Integrated Theory of Finite Element Methods

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PREFACE

The use of finite element methods for the analysis of structures and continua has become commonplace in many engineering fields—aeronautical, civil, automobile, mechanical, shipbuilding, etc. The early books on finite element methods more or less covered the whole field of achievement. However, it is noticeable that the more recent books have concentrated solely on one method, namely the Displacement Method. Research and development has made very rapid progress during the past decade and has branched out into the areas of statics, dynamics, stability, non-linear stress–strain, large displacements, optimization, etc. A person directly involved in finite element method research now finds it impossible to keep up with the whole field and quickly finds himself a specialist in a specialized area. It has now become impossible to write one book which adequately covers one of the areas, let alone the whole field. The present book concentrates on the area of elastostatics.

In Chapter 1 a review of finite element methods is presented. This covers Basic Relationships, Basic Force Method, Basic Displacement Method, Analogous Displacement Method, Direct Stiffness Method, Combined Methods I, II and III (the third being the method of Lagrange multipliers), Rank Technique for automatically selecting redundancies, Rank Force Methods I and II, Optimization of Redundancy Selection, Rank Force Method III, Eigen Force Method, Eigen Displacement Method (these last two methods solve static problems using an eigenvalue formulation), Stress Function Method, Unified Force–Deformation Method, Matrix Deformation Method and Static Force Method. All the methods are presented in a standardized manner so that the interrelationships between the various procedures can be readily appreciated. In fact, this standardized approach to all methods was predominant in the developments given in Chapter 2.

The second chapter develops the concept of characteristic matrices for any finite element for elastostatic analysis. This approach permits complete interchangeability of different types of elements between independently developed analysis programs irrespective of the analysis method or basic element assumption. The analogy between element derivation procedures and the equality of principal matrices is developed. A widespread adoption of this concept would have far-reaching implications, facilitating communications and the exchange of new developments. and it would speed up practical application in the field of finite elements.

The characteristic matrices of an element are: (1) the elastic matrix (natural, non-singular, stiffness or flexibility), (2) the initial deformation or force vector due to initial strains, (3) the assembly matrix and (4) the output matrix. These matrices have been derived using both strain and stress assumptions for various types of elements in Chapters 3 and 4. The strain-assumed elements are given in Chapter 3 and the stress-assumed elements in Chapter 4.

The first and second characteristic matrices form the natural or consistent force–deformation relationship for the element. The conventional force–deformation relationship of displacement (strain) elements is inconsistent, that is, the element stiffness matrix is singular. In the more recent stress function elements the force–deformation relationship is also inconsistent, that is, the element flexibility matrix is singular. In Chapter 5 these inconsistent relationships are developed for both displacement and stress function elements. The inconsistency is due to the presence of rigid body modes; a method of automatically extracting these is given that also leads to an automatic procedure for formulating the element characteristic matrices from inconsistent relations. The theory is demonstrated for both displacement and stress function elements.

In Chapter 6 the family of isoparametric strain (displacement) elements is presented. In these elements the displacement interpolation functions are assumed to be the same as the shape interpolation functions. This procedure enables elements to be developed with curved boundaries while still ensuring displacement continuity between adjacent element boundaries. This family of elements contains axials, membranes and solids but no plate-bending elements.

In Chapter 7 the family of isoparametric stress elements is presented. In these elements the stress interpolation functions are assumed to be the same as the shape interpolation functions. This procedure enables elements to be developed with curved boundaries whilst, in this case, ensuring continuity of forces between adjacent element boundaries. This family of elements contains axials, plates and solids but no membrane elements. The theory of Chapter 7 is new and leads to a more general definition of an isoparametric element; an isoparametric element is one whose elastic (stress or strain) interpolation functions are assumed to be the same as the shape interpolation functions. Such an element exists if the elastic (stress or strain) and shape coordinates are equal in number.

Chapter 8 redevelops, with some extensions and in the form of characteristic matrices, two combinations of axial and shear panel elements. The first is the well-established constant load axial, with constant area, and the four-node warped quadrilateral constant shear panel. The second is the linear load axial, with linear area, and the eight-node warped quadrilateral constant shear panel. The latter combination of two elements gives force continuity directly. They are very simple elements and are used extensively in aircraft design. They are ideal for initial analyses and structural optimization because of their simplicity.

In Chapter 9 the characteristic matrices of a curved beam element with constant curvature, uniform section properties, no distributed loading and six degrees of freedom at each end, are developed. The element is considered in three dimensions and can be used in any finite element program. Many structures, such as general frames and piping assemblies, contain curved beams. However, in an analysis, these are usually replaced by a series of straight beams. The curved beam element has existed for many years and is a useful addition to any finite element library. Unfortunately, very few analysis programs offer it at present.

The finite element analysis of a structure is carried out in three distinct stages, namely input preparation, solution, and output interpretation. The first stage requires that the actual structure be replaced by a simpler model. The simplified model is achieved by a 'lumping' procedure. In aircraft design, for example, a number of stringers plus an effective amount of skin are lumped together to form an equivalent axial element. The model is then analysed to give the stress distributions in the 'equivalent' structure. To obtain results for the actual structure (third stage), a 'delumping' procedure has to be carried out. The lumping and delumping stages of analysis are very time consuming, prone to error and costly. To minimize the lumping and delumping stages a new family of elements has been developed. These are referred to as 'Semi-Monocoque Elements'. In Chapter 10 a four-node warped semi-monocoque quadrilateral membrane element is developed and evaluated. The element is based on stress assumptions but can be employed in any of the finite element methods.

It should be noted that the shear panels of Chapter 8 and the membrane element of Chapter 10 are quadrilateral and warped and therefore very practical.

When applying finite element techniques to analyse structural problems which contain regions of high stress gradients it is necessary to increase the detail of the finite element model to obtain accurate results. In the case of structures containing a crack the stress gradients approach infinity in the immediate vicinity of the crack tip and conventional finite element modelling techniques are either inadequate or inefficient. The object is therefore to develop an element which contains steep stress gradients internally and actually contains the crack tip. Such an element would be very practical for investigating the effects of cracks in general structural configurations. To meet this need a family of cracked finite elements have been developed. These are given in Chapter 11. The characteristic matrices of a seven-node crack element, an eleven-node crack element and a symmetric crack element are given together with examples to check their subroutines and their incorporation into the main program.

The elements contained in Chapters 8 to 11 are very practical. Their characteristic matrices are given, which means that these elements can be readily incorporated into any analysis system by simply using the interchangeability relationships. When adding a new element to a system two main steps are involved; firstly an element subroutine has to be written and, secondly, it has to be placed in the overall program system. Obviously these steps have to be checked. Examples are therefore given for all the elements to check both the element subroutines and their correct incorporation into the system.

In the analysis of a large complex structure it is necessary, for several economic, computer capacity or organizational reasons, to divide it into a number of smaller substructures. As an example, the fuselage and wings of an aeroplane may be designed at different locations which are thousands of miles apart; indeed, they may be designed in different countries. Substructuring is essential in these cases. Each substructure is considered as though it were a separate problem. The various substructures are finally coupled together in such a way that equilibrium and continuity are satisfied at common boundaries. The substructure coupling procedure for the displacement method is given in Chapter 12.

Many lecturers and researchers have the misconception that there is nothing left to do in the area of elastostatics. It can only be assumed that such people have no practical experience in the application of finite element methods in a production environment where the analysis covers a period of about two years and each design iteration takes about two or three months. The biggest headache in static applications is the preparation of input and the interpretation of output. Considerable research is still required to minimize these two design phases.

This book contains a considerable amount of new material not to be found in any other book. It is hoped that its content will encourage new, interesting and practical research and development activities, particularly in the areas of force continuity, isoparametric stress elements and the family of semi-monocoque elements. The concept of element characteristic matrices and the interchangeability proposal should also be considered very seriously for new and future developments of computerized analysis systems. Also, from the point of view of standardization, it is felt that research should continue in the eigenvalue formulation of static problems. This may prove to be important in the future since it places statics, dynamics and stability on the same footing.

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