The Theory of Linear Economic Models

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Preface

This book is written at a time of revived activity in the field of applied mathematics. "Revived" is perhaps the wrong word to use in this connection, for the characteristic feature of the new applied mathematics is not an intensification of work on old problems but rather an attempt to extend the application of mathematical reasoning to entirely new kinds of situations. Information theory, cybernetics, game theory, theory of automata are but a few of the new disciplines. Naturally, much of the work in these subjects is of a tentative and experimental nature. On the other hand, there have been certain developments which after a decade's experience seem to be of permanent usefulness. One such is probably information theory. Another is linear programming and the related linear models. Being convinced that this latter subject is "here to stay," I felt it was appropriate to try preparing a suitable text. This book is the result.

Before asking the reader to plunge into the subject of linear models I shall, in accordance with a sensible custom, attempt in the few pages which follow to give some idea of what this subject is. An ideal preface is one which tells the reader in a few words exactly what the rest of the book contains and thus saves him the trouble of reading it. I regret that the writing of such a preface in the present case is beyond my powers of exposition. The best I can do is to describe in a general way the sort of problems we shall be concerned with, the approach we shall take to these problems, and the manner in which the relevant material will be organized.

The Subject Matter. The term "economic model" is admittedly a vague one, but for our purposes we may think of such a model as an abstraction and simplification of some typical economic situation. As an example, the first model we shall take up is that of linear programming, which in its abstract formulation is a certain kind of mathematical maximum or minimum problem. The importance of this model derives from the fact that many actual economic situations lead to precisely this problem after the appropriate simplifying assumptions have been made. Later we take up the two-person game model. This is again formulated in a purely abstract manner, but the significance of the model for us comes from the fact that it is designed to reflect the essential features of certain games of strategy, and thus indirectly certain aspects of economic competition. Other models to be treated concern patterns of exchange between countries or industries, alternative schemes of production, certain economic equilibrium situations, and so on. In each case the models will be introduced by first describing the economic situation, next stating what simplifications are to be made, and then giving the purely abstract formulation.

Having arrived at this abstractly formulated model, what do we intend doing with it? By way of answer let us first state clearly some of the things we *do not* intend doing. A very important question in relation to any model is that of applicability. Does the model really give a reasonable approximation to the situation which gave rise to it? Is it to be relied on in making decisions and predictions? To what extent have predictions based on the model been borne out experimentally? Such questions belong to pure economics and will not be touched on here. Indeed, the models we have chosen to discuss vary widely as regards applicability. At one extreme we have linear programming, which is already being used quite extensively in industrial planning. At the other we have topics like game theory and some of the equilibrium models, which are in no sense ready for practical application in their present stage of development.

But if applicability is not the criterion for selection how then have we decided which topics to discuss? The answer is this: We have tried to select those models which best illustrate the manner in which mathematical reasoning can be used to obtain information about idealized economic situations. In some instances we have had to make rather drastic simplifications. The resulting lack of realism is unfortunate but is to be expected in early attempts at understanding complex situations.

Having formulated our models, the rest of the task consists in analyzing them, that is, of deducing in a rigorous fashion the consequences of the assumptions which have been made. The procedure is quite analogous to deducing theorems from the axioms of, say, plane geometry. As in the case of geometry, some of the results we shall obtain could hardly have been guessed in advance. It is this fact which encourages one to believe that mathematical analysis may help to bring about new and significant advances in the understanding of economic phenomena.

We have restricted our presentation to the study of *linear* models, that is, roughly speaking, models in which the mathematical relations have the form of equations or inequalities of degree one. This restriction is due simply to limitations of space and time. An equal number of pages could have been devoted to nonlinear models. This would, however, have involved developing a great deal of additional mathematical machinery, and for this reason we chose to remain within the linear framework. A further justification for this decision was the fact that most of the nonlinear results make use of the linear theory. Much of this book may thus be regarded as foundation material for work on more advanced levels.

It might be thought from what has been said so far that we have gathered together a miscellaneous collection of problems whose only common features are an economic flavor and the occurrence of linear relations. Fortunately, this is not the case, for although there is considerable variety in the models to be studied, the mathematics involved will exhibit a noteworthy degree of unity. Most of our analysis will use the mathematical material developed in Chap. 2 on Real Linear Algebra or, in more everyday language, the theory of linear equations and inequalities in real numbers. The feature of this theory which plays the unifying role in most of the applications is the fundamental notion of *duality*. We shall not even attempt to define this term here but remark that it is the recurrent theme which ties together the various parts of the book into what may legitimately be called a theory.

The Approach. We have already remarked that this book is intended as a text. We hesitate to use the words "advanced text," for this suggests that preliminary familiarity with the subject matter is assumed, which is not the case. The book is advanced in the sense that it attempts to bring the reader to the frontiers of the subject, enabling him to understand and possibly contribute to current research in the field. In other words, we are trying primarily to fill the needs of the would-be specialist, be he mathematician, economist, business student, or engineer. But while our main objective is the training of experts, we have tried to arrange matters so that the book will also be useful to readers who wish to go into the subject less intensively. The less technical parts of the book, in particular Chapter 1 on linear programming and most of Chapter 6 on game theory, are designed to be usable in courses on these subjects on the level of an advanced undergraduate course in economics or engineering.

Concerning the use of the book as a basic text for a course, it should be explained that the book is itself based on a set of notes from a course given to a group of graduate students in pure and applied mathematics, and the treatment should be suitable for students at this level. We suspect the average graduate student in economics would have some difficulty in going through the book on his own, for we emphasize that this is a text not in economics but in applied mathematics. Nevertheless, the theorems we prove are about economics, are used by economists, and in many cases were first discovered by economists.

Concerning the use of this book by economists, a further word of caution is in order. It has been brought to my attention by Professor Dorfman that certain words and expressions mean quite different things to economists on the one hand and mathematicians on the other. It was both startling and illuminating to me to realize that the very first words of my title "The *Theory* of" belong to this category. By way of illustration, a mathematician or natural scientist on reading one of the important *theory of* books of economics, say Hicks or Keynes, might well remark "very interesting, but where is the theory?" The remark would imply no disparagement of these works but would simply point up a confusion of language, for the natural scientist expects a theory to consist of a large body of results derived from a small set of assumptions. What he has read consists instead of a careful formulation and detailed justification of a particular set of assumptions, with rather less formal deduction of implications than he would find in a theoretical treatise in the natural sciences. Analogously, an economist reading the present volume will undoubtedly feel that it has been misnamed in that most of the "theory" has been left out, and he will correctly point out that the book is teeming with economic assumptions for which little or no justification is given. We reply that the word "theory" is to be understood here as it is used in the natural rather than the behavioral sciences and is therefore not directly concerned with the justification of assumptions. We stress this point in order not to mislead the reader concerning our intentions.

It is our hope that our presentation of results will be useful to the economics student with exceptional aptitude for the mathematical approach. It should also be useful in the hands of a teacher of mathematical economics who can modify the exposition to suit the needs of his students, skimming over portions which present purely technical difficulties, elaborating on other parts in which our treatment has not been sufficiently detailed. As such this book might usefully supplement one of the texts in economics which covers the same material, such as "Linear Programming and Economic Analysis" by Dorfman, Samuelson, and Solow or "Mathematical Economics" by R. G. D. Allen.

Finally, we hope the book will be used as a reference for workers in the field of linear models who will find here a mathematically unified treatment of many important results which were previously available only in scattered sources in the economic and mathematical literature.

We come next to the question of mathematical prerequisites. It is customary to remark at this point that the only requirement for an understanding of what is to follow is a knowledge of elementary calculus. In the present case even this requirement may be waived, for calculus is never used. Our principal tool is matrix algebra, but no previous knowledge is required here either, as all necessary facts are developed in the text. What is required is the ability to follow a moderately involved mathematical argument, an ability which generally comes only with a fair amount of experience and is often characterized by the illusive phrase "mathematical maturity." Some of the proofs we shall present are quite difficult. Even the proof of the "theorem of the separating hyperplane," which is the key mathematical result of the book, is not entirely straightforward. There is no way around this difficulty, for most of the results we wish to present are not mathematical trivialities, and one cannot make things easy without omitting proofs altogether, which would defeat our main purpose. We shall, of course, use all the available devices to help the reader's understanding such as geometric pictures, plausibility arguments, and numerical examples.

We may summarize what has been said in the foregoing paragraphs by remarking that a course based on this book would occupy a position somewhat analogous to a course in mathematical statistics. Such courses are generally given in a mathematics department but are available to students in other fields with the necessary mathematical qualifications.

The Organization. How to Use the Book. We envision four possible courses which could be based on this book.

1. A full-year course covering the entire nine chapters. It would not be necessary to take them up in order, as will be seen from the diagram on page ix.

2. A one-semester course on linear programming. This would cover the first five chapters of the book.

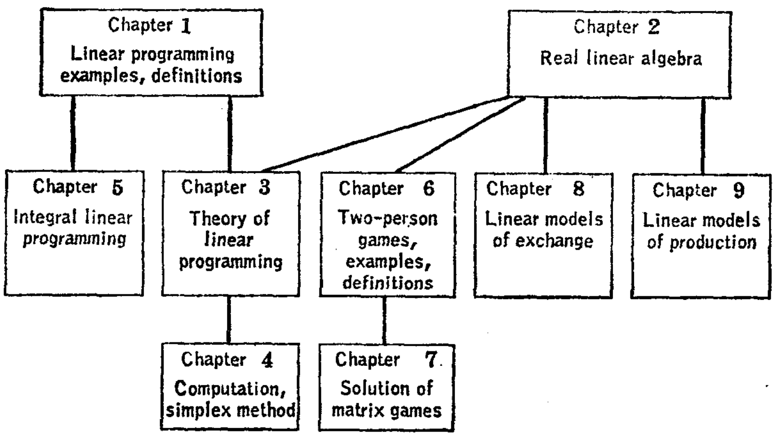
3. A one-semester course in linear programming and game theory. This would consist of Chaps. 1, 2, 3, 6, and 7, omitting Sec. 2 of Chap. 7.

4. A one-semester course in linear economic models. This would cover Chaps. 1, 2, 3, 8, and 9.

The schematic diagram on page ix shows how the various chapters depend on each other.

As the figure shows, Chap. 2 on Real Linear Algebra is necessary for all later chapters. However, the second half of the chapter, from Sec. 5 on, is used only occasionally in subsequent chapters. The instructor may wish, therefore, to take up only the first four sections of this chapter, which are sufficient for all the applications in Chaps. 3, 4, 6, and 8.

From a logical point of view it would have been most natural to begin with Chap. 2, in which the mathematical machinery is developed. This procedure would have the disadvantage, however, of requiring the reader to absorb a considerable amount of abstract material without knowing what it was to be used for. For this reason it seemed preferable to start with the applications, in this case linear programming, and state the main theoretical results without proof in order to motivate further study in the algebraic foundations. Chapter 1 is therefore devoted to describing the linear programming problem first by means of a set of illustrative examples, then by a formal definition. The discussion of the next section leads up to the statement



ment (but not the proof) of the fundamental duality theorem, which is then illustrated in specific cases. Assuming the duality theorem we then prove the important "equilibrium theorem" and give applications. The chapter, like all the others, ends in a short set of bibliographical references and a somewhat longer set of exercises of varying degrees of difficulty.

The first sections of Chap. 2 are devoted to introducing vectors and matrices and developing the classical theory of linear equations in a rapid but complete and self-contained manner. The mathematical heart of the chapter, and, in fact, of the book, is in Secs. 3

and 4, in which we develop the not so classical theory of real linear equations and linear inequalities. The latter half of the chapter is devoted to a more detailed and somewhat geometric analysis of the solutions of inequalities.

The reader who is acquainted with linear algebra may be struck by the fact that certain popular topics in this subject are conspicuous by their absence, among them the theory of determinants and of characteristic roots or eigen-values. The reason for this omission is simply that we know of no cases in which these particular algebraic objects are useful in drawing conclusions about economic models, and therefore there is no reason why the reader should spend time trying to master these somewhat intricate topics.[1]

In Chap. 3 we return to linear programming problems, which are now defined in complete generality. Using the algebraic apparatus developed in Chap. 2 it is possible to give a complete treatment of the duality and equilibrium theorems as well as the important result on basic solutions. The last part of the chapter is concerned with a most important economic application of linear programming theory, namely, the solution of the problem of optimal resource allocation by the method of price equilibrium under free competition.

Chapter 4 is devoted primarily to an exposition of the simplex method of Dantzig and its application not only to linear programming but also to such general problems as solving systems of inequalities and finding nonnegative solutions of linear equations. Our approach has been to show that the simplex method may be looked upon as an extension of the ordinary "high-school method of elimination" for solving sets of simultaneous linear equations. In vector language the

[1] In view of the rather frequent occurrence in the economic literature of results involving determinants and eigen-values, this statement perhaps calls for some amplification. An example will perhaps illustrate the point. It is a true theorem that a Leontief model is capable of producing a positive bill of goods if and only if the principal minors of the production matrix are all positive. This fact, however, gives us no new economic insight into the properties of Leontief models because there is no economic interpretation to be attached to these principal minors. Contrast this result with the theorem that if a Leontief model can produce one positive bill of goods it can produce any positive bill of goods. The latter statement is a useful and interesting result about the model itself, since both the hypothesis and the conclusion have an obvious economic meaning.

elimination of a variable becomes the replacement of a vector in a basis, and it is this *"replacement operation"* which becomes the basic computational unit in our presentation. The final section of the chapter presents the generalized simplex method of Dantzig, Orden, and Wolfe for resolving the problem of degeneracy.

Chapter 5 is devoted to the very important class of linear programs, including transportation problems, which always have integral solutions if the initial data are integral. As indicated by our schematic diagram, the material of this chapter is essentially independent of the previous theory. We begin by presenting the network-flow theory of Ford and Fulkerson which, together with the method of Kuhn for the optimal-assignment problem, provides us with a complete and elegant theory for a wide class of integral problems. The relationship of this theory to the classical notion of price equilibrium is given in Sec. 6. The Hitchcock transportation problem is treated in detail as well as various other applications. Again in this chapter it is the duality concept which does the work.

In Chap. 6 we introduce two-person zero-sum games by a sequence of examples which lead first to the statement and then the proof of von Neumann's minimax theorem. The proof is that of Gale, Kuhn, and Tucker using the symmetrization of a game of von Neumann.

The "equivalence" of linear programming and matrix games is the first topic of Chap. 7, and it is shown that the minimax theorem can be derived as a special case of the fundamental duality theorem of linear programming. A short section is devoted to solving games by the simplex method. Several sections are then devoted to a detailed analysis of the structure of the sets of optimal strategies of a matrix game. The final sections are devoted to a description of the method of fictitious play of Brown and to Robinson's proof that the method converges.

Chapter 8 is concerned first with a linear exchange model, equivalent versions of which seem to have been discovered independently by Frisch, Remak, and Bray. A complete analysis is given of the equilibria of such models. A dynamic theory of linear trade models is then treated along the lines of some work of Solow. The theorems here are exactly the same as those which occur in the theory of Markov chains in probability theory. The final sections of the chapter treat a particular model of price equilibrium.

Among the topics treated in the final chapter are Leontief models, including the Samuelson-Koopmans-Arrow substitutability theorem, the work of Koopmans on the relation between efficiency and profit maximization, and von Neumann's expanding linear model.

Terminology, Notation, Bibliography. We shall, of course, define all technical terms and symbols as they are introduced. For the most part we have adhered to standard terminology and notations when such things existed. On the other hand, we have exercised the mathematical equivalent of poetic license to institute an occasional "improvement," mostly in the interests of typographical simplicity. Thus the scalar product of two vectors is simply indicated by their juxtaposition, no unnecessary dots, parentheses, commas, or brackets. Also, we do not make the distinction between row and column vectors, though this seems still to be the vogue in many quarters, for what reason we cannot imagine. Perhaps we are carrying typographical economy too far when we denote the vector x with coordinates from ξ_1 to ξ_n by the symbol (ξ_i) instead of the conventional $(\xi_1, \ldots, \xi_n)$, but why not? After all, nobody objects to indicating a matrix A in terms of its coordinates by the symbol (α_{ij}). We have gone to considerable length to avoid hanging subscripts on subscripts. The general philosophy has been that a clean-looking page of symbols will have a good psychological effect on the reader, or to put it the other way, a tangled symbolism suggests a tangled argument and is likely to frighten rather than entice.

About the most radical innovation in terminology is the replacement of the universally used "nonsingular" by "regular" in describing a square matrix of maximal rank. We just didn't like the sound of the double negative. Vector spaces have a certain "rank" rather than "dimension" simply because there is no reason to use two words for the same thing. "Polyhedral cone" hasn't been around very long yet. Perhaps we can persuade others to join us in calling them "finite cones." It does sound better, and as Professor Coxeter has pointed out, "polyhedron" belongs to the 3-space just as "polygon" belongs to the plane. The correct n-dimensional word is "polytope," and this is the word that will be encountered here.

Our system for numbering displayed relations is admittedly unorthodox. In each proof we start numbering the relations from the beginning starting with (1). Thus, if we argue that a certain conclusion follows from (3) we are referring to (3) in that same proof.

If the reader disagrees with some of the liberties we have taken we hope he will simply attribute them to temperament and forgive us. To ensure against the possibility of serious confusion we have included a table of notations at the front of the book and an index of terms at the back.

Finally, a word concerning the bibliography. We have listed conscientiously at the end of each chapter all sources which were actually used in its preparation. We have, however, made no attempt at bibliographical completeness, as this is not generally done in textbooks. The people whose names appear in the bibliography at the end of the book represent but a fraction of those who have made significant contributions to the subject—an ever-dwindling fraction since new investigators are constantly entering the field. For the reader who is interested in bibliographical matters we recommend the very complete "Bibliography on Linear Programming and Related Techniques" by Riley and Gass (Johns Hopkins Press, Baltimore, 1958).

Acknowledgments

This book evolved in three distinct stages. The initial stage involved a course given in the academic year 1956–1957 to a group of graduate students in pure and applied mathematics at Brown University. From this course my assistant Edmund Eisenberg and I assembled a set of mimeographed notes which were made available to the public in a limited supply. This enterprise was carried out in part under a contract with the Logistics Branch of the Office of Naval Research, to which I am indebted not only for financial support but also for encouragement and interest in the project.

Because of the favorable response to the course notes I decided to expand them into a textbook. Most of this work was done while I was working as a consultant to the Mathematical Analysis Division of The RAND Corporation in 1957–1958. RAND not only supplied me with all the physical equipment needed for this operation but, even more

important, enabled me to do the work in the place having the highest concentration of contributors to the subject about which I was writing. For this stimulating atmosphere I am ultimately indebted to the United States Air Force, whose Project RAND contract has enabled The RAND Corporation to undertake its broad program of scientific research. Part of this volume was given limited circulation in three Project RAND research memoranda. I will not even attempt to list all the people at RAND who have helped me in one way or another on various portions of the exposition, but should like to give special thanks to J. D. Williams, head of the Mathematics Division, who made it possible for me to come to RAND.

The final stages of writing were completed at Brown University in the fall of 1958, again with the support of the Logistics Branch of the Office of Naval Research.

Finally, I should like to thank Professor E. Barankin of the University of California whose suggestions based on a critical reading of the mimeographed course notes led me to make fairly extensive revisions in my original organization of material.

To all the above groups and individuals let me convey my gratitude and express the hope that the finished product presented herewith will to some extent justify their support.

David Gale

List of Notations

Below are listed the principal mathematical notations used in this book. The notations are listed in the order in which they occur in the text.

$\alpha, \beta, \gamma, \ldots, \xi, \eta, \zeta$, and other Greek letters represent numerical quantities also referred to as *scalars*

$a, b, c, \ldots, x, y, z$, and other italic letters represent vector quantities

$x = (\xi_i)$ the vector whose ith coordinate is ξ_i

$b = (\beta_j)$ the vector whose jth coordinate is β_j

$y = (\eta_1, \ldots, \eta_n)$ the vector whose coordinates are $\eta_1, \ldots, \eta_n$

F^n the set of all n-vectors over the field F

R^n *n space*, the set of all real n-vectors

u, v the *unit vectors* all of whose coordinates are one

$u_i, (v_j)$ the *ith* (*jth*) *unit vector* whose ith (jth) coordinate is one and whose other coordinates are zero

λx product of scalar λ with vector x

ε symbol for set-theoretic membership, "is an element of"

xy *scalar product* of vectors x and y

$A = (\alpha_{ij})$ the *matrix* whose ijth coordinate is α_{ij}

$a_i = (\alpha_{i1}, \ldots, \alpha_{in})$ the ith *row vector* of the matrix A

$a^j = (\alpha_{1j}, \ldots, \alpha_{mj})$ the jth *column vector* of the matrix A

$xA, (Ay)$ the product of the matrix A with the vector $x(y)$

L linear subspace of a vector space

L^* orthogonal or *dual* subspace of L

$x \geqq 0$ vector x is *nonnegative*

$x \geq 0$ vector x is *semipositive*

$x > 0$ vector x is *positive*

$M = \{1, \ldots, m\}$, $N = \{1, \ldots, n\}$ the set of positive integers from 1 to m and 1 to n, respectively

$\subset, \supset$ set-theoretic inclusion, "is contained in" and "contains," respectively

$\{x|P\}$ the set of all x such that x has property P

$\cup$ set-theoretic *union*

$\cap$ set-theoretic *intersection*

C convex cone

$C_1 + C_2$ algebraic *sum* of convex cones

C^* *dual cone* of C

P the *positive orthant*, all nonnegative vectors

(b) the *halfline* generated by the vector b

$(b)^*$ the *halfspace* generated by the vector b

$(a_1) + \cdots + (a_m)$ the finite cone generated by $a_1, \ldots, a_m$

K convex set

$<X>$ the *convex hull* of the set X

$<x_1, \ldots, x_n>$ the convex hull of vectors $x_1, \ldots, x_n$

$I = (\delta_{ij})$ the *identity matrix*

A^{-1} the *inverse* of the matrix A

A^* the *transpose* of the matrix A

$x \succ 0$ x is *lexicographically positive*

(N, k) *capacitated network* with *nodes* N and *capacity function* k

(x, y) *edge* from node x to node y

$g(A)$ values of function on nodes A of N given by $g(A) = \sum_{x \epsilon A} g(x)$

$h(A, B)$ value of function on edges from A to B given by $h(A, B) = \sum_{x \epsilon A, y \epsilon B} h(x, y)$

s *source* in a network

s' *sink* in a network

(S, S') a *cut* in a network

∞ symbol for infinity

Γ two-person zero-sum game