e^{j\omega_0 t}$$

differentiation with respect to time gives $\mathbf{v} = j\omega_0\mathbf{x}$, and hence the complex vector representing velocity *leads* that representing displacement by j, i.e., by a phase angle of 90°. The projection of this vector on the axis of reals then represents the instantaneous velocity of motion, the velocity amplitude being $\omega_0 A$. A further differentiation shows that the complex vector representing acceleration is out of phase with the displacement vector by -1, or 180°.

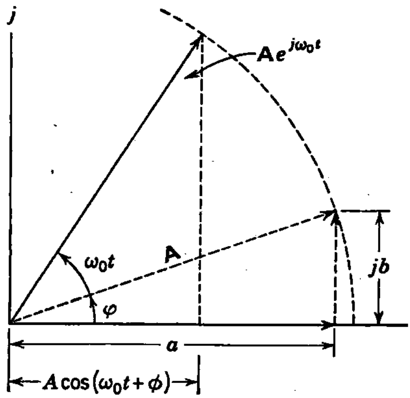

FIG. 1.3. Physical representation of a rotating complex vector $\mathbf{A}e^{j\omega_0 t}$.

1.8 Energy of Vibration. The energy of a mass oscillating with simple harmonic motion of amplitude A and angular frequency ω_0 is the sum of the system's potential energy E_p and its kinetic energy E_k. The potential energy is the work done in distorting the spring as the mass moves from its position of static equilibrium. Since the force exerted by the mass on the spring is in the direction of the displacement and equals $+sx$, the potential energy E_p stored in the spring is

$$E_p = \int_0^x sx\,dx = \tfrac{1}{2}sx^2 = \tfrac{1}{2}m\omega_0^2x^2 \tag{1.9}$$

An alternative form of this equation may be obtained if the value of x as given by equation 1.5 is substituted in equation 1.9. Then

$$E_p = \tfrac{1}{2}m\omega_0^2A^2\cos^2(\omega_0 t + \phi) \tag{1.9a}$$

Using the usual expression for kinetic energy, we have

$$E_k = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega_0^2A^2\sin^2(\omega_0 t + \phi) \tag{1.10}$$

The total energy E of the system at all times is therefore

$$E = E_p + E_k = \tfrac{1}{2}m\omega_0^2A^2[\cos^2(\omega_0 t + \phi) + \sin^2(\omega_0 t + \phi)]$$

or

$$E = \tfrac{1}{2}m\omega_0{}^2A^2 = \tfrac{1}{2}sA^2 \tag{1.11}$$

so that the total energy is constant. Since the system was assumed to be nondissipative, i.e., to have no frictional losses, this result is to be expected. The magnitude of the total energy is seen to be equal to the potential energy ($\frac{1}{2}sA^2$), when the mass has its greatest displacement, and is also equal to the kinetic energy ($\frac{1}{2}m\omega_0{}^2A^2$) when the mass has its greatest velocity. Expressed in terms of ω_0 and A, it is to be noted that E depends on the product of the squares of these two quantities. This particular dependence of energy on frequency and displacement amplitude recurs frequently in acoustics, both for sound sources and sound waves. For instance, a specified acoustic output may be obtained at high frequencies with an amplitude of vibration of the sound source that is small as compared to that required at low frequencies.

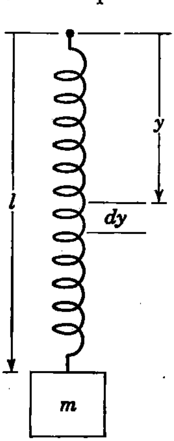

FIG. 1.4. Effect of mass of spring.

1.9 Effect of Including Mass of Spring. If the mass m_s of the spring is not negligible as compared with the mass m attached to the spring, it is to be expected that this additional inertia of the system will result in a reduced frequency of vibration. Let the length of the spring be l, and assume the velocity of any element dy of the spring, Fig. 1.4, to be proportional to its distance y from the fixed end of the spring. Then the velocity of this element is given by vy/l, where v is the velocity of the free end of the spring to which the mass is attached. The total kinetic energy of the spring can be obtained by integrating the kinetic energy of a length dy, along the entire spring. Then

$$E_k \text{ of spring} = \frac{1}{2}\int_0^l \left(\frac{m_s}{l}\,dy\right)\left(\frac{y}{l}\,v\right)^2 = \tfrac{1}{6}m_sv^2$$

and hence the total kinetic energy of the system is given by

$$E_k \text{ of system} = \frac{1}{2}\left(m + \frac{m_s}{3}\right)v^2$$

Assuming that the stiffness constant s is measured with the spring hanging in a vertical position, the potential energy ($\frac{1}{2}sx^2$) is the same as for a massless spring.

Since the system is nondissipative the total energy must be constant. Therefore

$$E = \frac{1}{2}\left(m + \frac{m_s}{3}\right)v^2 + \frac{1}{2}sx^2 = \text{constant} \tag{1.12}$$

Setting $v = dx/dt$ and differentiating with respect to time we have

$$\left(m + \frac{m_s}{3}\right)\frac{d^2x}{dt^2} + sx = 0 \tag{1.12a}$$

as the differential equation representing the motion. Upon comparing this equation with equation 1.3, it is evidently equivalent if ω_0 is now given by

$$\omega_0{}^2 = \frac{s}{m + (m_s/3)} \tag{1.12b}$$

When the mass of the spring is not negligible, the frequency of vibration may therefore be determined by adding to the suspended mass one-third of the mass of the spring.

1.10 Linear Combinations of Simple Harmonic Vibrations. In many important situations that arise in acoustics the motion of a body is a linear combination of the vibrations induced separately by two or more simple harmonic motions. The displacement of the body is then the *algebraic* sum of the individual displacements.

One important example is the combination of two such motions having the same angular frequency ω. Thus, if the two individual displacements are given by

$$x_1 = A_1 \cos(\omega t + \phi_1) \quad \text{and} \quad x_2 = A_2 \cos(\omega t + \phi_2)$$

then their linear combination along one direction is $x = x_1 + x_2$. By the use of the familiar trigonometric relations involving the sums and differences of angles it is possible to convert this expression for the displacement into the more convenient form

$$x = A \cos(\omega t + \phi) \tag{1.13}$$