Electromagnetic Fields

REVISED PRINTING

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Preface to the Revised Printing

Electromagnetic Fields has been out of print for fifteen years now. When the book first appeared in 1964, its main objective was to apply to electromagnetic problems such powerful mathematical tools as variational principles and Green's dyadics. Though widely used in mathematical physics, they were not particularly familiar to the electromagneticist of the sixties. Some of these methods have since made great progress in their numerical implementation, while new analytical procedures have come to the fore. The original version, however, is believed to contain enough fundamental data and reference material to warrant its present reappearance, which was made possible by the efforts of both Dr. Carl Baum and the Hemisphere Publishing Corporation. The author hopes that the book will remain useful as a text for graduate courses in physics or electrical engineering, and as a research tool for scientists in government, industrial and university laboratories. May this opinion be shared by the electromagnetics community!

Ghent, December 1984 *J. Van Bladel*

Preface

The present book deals with the calculation of electric and magnetic fields in the presence of ponderable bodies at rest. This is an old subject, but one which has enjoyed a considerable resurgence of interest in the last two decades, mainly because of the development of microwave devices. The radio engineer, faced with the necessity of understanding the operation of these devices, soon discovered that his traditional background was insufficient for the purpose. In fact, the solution of the exciting new theoretical problems which stemmed from the early development of radar was often effected by nuclear physicists, already well-trained in the necessary mathematical techniques. The use of these techniques is now a matter of routine in the radio technical literature. It is against this background that the present book has been prepared. Its writing has been guided by the following two thoughts:

1. The desirability of injecting lively illustrative examples from the very recent literature;
2. The decision to use eigenfunction expansions and other mathematical methods throughout the book, even in such timeworn subjects as electrostatics and magnetostatics.

The subject of this volume in all of its modern developments is a broad one, and the author has been compelled to prune severely, often at the expense of completeness. In the selection of topics, an attempt has been made to aim at an "intermediate" level; that is, elementary applications such as plane waves are treated very lightly, while some of the more advanced topics are not included. A few of these are Wiener-Hopf techniques, fields in periodic structures, multibody scattering, and applications of the reaction method. A number of important subjects, such as conformal mapping and geometrical optics, are given the briefest of treatments. Also largely ignored is the problem of obtaining macroscopic field quantities from microscopic fields.

A text on mathematical physics such as this considers mathematics as a tool and not as a goal. Thus, rigorous justification of a certain number of steps is omitted (particularly in potential theory), but not without inclusion of suitable references. Occasionally, and this is the case for Dirichlet's and Neumann's problems, an existence proof is given because it is suggestive of a useful numerical method of solution. Detailed calculations of complex integrals are generally avoided, because of their length, but here again references to the original papers are provided. Numerical results are frequently included to add flavor to a formal solution. This is done, in particular, for the integral equations of potential theory and for those of two-dimensional scattering.

Part of the material included in this volume has been used in a graduate course in electrical engineering, taught both at the University of Wisconsin and, during the academic year 1962–1963, at the Royal Institute of Technology, Stockholm. As more and more schools upgrade their undergraduate curricula, graduate courses in electromagnetic fields are permitted to increase their mathematical content. The author hopes that the present book will be of service during this period of evolution and that it will also find an audience as a companion volume to textbooks used in courses in theoretical physics and applied mathematics. It is also hoped that physicists and electrical engineers employed in industrial and government laboratories will find the text valuable as a tool in their research. For such use, the book has been made self-contained (to the extent of including a summary of complex-variable theory) and has been enriched by a fairly abundant collection of formulas, selected admittedly on the basis of the author's personal experience.

The author wishes to acknowledge the assistance given by Professors Thomas J. Higgins and John L. Asmuth, who read the manuscript in its initial stages. Thanks should also be expressed to Professor Calvin H. Wilcox, who checked the accuracy of the mathematical statements in some of the chapters. The final version of the book was prepared while the author was supported by the Guggenheim Foundation and by the Research Committee of the University of Wisconsin. The help of these organizations is gratefully acknowledged.

J. Van Bladel

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1

Linear analysis

1.1 Linear spaces

The linear equations of mathematical physics can be solved by methods which are independent of the branch of physics (electrostatics, hydrodynamics, acoustics, etc.) in which the equations are encountered. It is instructive, therefore, to describe these methods in very general and abstract terms. Such an approach is both intellectually satisfactory and economical, for it avoids tedious repetition of steps that are essentially the same for each new equation which is encountered. To illustrate the abstract concepts of linear analysis, reference will often be made to the equations of the flexible string, which constitute a simple and almost trivial example of a linear problem. The static displacement of a string subjected to a uniform longitudinal tension T and a vertical force density $g(x)$ satisfies the differential equation (see Fig. 1.1)

$$\frac{d^2y}{dx^2} = -\frac{g(x)}{T} \tag{1.1}$$

Two different types of boundary conditions will be considered. They correspond to

1. The *clamped* string, where the displacement $y(x)$ vanishes at both ends, $x = 0$ and $x = l$

2. The *sliding* string, which is free to slide vertically at both ends but is constrained to keep zero slope there.

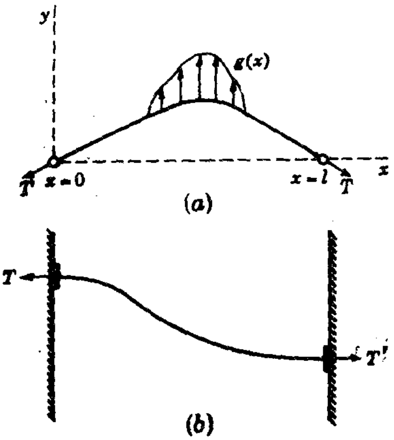

Figure 1.1 Clamped string and sliding string

Occasional reference will also be made to the equations of the lossless transmission line, in which voltage and current satisfy

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} + v_a \qquad \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \tag{1.2}$$

Here L and C denote the linear inductance and capacitance of the line, and v_a denotes the applied voltage per unit length (see Fig. 1.2). When phenomena are harmonic in time, $v(x,t)$ and $i(x,t)$ can be obtained from a knowledge of the phasors $V(x)$ and $I(x)$. Typically,

$$v(x,t) = \operatorname{Re}\,[V(x)\,e^{j\omega t}]$$

The phasor voltage satisfies the equation

$$\frac{d^2V}{dx^2} + \omega^2LCV = \frac{dV_a}{dx} \tag{1.3}$$

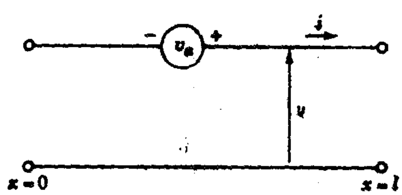

Figure 1.2 Open transmission line with series voltage source

which must be supplemented by the end conditions $I = 0$ and $dV/dx = 0$ for an open line, and $dI/dx = 0$ and $V = 0$ for a short-circuited line.

The field quantities which appear in a linear problem possess mathematical properties dictated by the physical nature of the phenomenon under investigation. The displacement of a string, for example, must be a continuous function of x. The electric field near a metallic edge must be square-integrable. In general, the nature of the problem requires the field quantities to belong to a *linear space* $\mathscr{S}$, that is, to a collection of elements f for which addition and multiplication by a scalar have been defined in such a manner that

1. Addition and multiplication are commutative and associative.
2. These operations create an element which is in $\mathscr{S}$.
3. The product of f and the scalar 1 reproduces f.
4. The space contains a unique null or zero element 0 such that $f + 0 = f$ and $f \cdot 0 = 0$.
5. To each f there corresponds a unique element $-f$ such that $f + (-f) = 0$.

The space of three-dimensional Euclidean vectors is obviously a linear space. Another is the space of complex-valued functions which are Lebesgue-measurable and square-integrable in a given domain. This space is denoted by the symbol $\mathscr{L}^2$ (Ref. 59).

In an abstract space $\mathscr{S}$, the *scalar product* $\langle f,g\rangle$ of two elements is a scalar-valued function which has the properties

$$\langle f,g\rangle = \langle g,f\rangle^*$$
$$\langle af_1 + bf_2, g\rangle = a^*\langle f_1,g\rangle + b^*\langle f_2,g\rangle \tag{1.4}$$
$$\langle f,f\rangle > 0 \quad \text{if} \quad f \neq 0$$

where an asterisk denotes complex conjugate. The integral $\int_0^l y_1 y_2 \, dx$, for example, is a suitable scalar product for the real displacements of a string; the complex integral $\int_0^l V_1^* V_2 \, dx$ is a suitable product for the complex voltages along a transmission line. Two elements are said to be *orthogonal* when their scalar product is zero.

The next step in establishing the structure of a linear space is to define the *norm* of an element. The explicit definition of the norm depends on the kind of space under consideration; a frequent choice is $\|f\| = \langle f,f\rangle^{\frac{1}{2}}$. The *distance* between two elements f and g is the norm $\|f - g\|$ of their difference. A sequence of elements f_n belonging to $\mathscr{S}$ converges in the Cauchy sense when the distance between any of its elements approaches zero as the index n approaches infinity. The space is *complete* when every Cauchy sequence converges to a limit which is also in $\mathscr{S}$.

Let us apply these concepts to the problem of the flexible string. A suitable norm for this problem is $\left(\int_0^l y^2 \, dx\right)^{\frac{1}{2}}$, and the condition for convergence of y_n to y is that $\left(\int_0^l (y - y_n)^2 \, dx\right)^{\frac{1}{2}}$ approaches zero for large values of n. With this type of norm the space of continuous real functions is not complete (because the limit function can be discontinuous), but the space $\mathscr{L}^2$ of Lebesgue-square-integrable functions is complete (Ref. 59). A scalar-product space which is infinite-dimensional and complete is termed a *Hilbert space*. This is the type of space which will be considered in this book.

Two additional definitions are worth noting here. A subset of elements of $\mathscr{S}$ forms a *linear manifold* when it contains all linear combinations of any two of its elements. It is, in addition, a *subspace* when it contains the limit of any converging sequence of its elements. For example, the displacements of a string which are symmetric with respect to the center of the string constitute a subspace. It is interesting to remark that an arbitrary displacement can always be decomposed into a symmetric part and an antisymmetric part. This decomposition is unique, and the two terms of the "splitting" are mutually orthogonal. This property represents a particular case of the more general *projection thereom*, which states that if $\mathscr{M}$ is a subspace of $\mathscr{S}$, and f is not in $\mathscr{M}$, there exists a splitting $f = v + w$ such that v (termed the *projection* of f on $\mathscr{M}$) belongs to $\mathscr{M}$, and w is orthogonal to all elements of $\mathscr{M}$.

1.2 Linear transformations

The basic problem associated with the clamped string is to determine the displacement $y(x)$ due to a given forcing function $g(x)$. We shall assume that $g(x)$ is piecewise-continuous. The string problem is a particular case of the more general problem

$$\mathscr{L}f = g \tag{1.5}$$

where $\mathscr{L}$ is an operator mapping the space of elements f (the domain) into the space of elements $\mathscr{L}f$ (the *range*). This mapping is a *transformation*. In the clamped-string problem, the domain consists of those functions which are continuous in $(0,l)$, vanish at $x = 0$ and $x = l$, and have piecewise-continuous second derivatives in $(0,l)$. A transformation is *linear* when it is additive and homogeneous, that is, when $\mathscr{L}(f_1 + f_2) = \mathscr{L}f_1 + \mathscr{L}f_2$ and $\mathscr{L}(af) = a\mathscr{L}f$. These properties imply (1) that the operator is linear and (2) that the domain is a linear manifold. The transformation associated with the clamped-string problem is obviously linear. The transformation associated with the inhomogeneous boundary conditions $y = 1$ at $x = 0$ and $y = 3$ at $x = l$ is *not* linear. The reason is clear: The sum of two possible displacements takes the values $y = 2$ at $x = 0$ and $y = 6$ at $x = l$. These are not the values required by the boundary condition; hence the latter does not define a linear manifold.

Consider again the linear problem defined by Eq. (1.5), together with the requirement that f belong to a linear manifold $\mathscr{F}$. The solution of this problem is greatly facilitated when a linear operator $\mathscr{M}$, a scalar product $\langle f,g\rangle$, and a domain $\mathscr{H}$ can be found such that

$$\langle \mathscr{L}f,h\rangle = \langle f,\mathscr{M}h\rangle \tag{1.6}$$

whenever h belongs to $\mathscr{H}$. The linear transformation defined by operator $\mathscr{M}$ and domain $\mathscr{H}$ is the *adjoint* of the original one. In the case of the clamped string, the left-hand member of Eq. (1.6) can be transformed by *integrating by parts.* One obtains

$$\langle \mathscr{L}f,h\rangle = \int_0^l \frac{d^2f}{dx^2}\,h\,dx = \int_0^l f\,\frac{d^2h}{dx^2}\,dx + \left[h\,\frac{df}{dx} - f\,\frac{dh}{dx}\right]_0^l \tag{1.7}$$

It is seen that Eq. (1.6) is satisfied if one chooses $\mathscr{M}$ to be the differential operator d^2/dx^2, and the domain $\mathscr{H}$ to consist of functions which are zero at $x = 0$ and $x = l$ [whereby the bracketed term in Eq. (1.7) vanishes] and which possess piecewise-continuous second derivatives. Clearly, the adjoint of the clamped-string transformation is the transformation itself, which is therefore termed *self-adjoint*. Self-adjoint transformations occur very frequently in mathematical physics. They have remarkable properties which will be examined in more detail in subsequent paragraphs.†

The pattern suggested by Eq. (1.7) is frequently encountered in mathematical physics. In general, the scalar product is an n-dimensional integral. The equivalent of Eq. (1.7) is obtained by using a suitable Green's theorem in n-dimensional space, in which the bracketed term is replaced by an $(n - 1)$-dimensional integral, linear in f and h, which is termed the *bilinear concomitant*. The domain $\mathscr{H}$ is determined by the requirement that the bilinear concomitant vanish. Note that scalar product $\langle \mathscr{L}f,h\rangle$ is also linear in f and h, which implies that $\langle \mathscr{L}f,f\rangle$ is a quadratic form in f. This can easily be checked for the clamped string, where

$$\langle \mathscr{L}f,f\rangle = \int_0^l \frac{d^2f}{dx^2}\,f\,dx = \left[f\,\frac{df}{dx}\right]_0^l - \int_0^l \left(\frac{df}{dx}\right)^2 dx = -\int_0^l \left(\frac{df}{dx}\right)^2 dx \tag{1.8}$$

In this case, the quadratic form is real. This property holds for all self-adjoint transformations in a Hilbert space. We note, indeed, that the properties of the scalar product in a Hilbert space imply that

$$\langle \mathscr{L}f,f\rangle = \langle f,\mathscr{L}f\rangle^*$$

On the other hand,

$$\langle f,\mathscr{L}f\rangle^* = \langle \mathscr{L}f,f\rangle^*$$

because of the self-adjoint character of the transformation. Comparison of these two equations shows that the quadratic form is equal to its conjugate, hence that it is real.

The quadratic form of the clamped string has the additional property, evident from Eq. (1.8), that it is negative or zero. The same is true for the quadratic form of the sliding string. The corresponding transformations are termed *nonpositive*. In the case of the string the vanishing of the quadratic form $\langle \mathscr{L}f,f\rangle$ implies that $df/dx = 0$ at all points of the interval $(0,l)$. This, in turn, requires $f(x)$ to be a constant. For the clamped string, this constant must be zero because of the end conditions. The corresponding transformation is termed *negative-definite*, which means that it is a nonpositive transformation whose $\langle \mathscr{L}f,f\rangle$ is always negative for nonzero elements

† Non-self-adjoint transformations are more difficult to treat. See, e.g., C. L. Dolph, Recent Developments in Some Nonself-adjoint Problems of Mathematical Physics, *Bull. Am. Math. Soc.*, **67**(1):1–69 (1961).

f and vanishes for, and only for, the zero element. The transformation associated with the sliding string is *not* definite, because $\langle \mathscr{L}f,f\rangle = 0$ is satisfied by $f = \text{const}$, a nonzero function which belongs to the domain of the transformation.

1.3 The inversion problem

A very fundamental problem concerning a linear transformation is *inversion*, that is, finding an element f of the domain such that $\mathscr{L}f = g$, g being given. This inverse transformation can be represented symbolically as $f = \mathscr{L}^{-1}g$. Two questions immediately arise:

1. Is there an inverse?
2. Is that inverse unique?

The question of uniqueness can be answered simply. Assume that there are two distinct solutions, f_1 and f_2. These solutions satisfy the equations

$$\mathscr{L}f_1 = g \qquad \mathscr{L}f_2 = g$$

Subtraction of corresponding members shows that the difference $f_0 = f_1 - f_2$ must be a solution of the homogeneous problem

$$\mathscr{L}f_0 = 0$$

If this problem does not possess a nonzero solution, f_1 and f_2 must be equal, and the solution of the original inhomogeneous problem is unique. If, on the other hand, the homogeneous problem has linearly independent solutions $f_{01}, f_{02}, \ldots, f_{0n}$, the solution of $\mathscr{L}f = g$ is determined to within an arbitrary linear combination of the f_{0i}'s.

To apply these notions to the clamped string, consider the homogeneous problem

$$\frac{d^2y_0}{dx^2} = 0 \qquad y_0 = 0 \text{ at } x = 0 \text{ and } x = l$$

In this simple example, two successive integrations are sufficient to show that the only solution is $y_0 = 0$. The same result can be obtained in an indirect manner, frequently used for nonpositive or nonnegative transformations. The method consists in evaluating $\langle \mathscr{L}f_0, f_0\rangle$. For the clamped string,

$$\langle \mathscr{L}f_0, f_0\rangle = \int_0^l \frac{d^2y_0}{dx^2}\, y_0\, dx = \left[y_0 \frac{dy_0}{dx}\right]_0^l - \int_0^l \left(\frac{dy_0}{dx}\right)^2 dx = -\int_0^l \left(\frac{dy_0}{dx}\right)^2 dx$$

Clearly, $\mathscr{L}f_0 = 0$ implies that $\langle \mathscr{L}f_0, f_0\rangle = 0$, which in turn requires the first derivative dy_0/dx to vanish. For a clamped string, this implies that y_0 is zero. The physical interpretation is obvious: The clamped string without forcing function remains stretched along the x axis. For the sliding string, on the contrary, the homogeneous problem has the nonzero solution $y_0 = \text{const}$. This means that the average height of the string is not defined or,