# Nonlinear Filtering and Smoothing

## An Introduction to Martingales, Stochastic Integrals and Estimation

Venkatarama Krishnan

# NONLINEAR FILTERING AND SMOOTHING:

## AN INTRODUCTION TO MARTINGALES, STOCHASTIC INTEGRALS AND ESTIMATION

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# PREFACE

This book is the outcome of a course on martingales and estimation theory being given since the fall of 1977 by the author at the Indian Institute of Science, Bangalore, to engineering graduate students with a basic knowledge of probability theory. The continued rapid advances in the martingale approach to filtering and smoothing problems made it necessary to give the engineering students a clear physical understanding of the fundamental concepts in this area. As a consequence, applicational aspects have been stressed throughout the book. Starting with the basic concepts of probability and stochastic processes in Chapters 1 and 2, martingales and square integrable martingales have been introduced in Chapters 3 and 4. Chapter 5 covers white noise and white noise integrals with an introduction to Fourier transforms and spectral measures. Chapters 6 and 7 deal with stochastic integrals and stochastic differential equations and the associated Ito calculus and extensions to the Ito calculus. Differences between white noise differential equations and the corresponding stochastic differential equations have been clearly brought out. After having defined the Stratonovich integral, the correction terms needed for computational purposes to convert the Ito stochastic differential equation to the Stratonovich form have been derived. Chapter 8 contains the derivation of optimal nonlinear filtering representation in a form slightly different from that of the classic work of Fujisaki, Kallianpur, and Kunita (15). At this stage it was felt necessary that some time ought to be spent on the classical Kalman filter (optimal linear Gaussian nonstationary filter), the heuristic derivation of which is contained in Chapter 9. In the same chapter the Kalman filter has been derived as a special case of the general nonlinear filtering representation. In Chapter 10 fault detection problems using the nonlinear filtering representation are considered, and Chapter 11 contains some of the results of the work on smoothing problems carried out by the author and his students during the early seventies.

This book is written by an engineer for engineers. As far as possible, the physical understanding of the problem has been stressed, and as a result rigorous mathematical proofs have in some cases given way to heuristic proofs. Rigorous proofs have also been given and in some of those cases they lead to a better physical understanding of the problem. In some other cases proofs have been referred to other textbooks. This book has been class tested for the past several years, and the generous feedback from colleagues and graduate students from two continents has helped the author to present it in this particular form.

During the preparation of this book the two-volume work by Lipster and Shiryaev (34) and the book by Kallianpur (29) have appeared on the market. This book follows the same martingale approach to filtering and smoothing problems as these other two books, but the presentation is kept at a lower level. A third book by Bremaud (5) on point processes has also appeared on the market. This book also carries some aspects of point processes, but again at a lower level of presentation. It has been the intent of the author to give a concise physical understanding of the principles of martingales, stochastic integrals, and estimation theory from an applicational point of view at a level where an engineering student with a basic probability theory background can comprehend. For more intensive studies, including mathematical rigor, the student can refer to the books mentioned above. Selected problems have also been included at the end of every chapter to enhance the utility of the book.

The references given at the end are by no means exhaustive, but only reflect the relevance they bear to the material in the book.

In writing this book the author has been greatly influenced by the now classic works of Wong, Kailath, Kallianpur, Segall, and Lipster and Shiryaev. He has freely drawn on their works and would like to express his scientific debt of gratitude to these authors.

This book could not have been written but for the direct and indirect support obtained from many sources in India and in the U.S.A. The author is thankful to the students who took the course and suggested many improvements. In particular, he would like to mention J. Viswanathan, C. E. Venimadhavan, K. R. Ramakrishnan, and H. S. Jamadagni for technical discussions on the material of the book. S. L. Yadav of the Tata Institute of Fundamental Research, Bangalore, gave him suggestions for improving the clarity of presentation of earlier chapters. He is thankful to Professor Joseph L. Hibey of the University of Notre Dame and Dr. Wolf Jachimowicz of WBC Extrusion Products, Lowell, for the long hours spent in formulating the fault detection problem and the painstaking discussions on the clarity of its presentation. He is indebted to Professor Harold J. Kushner and Professor Thomas Kailath for their critical review and excellent suggestions for improving the quality of presentation. He appreciates the support given to him by Herb Sandberg and Allen Dushman by extending him facilities at the Dynamics Research Corporation, Wilmington, Massachusetts, for finishing parts of the book. He also appreciates the facilities extended to him at the University of

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During the preparation of the final manuscript I faced extremely difficult times. During this difficult and trying period, my wife, Kamala, my daughters, Gayathri and Hemalekha, and my mother were a constant source of inspiration to me with their unwavering support, without which this book would never have seen the light of day. It is fitting that I dedicate this book to them. I am also grateful to Professor S. V. Rangaswamy of the Indian Institute of Science, who helped me over some of the difficulties.

Finally, I acknowledge with great pleasure the constant encouragement given to me by David Kaplan and the Wiley staff; their skills in transforming a rough manuscript into a finished book amazes me.

दृष्ट्वेदं मानुषं रूपं तव सौम्यं जनार्दन।
इदानीमस्मि संवृत्तः सचेताः प्रकृतिं गतः ॥

Bhagavad-Gita
Canto XI

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# CONTENTS

# 1 BASIC CONCEPTS OF PROBABILITY THEORY

## 1.1 INTRODUCTION

Probability theory is the mathematical study of phenomena occurring due to chance mechanism. If we toss a coin, we cannot say a priori whether we will get heads or tails. Outcomes of a random experiment can be analyzed or modeled only in an abstract manner. A random experiment or a mathematical experiment is one in which the possible outcomes may be finite or infinite. In the experiment of tossing a coin there are two outcomes, heads and tails. In the tossing of a die there are six outcomes. On the other hand, the weight of a full-term new-born baby may vary continuously from 4 to 10 pounds. Each of these outcomes is known as an *elementary outcome*. The collection of all elementary outcomes of a random experiment is called *sample space* and is denoted by $\Omega$. In set terminology the sample space is termed the *universal* set. Thus, the sample space $\Omega$ is a set consisting of mutually exclusive, collectively exhaustive listing of all possible outcomes of a random experiment. That is, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ denotes the set of all finite outcomes, $\Omega = \{\omega_1, \omega_2, \ldots\}$ denotes the set of all countably infinite outcomes, and $\Omega = \{0 \le t \le T\}$ denotes the set of uncountably infinite outcomes.

## 1.2 ALGEBRA OF SETS

Let $\Omega$ represent the sample space which is a collection of $\omega$-points as defined earlier. The various set operations are (1) *complementation*, (2) *union*, and (3) *intersection*. Let $A$ and $B$ be two subsets of the sample space $\Omega$, denoted by

$A \subset \Omega$, $B \subset \Omega$. The complement of $A$, denoted by $A^c$, represents the set of all $\omega$-points not contained in $A$:

$$A^c = \{\omega: \quad \omega \notin A\} \tag{1.2.1}$$

Evidently the complement of $\Omega$ is the empty set $\varnothing$. Two sets $A$ and $B$ are equal if and only if $A$ is contained in $B$ and $B$ is contained in $A$:

$$A = B \Leftrightarrow A \subset B \quad \text{and} \quad B \subset A \tag{1.2.2}$$

The union of sets $A$ and $B$, denoted by $A \cup B$ or $A + B$, represents the occurrence of $\omega$-points in either $A$ or $B$. Similarly, the intersection of sets $A$ and $B$, denoted by $A \cap B$ or $AB$, represents the occurrence of $\omega$-points in $A$ and $B$. Clearly, if there is no commonality of $\omega$-points in $A$ and $B$, then $A \cap B$ is the empty set $\varnothing$.

$$\begin{aligned} A \cup B &= \{\omega: \quad \omega \in A \quad \text{or} \quad \omega \in B\} \\ A \cap B &= \{\omega: \quad \omega \in A \quad \text{and} \quad \omega \in B\} \end{aligned} \tag{1.2.3}$$

**Example 1.2.1**

Let $\Omega$ be the $\omega$-points on the real line $R$.

$$\Omega = \{\omega: \quad -\infty < \omega < \infty\}$$

Define

$$\begin{aligned} A &= \{\omega: \quad \omega \in (-\infty, a)\} = \{\omega < a\} \\ B &= \{\omega: \quad \omega \in (b, c)\} = \{b < \omega < c\} \end{aligned}$$

Then the set operations yield

$$\begin{aligned} A^c &= \{a \le \omega < \infty\} \\ B^c &= \{-\infty < \omega \le b\} \cup \{c \le \omega < \infty\} \end{aligned}$$

$$A \cup B = \begin{cases} \{\omega < a\} & c < a \\ \{\omega < c\} & b < a < c \\ \{\omega < a\} \cup \{b < \omega < c\} & a < b \end{cases}$$

$$A \cap B = \begin{cases} \{b < \omega < c\} & c < a \\ \{b < \omega < a\} & b < a < c \\ \varnothing & a < b \end{cases}$$

The unions and intersections of an arbitrary collection of sets are defined by

$$\begin{aligned} \bigcup_{n \in N} A_n &= \{\omega: \quad \omega \in A_n \text{ for some } n \in N\} \\ \bigcap_{n \in N} A_n &= \{\omega: \quad \omega \in A_n \text{ for all } n \in N\} \end{aligned} \tag{1.2.4}$$

where $N$ is an arbitrary index set which may be finite or countably infinite.

The unions and intersections follow the reflexive, commutative, associative, and distributive laws.

The complements $(\bigcup_{n\in N} A_n)^c$ and $(\bigcap_{n\in N} A_n)^c$ are given by de Morgan's laws as follows:

$$\left(\bigcup_{n\in N} A_n\right)^c = \{\omega: \ \omega \text{ does not belong to any } A_n, n \in N\}$$
$$= \{\omega: \ \omega \notin A_n \text{ for all } n \in N\}$$
$$= \bigcap_{n\in N} A_n^c \tag{1.2.5}$$

$$\left(\bigcap_{n\in N} A_n\right)^c = \{\omega: \ \omega \text{ does not belong to each and every } A_n, n \in N\}$$
$$= \{\omega: \ \omega \text{ does not belong to some } A_n, n \in N\}$$
$$= \bigcup_{n\in N} A_n^c \tag{1.2.6}$$

### Sequences

A sequence of sets $A_n$, $n \in N$, is *increasing* if $A_{n+1} \supset A_n$ and *decreasing* if $A_{n+1} \subset A_n$ for every $n \in N$.

A sequence which is either increasing or decreasing is called a *monotone* sequence. We can write the limits ($N$ countably infinite) of monotone sequences as

$$\begin{aligned} \lim_{n\to\infty} A_n &= \lim_n A_n = \bigcup_{n=1}^{\infty} A_n = A \quad \text{increasing} \\ \lim_{n\to\infty} A_n &= \lim_n A_n = \bigcap_{n=1}^{\infty} A_n = A \quad \text{decreasing} \end{aligned} \tag{1.2.7}$$

The limit of monotone sequences $\{A_n\}$ is written as $A_n \uparrow A$ when it is increasing and $A_n \downarrow A$ when it is decreasing.

### Example 1.2.2

Let $\Omega$ be the real line $R$. If $A_n = \{\omega: \ 0 < \omega < a - 1/n\}$, then $A_n \uparrow A = \{\omega: \ 0 < \omega < a\}$. On the other hand, if $B_n = \{\omega: \ 0 < \omega < a + 1/n\}$, then $B_n \downarrow B = \{\omega: \ 0 < \omega \leq a\}$.

We can define a *superior limit* and an *inferior limit* for any sequence $\{A_n\}$ not necessarily monotone. We first define sequences $\{B_n\}$ and $\{C_n\}$ derived from $\{A_n\}$ as follows:

$$B_n = \sup_{k \geq n} A_k = \bigcup_{k=n}^{\infty} A_k$$

$$= \{\omega: \quad \omega \text{ belongs to at least one of } A_n, A_{n+1}, \ldots\} \qquad (1.2.8)$$

$$C_n = \inf_{k \geq n} A_k = \bigcap_{k=n}^{\infty} A_k$$

$$= \{\omega: \quad \omega \text{ belongs to all } A_k \text{ except } A_1, A_2, \ldots, A_{n-1}\} \qquad (1.2.9)$$

Clearly the sequences $\{B_n\}$ and $\{C_n\}$ are monotone and decreasing and increasing, respectively. We can now define a limit from eq. 1.2.7 for these monotone sequences:

$$B = \lim_{n \to \infty} B_n = \lim_n B_n = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$$= \lim_{n \to \infty} \sup A_n = \lim \sup_n A_n$$

$$= \{\omega: \quad \omega \text{ belongs to infinitely many } A_n\}$$

$$C = \lim_{n \to \infty} C_n = \lim_n C_n = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} A_k$$

$$= \lim_{n \to \infty} \inf A_n = \lim \inf_n A_n$$

$$= \{\omega: \quad \omega \text{ belongs to all but a finite number of } A_n\}$$

Hence

$$\lim \sup_n A_n \supset \lim \inf_n A_n$$

If $\lim \sup_n A_n = \lim \inf_n A_n$, then $\{A_n\}$ is a convergent sequence and $\lim_n A_n = A$, say, exists, that is,

$$\lim \sup_n A_n = \lim \inf_n A_n = \lim_n A_n = A \qquad (1.2.10)$$

**Example 1.2.3**

Let $A_k$ be the set of points $(x, y)$ of the Cartesian plane $R^2$ in the region $\{0 \leq x < k, 0 \leq y < 1/k\}$, that is,

$$A_k = \left\{x, y \in R^2: \quad 0 \leq x < k, 0 \leq y < \frac{1}{k}\right\}$$

Here $\{A_k\}$ does not belong to the monotone class.

But

$$B_n = \bigcup_{k=n}^{\infty} A_k = \left\{x, y \in R^2: \quad 0 \le x < \infty, 0 \le y < \frac{1}{n}\right\}$$

is a decreasing sequence, and hence

$$B = \lim_n B_n = \bigcap_{n=1}^{\infty} B_n = \{x, y \in R^2: \quad 0 \le x < \infty, y = 0\} = \limsup_n A_n$$

Similarly,

$$C_n = \bigcap_{k=n}^{\infty} A_k = \{x, y \in R^2: \quad 0 \le x < n, y = 0\}$$

is an increasing sequence, and hence

$$C = \lim_n C_n = \bigcup_{n=1}^{\infty} C_n = \{x, y \in R^2: \quad 0 \le x < \infty, y = 0\} = \liminf_n A_n$$

Since $\limsup_n A_n = \liminf_n B_n = \{x, y \in R^2: \quad 0 \le x < \infty, \; y = 0\}$, we have $\lim_n A_n = B = C = \{x, y \in R^2: \quad 0 \le x < \infty, y = 0\}$.

**Example 1.2.4**

Let $\Omega$ be the positive real line $R^+$. Consider the sequence

$$A_n = \left\{\omega \in \Omega: \quad 0 < \omega < a + \frac{(-1)^n}{n}\right\}$$

Here

$$B_n = \begin{cases} \left\{\omega \in \Omega: \quad 0 < \omega < a + \dfrac{1}{n}, n \text{ even}\right\} \\ \left\{\omega \in \Omega: \quad 0 < \infty < a + \dfrac{1}{n+1}, n \text{ odd}\right\} \end{cases}$$

$$\limsup_n A_n = \{\omega \in \Omega: \quad 0 < \omega \le a\}$$

Similarly

$$C_n = \begin{cases} \left\{\omega \in \Omega: \quad 0 < \omega < a - \dfrac{1}{n}, n \text{ odd}\right\} \\ \left\{\omega \in \Omega: \quad 0 < \omega < a - \dfrac{1}{n+1}, n \text{ even}\right\} \end{cases}$$

$$\liminf_n A_n = \{\omega \in \Omega: \quad 0 < \omega < a\}$$

Since $\limsup_n A_n \neq \liminf_n A_n$, the sequence $\{A_n\}$ does not converge and has no limit.

Even though the limit of a sequence may not exist, superior and inferior limits will always exist, as is evident from the definitions.

## 1.3 FIELDS, σ-FIELDS, AND EVENTS

We define $\mathcal{Q}$ as the nonempty class of subsets drawn from the sample space $\Omega$. We say that the class $\mathcal{Q}$ is a *field* or an *algebra* of sets in $\Omega$ if it satisfies the following definition.

### Definition 1.3.1 Field or Algebra

A class of a collection of subsets $A_j \subset \Omega$ denoted by $\mathcal{Q}$ is a field when the following conditions are satisfied:

1. If $A_i \in \mathcal{Q}$, then $A_i^c \in \mathcal{Q}$.
2. If $\{A_i = i = 1, 2, \ldots, n\} \in \mathcal{Q}$, then $\bigcup_{i=1}^n A_i \in \mathcal{Q}$. (1.3.1)

Given the above two conditions, de Morgan's law ensures that finite intersections also belong to the field. Thus a class of subsets is a field if and only if it is closed under all finite set operations like unions, intersections, and complementations. Since every Boolean algebra of sets is isomorphic to an algebra of subsets of $\Omega$, we can also call the field a *Boolean field* or *Boolean algebra*. Every field contains as elements the sample space $\Omega$ and the empty set $\varnothing$.

### Example 1.3.1

Let $\Omega = R$ and consider a class $\mathcal{Q}$ of all intervals of the form $(a, b]$, that is, $\{x \in R:\ a < x \le b\}$:

$$
\begin{aligned}
(a, b] \cap (c, d] &= \varnothing & a < b < c < d \\
&= (c, b] & a < c < b < d \\
&= (a, d] & c < a < d < b \\
&= (c, d] & a < c < d < b \\
&= (a, b] & c < a < b < d
\end{aligned}
$$

Clearly the class $\mathcal{A}$ is closed under intersections. However,

$$(a, b]^c = (-\infty, a] \cup (b, \infty) \notin \mathcal{A}$$

$$(a, b] \cup (c, d] \notin \mathcal{A} \qquad \text{if } a < b < c < d$$

The class $\mathcal{A}$ is not a field.

**Example 1.3.2**

The smallest field containing $A \subset \Omega$ is

$$\mathcal{A} = \{\Omega, \varnothing, A, A^c\}$$

If a class of subsets is closed under finite set operations, it does not necessarily mean that it is also closed under countably infinite set operations. Very often we come across the sequence of sets $\{A_n^c\}$ as $n \to \infty$ and the convergence of such sequences ($\limsup_n A_n, \liminf_n A_n$). In Example 1.3.1 the class $\mathcal{A}$ is closed under finite intersections. If we now take countably infinite intersections, $\bigcap_{n=1}^{\infty}(a - 1/n, b] = [a, b] \notin \mathcal{A}$. If a class of sets $\mathcal{F}$ drawn from the sample space $\Omega$ is closed under all countably infinite set operations, then that class $\mathcal{F}$ is called a *σ-field* or *σ-algebra*.

**Definition 1.3.2 σ-Field or σ-Algebra**

A class of a countably infinite collection of subsets $A_j \subset \Omega$ denoted by $\mathcal{F}$ is a σ-field when the following conditions are satisfied:

1. If $A_i \in \mathcal{F}$, then $A_i^c \in \mathcal{F}$.
2. If $\{A_i, i = 1, 2, \ldots\} \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$. (1.3.2)

In general a σ-field is a field, but a field may not be a σ-field.

**Example 1.3.3**

Let $\Omega = R$ and $\mathcal{A}$ be the class of all intervals of the form $(-\infty, a]$, $(b, c]$, and $(d, \infty)$:

$$(b, c]^c = (-\infty, b] \cup (c, \infty) \in \mathcal{A}$$

$$(d, \infty)^c = (-\infty, d] \in \mathcal{A}$$

$$(-\infty, a]^c = (a, \infty) \in \mathcal{A}$$

From Example 1.3.1 the class $\mathcal{A}$ is closed under finite intersections. Similarly it can also be shown that $\mathcal{A}$ is closed under finite unions. Hence the class $\mathcal{A}$ is a field. However, for infinite intersections of the form

$$\bigcap_{n=1}^{\infty}\left(b-\frac{1}{n}, c\right) = [b, c) \notin \mathcal{A}$$

the class $\mathcal{A}$ is not a $\sigma$-field.

**Proposition 1.3.1 Intersection of $\sigma$-Fields**

The *intersection* of any nonempty but arbitrary collection of $\sigma$-fields in $\Omega$ is a $\sigma$-field in $\Omega$.

In general the arbitrary *union* of a collection of $\sigma$-fields may not be a $\sigma$-field. Many of the examples given above illustrate that an arbitrary class $\mathcal{A}$ of subsets of $\Omega$ may or may not be a $\sigma$-field. However, we can always construct the smallest $\sigma$-field over $\mathcal{A}$ which will contain $\mathcal{A}$ and will be denoted by $\sigma(\mathcal{A}) = \mathcal{F}$. This will always exist since $\sigma(\mathcal{A})$ can be defined as the intersection of all $\sigma$-fields containing $\mathcal{A}$. If $\sigma_1(\mathcal{A}), \sigma_2(\mathcal{A}), \ldots$ are all $\sigma$-fields containing $\mathcal{A}$, then

$$\sigma(\mathcal{A}) = \bigcap_{i=1}^{\infty} \sigma_i(\mathcal{A})$$

Further the minimal $\sigma$-field thus generated is unique. We shall call $\sigma(\mathcal{A})$ the $\sigma$-field generated by $\mathcal{A}$.

**Example 1.3.4**

Let the sample space $\Omega$ contain $\omega$-points of the toss of a die. $\Omega$ is the set $\{1, 2, 3, 4, 5, 6\}$. We shall now define a class of sets

$$\mathcal{A} = \{\varnothing, \Omega, \{1,3,5\}, \{2,4,6\}, \{2,4\}\}$$

Clearly $\mathcal{A}$ is not a field since $\{1,3,5\} \cup \{2,4\} = \{1,2,3,4,5\}$ is not in $\mathcal{A}$. However, we can generate the field containing $\mathcal{A}$ by

$$\sigma(\mathcal{A}) = \mathcal{F} = \{\mathcal{A}, \{1,3,5,6\}, \{6\}, \{1,2,3,4,5\}\}$$

which is indeed a $\sigma$-field, and we can show that it is the minimal $\sigma$-field generated by $\mathcal{A}$.

So far we have not considered the nature of the sample space $\Omega$, except that it is nonempty. A set $A$ and a class of subsets of $A$ called *open sets* of $A$, such that this class contains $\varnothing$ and $A$, and closed under finite intersections and arbitrary unions, is called a *topological space*. If $\Omega$ is a topological space, an