SIGNALS AND LINEAR SYSTEMS

Third Edition

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PREFACE

to the Third Edition

In response to suggestions from colleagues who have used the second edition of *Signals and Linear Systems* in their courses and from students in our own classes, we have made several changes in the presentation of topics in this text. As with previous editions, we emphasize a unified view of linear systems analysis and signal representations. It is our aim that the various approaches used to characterize the interaction of signals with systems be interrelated to stress their commonality, and not be presented as disjoint topics. Thus, we have retained the overall organization of the second edition and continue to contrast difference/differential equation models, convolution, and state-variable formulations in the presentation of continuous- and discrete-time systems. Likewise, we present transform methods with considerable attention given to relating them to the corresponding time-domain techniques.

Additional material has been devoted to applications of the theoretical material in physical problems. Our aim in this respect is to help motivate classroom discussions of the relevance of this material to the students design activities. Discussions of deconvolution were added to both the time-domain and transform-domain treatment of discrete-time systems, with examples relating its relevance to signal processing problems. The Fourier analysis material in Chapter 5 was augmented to emphasize the interrelationships among the discrete Fourier transform, Fourier series, and the more general Fourier transform.

In the course of choosing the material and methods of presentation in this text, we have carefully considered the advice of colleagues and reviewers. For example, in the treatment of forced difference and differential equations, we considered both the method of undetermined coefficients and the annihilator operator approach. We chose the latter because it directly relates the solution of non-homogeneous equations to the previously solved homogeneous case; it explains well the special case of systems forced at their natural modes, leading to repeated roots of the characteristic equation. It also ties the treatment together in a consistent form (the solution mechanics of the two approaches involve, of course, essentially identical calculations). We realize that individual instructors may prefer alternative approaches in various topics; the chapter organization was designed to make this substitution easy to accomplish.

It has also been suggested that we highlight Section 7.5, which deals with hybrid systems of ADCs, DACs, and digital filters. This section is unique in treating systems that contain both continuous- and discrete-time elements. We generally cover this material in a summary lecture toward the end of our course, with the remainder of Chapter 7 left to the students reading.

Finally, we have renumbered the examples and equations to better relate them to the corresponding discussions. As previously, we are indebted to students and faculty who have forwarded comments and suggestions concerning the presentation of this material. We welcome a continuing dialogue.

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PREFACE

to the Second Edition

After communicating with a number of instructors who have adopted the first edition of this book and after using it ourselves for several semesters we have concluded that certain modifications would improve the presentation and have incorporated them in this new edition. As in the first edition, we stress the relationships among the various representations of a linear system: the difference or differential equation model, the block diagram or flow graph form, the impulse-response description, the state-variable formulation, and the transfer-function characterization. We emphasize throughout the book that these representations are tightly related and may be employed to great conceptual and computational advantage in the analysis and synthesis of linear systems. The presentation of the material is organized to enable the various topics to reinforce one another. New results are related to those that have been mastered previously. Additional examples have been incorporated to further illustrate the material as it is developed. Several problems are presented using comparative solution approaches, so that students may experience the various solution methods for representative applications. As previously, the material is general and chosen to lead naturally into succeeding courses in communication systems, control systems, and other areas that use these basic techniques in specific advanced applications. Prerequisites assumed of the student are a sophomore-level course covering differential equations and a course in the student's major area (such as circuits or mechanics) that deals with the derivation of mathematical models of physical systems.

The most pronounced change in the second edition is the reorganization of the time-domain material in Chapters 1 to 3. Whereas the first edition emphasized most strongly the similarities between discrete- and continuous-time system analysis, with different approaches in separate chapters, we have found the presentation to flow more smoothly if the various analysis approaches are contrasted within a body of material, with discrete- and continuous-time systems developed in separate chapters. It is our experience and that of our colleagues who have presented the material in both forms that students more readily perceive the similarities between discrete- and continuous-time system analysis than they do the relationships between, for example, analyses based on the system equation solution, the convolution sum or integral, and the state-variable description. This modified organization also allows the instructor to begin with either discrete-time or continuous-time systems. (Our suggestion, reflected in the chapter ordering, is to treat discrete-time systems first).

As an aid in motivating and interrelating the various approaches, we have introduced the concept of the system frequency response at an early point in the discussion. Evaluation of the frequency response is demonstrated with successive system models as they are developed, rather than being relegated to a later section of the transform domain discussion. This, we feel, is a major feature of the presentation offered here, and one that has greatly enlivened our classroom discussions.

We discovered that adoptions of the first edition were primarily for electrical engineering courses. Therefore and in line with the suggestions from users, we have changed the Z-transform notation of Chapter 4 to correspond with the standard used for publication in this discipline. Additional material concerning the frequency response of digital filters has been included and related to the preceding treatment.

Chapter 5 on Fourier analysis has been significantly revised and expanded to include the Fourier analysis of discrete-time signals and systems. It also includes new material on the implications of using windows in FIR filter synthesis, numerical computation of Fourier transforms, and intelligent use of the Fast Fourier Transform. This material serves to further integrate the student's perceptions of discrete- and continuous-time system analysis.

Chapter 7 on digital filter synthesis has been reorganized to more clearly develop and contrast the standard design approaches. The material on mixed continuous- and discrete-time systems has been placed to summarize the principal developments in the text.

This book is used in a one-semester junior course in linear systems analysis at the University of Colorado. The course meets for a total of 40 lecture hours.

In this time we cover most of Chapters 1 to 6 and selected portions of Chapter 7 as time permits. We present the material in the order given here, although other choices are certainly possible. For example, Chapters 2 and 3 may be interchanged if an individual would prefer to begin with continuous-time systems. Likewise, Chapters 5 and 6 may be interchanged, and Z-transforms may be introduced after Fourier and Laplace transforms, if desired. If time is a major factor, we recommend that Fourier analysis be emphasized at the expense of Z-transforms and Laplace transforms.

We thank sincerely the numerous instructors and students who have used the first edition and who have given us suggestions for improvements. In particular, we thank our colleagues, Llyod Griffiths and Tom Mullis, who have discussed the material and its presentation with us at length and have contributed many of the problems and examples.

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CHAPTER

1

LINEAR SYSTEMS

1.1 INTRODUCTION

The study of linear systems has been an essential part of formal undergraduate training for many years. Linear system analysis is useful because, even though physical systems are never completely linear, a linear model is often appropriate over certain ranges of input–output values. A large body of mathematical theory is available for engineers and scientists to use in the analysis of linear systems. In contrast, the analysis of nonlinear systems is essentially *ad hoc*. Each nonlinear system must be studied as one of a kind, since there are few general methods of analysis.

The analysis of linear systems is often facilitated by the use of a particular class of input signals. Thus, it is natural to include a study of signals and their various representations in our study of linear systems. We shall find sinusoidal and impulsive signals especially useful as system inputs.

As engineers, we are interested in the synthesis as well as the analysis of systems. In fact, it is the synthesis or design of systems that is the really creative portion of engineering. Yet, as in so many creative efforts, one must learn first how to analyze systems before one can proceed with system design. The work in this book is directed primarily toward the analysis of certain classes of linear systems. However, because design and analysis are so intimately connected, this material will also provide a basis for simple design.

We can divide the analysis of systems into three aspects:

1. The development of a suitable mathematical model for the physical problem of interest. This portion of the analysis is concerned with obtaining the "equations of motion," boundary or initial conditions, parameter values, and so on. This is the process of combining judgment, experience, and experiments to develop a suitable model. This first step is the hardest to develop formally.
2. After a suitable model is obtained, one then solves the resultant equations to obtain solutions in various forms.
3. One then relates or interprets the solution to the mathematical model in terms of the physical problem. Of course, meaningful interpretations and predictions concerning the physical system can be made only if the development in the first step has been accurate enough.

The primary emphasis of this work is on the second and third aspects just mentioned. The first step is essential but is accomplished more completely and appropriately within a particular discipline. Thus, chemical engineers will learn to write equations of motion for chemical processes, electrical engineers for electrical circuits, and so on. After a model is obtained, one can consider various techniques for its analysis and provide a basis for its mathematical interpretation.

Because linear models are so often used in all disciplines of engineering and science, this material is very useful. Perhaps the best way to point out this fact is to present examples from various physical problems. The only drawback to this method is that the reader may not always possess the necessary background to perform the first step in the analysis, that is, to write the equations of motion. This problem is to be expected. As one gains familiarity with a given discipline, this first step becomes natural. We shall use linear models based on electrical engineering applications for the most part. Certain problems at the ends of the chapters present physical examples from other disciplines.

We shall present several models for analyzing linear systems. Each model is useful in its own right, but together they present a more complete view of linear systems. By considering these different techniques, we hope to unify the reader's view of the subject.

1.2 CLASSIFICATION OF LINEAR SYSTEMS

A system is a mathematical model or abstraction of a physical process that relates inputs or external forces to the output or response of the system. Input and output share a cause–effect relationship. There are several classifications or types of systems.

A *causal* or *nonanticipatory* system produces an output that at any time t_0 is a function of only those input values that have occurred for times up to, and including, t_0. In other words, the system does not respond to input values until they have been actually applied to the system. Stated in this way, it appears that all real physical systems are causal. We shall show, however, that one can use noncausal systems in many applications.

The *state* of a system is a fundamental concept. The state is a minimal set of variables chosen such that if their values are known at time t_0 and all inputs are known for times greater than t_0, one can calculate the outputs of the system for times greater than t_0. The state of a system can be thought of as the system's memory. The memory at any time t_0 summarizes the effect of all past inputs and any initial state or memory.

The input, state, and output are, in general, sets of variables that we shall represent as vector quantities. For example, an n-variable input is written as

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} \tag{1.2.1}$$

We use $\mathbf{u}$, $\mathbf{y}$, and $\mathbf{x}$ to denote input, output, and state variables, respectively. Different vectors of the same class are distinguished by superscripts, for example, $\mathbf{u}^1$, $\mathbf{u}^2$, and so on.

We shall be concerned here with both continuous and discrete-time systems. Continuous-time systems are systems for which the input, output, and state are all functions of a continuous real variable t. We shall also study systems in which the time variable is defined only for discrete instants of time t_k, where k is an integer. In this case, the system is called a *discrete-time system.*

We shall denote a function of continuous time by $f(t)$ or, in cases where no confusion would result, merely by $\mathbf{f}$. Its value at t is $f(t)$. Similarly, a discrete-time function is denoted as $f(k)$ (or $\mathbf{f}$), and its value at $t = t_k = kT$ by any of the notations $f(t_k) \equiv f(kT) \equiv f(k) \equiv f_k$. Notice that $f(k)$ is itself a continuous variable—only its argument is discrete. Discrete-time functions are commonly called *time sequences* or merely *sequences.* Thus, an input sequence of n variables may be denoted as

$$\mathbf{u}(k) \equiv \mathbf{u} \equiv \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_n(k) \end{bmatrix} \tag{1.2.2}$$

Our models of physical systems are restricted to constant parameter systems. That is, we shall assume that parameters of the system do not change with time.

This leads to the idea of time-invariant and shift-invariant systems. A time-invariant, continuous-time system can be characterized as follows. If an input $u(t)$ gives rise to an output $y(t)$, then a shifted version of the input $u(t \pm \tau)$ gives rise to an output $y(t \pm \tau)$. Similarly, for a shift-invariant, discrete-time system, if an input $u(k)$ produces an output $y(k)$, then $u(k \pm n)$ produces an output $y(k \pm n)$. This is another way of saying the system response does not depend on the time origin but only on the form of the input.

We shall, in the sequel, consider linear, time- (or shift-) invariant systems. These systems are most often characterized by linear differential (or difference) equations with constant coefficients. These are our basic models for continuous and discrete-time systems, respectively.

1.3 LINEARITY

We have classified systems in several ways. One of the most important concepts in system theory is linearity. What precisely is a linear system? Linear systems possess the property of *superposition*. That is, if $\mathbf{u}^1$ produces an output $\mathbf{y}^1$ and $\mathbf{u}^2$ produces an output $\mathbf{y}^2$, then an input $(\mathbf{u}^1 + \mathbf{u}^2)$ produces the output $(\mathbf{y}^1 + \mathbf{y}^2)$. In symbols, if

$$\mathbf{u}^1 \rightarrow \mathbf{y}^1$$

and

$$\mathbf{u}^2 \rightarrow \mathbf{y}^2 \tag{1.3.1}$$

then

$$\mathbf{u}^1 + \mathbf{u}^2 \rightarrow \mathbf{y}^1 + \mathbf{y}^2$$

for some class of inputs $\mathbf{u}^j$, $j = 1, 2, \ldots$. Superposition also implies that if

$$\mathbf{u} \rightarrow \mathbf{y}$$

then

$$\alpha\mathbf{u} \rightarrow \alpha\mathbf{y}, \qquad \alpha \text{ a rational number} \tag{1.3.2}$$

This latter property is called *homogeneity* if it is true for all α.

A convenient notation for the arrows in (1.3.1) and (1.3.2) is to use functional notation and represent the system as a transformation T of inputs $\mathbf{u}$ into outputs $\mathbf{y}$. A system is linear if T satisfies

$$T(\alpha\mathbf{u}^1 + \beta\mathbf{u}^2) = \alpha T(\mathbf{u}^1) + \beta T(\mathbf{u}^2) \tag{1.3.3}$$

where α and β are arbitrary constants. Some examples of how to verify whether or not a system is linear follow.

■ **Example 1.3.1**

Suppose a system has an input–output relation given by the linear equation

$$\mathbf{y} = a\mathbf{u} + b, \qquad a, b \text{ constants} \tag{1.3.4}$$

Does this equation represent the input–output relation of a linear system? We can write (1.3.4) in the form

$$\mathbf{y} = T(\mathbf{u}) = a\mathbf{u} + b$$

Consider two inputs $\mathbf{u}^1$ and $\mathbf{u}^2$. The corresponding outputs are

$$T(\mathbf{u}^1) = a\mathbf{u}^1 + b$$
$$T(\mathbf{u}^2) = a\mathbf{u}^2 + b$$

Now apply an input $(\mathbf{u}^1 + \mathbf{u}^2)$. The output is

$$T(\mathbf{u}^1 + \mathbf{u}^2) = a(\mathbf{u}^1 + \mathbf{u}^2) + b$$

But notice that

$$\begin{aligned} T(\mathbf{u}^1) + T(\mathbf{u}^2) &= a\mathbf{u}^1 + b + a\mathbf{u}^2 + b \\ &= a(\mathbf{u}^1 + \mathbf{u}^2) + 2b \\ &\neq T(\mathbf{u}^1 + \mathbf{u}^2) \end{aligned}$$

Therefore, the system is not linear! The problem is that b is added to $a\mathbf{u}$. This offset at the origin destroys the superposition property. ■

■ **Example 1.3.2**

Consider the discrete-time system represented in Figure 1.3.1. The block diagram contains a unit-delay element, a multiplier of value $\frac{1}{2}$, and a summer. The delay element is a device that holds the previous value fed to it, in this case the previous value of the input. The equation for the output is

$$y(k) = \frac{1}{2}u(k) + u(k-1) \tag{1.3.5}$$

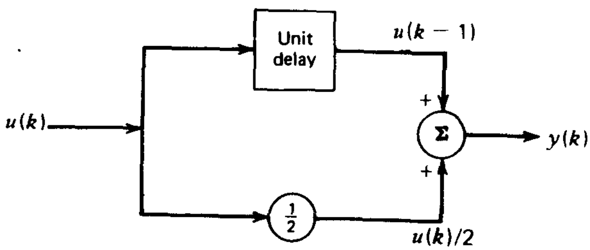

FIGURE 1.3.1 Block diagram for example 1.3.2.